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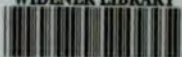
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DEPARTMENT OF COMMERCE AND LABOR

BULLETIN
OF THE
BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

VOLUME 3

(Nos. 1, 2, 3, 4)

1907



WASHINGTON
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1907

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DEPARTMENT OF COMMERCE AND LABOR

Vol. 3

BULLETIN

No. 1

OF THE

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S. W. STRATTON, DIRECTOR

APRIL, 1907

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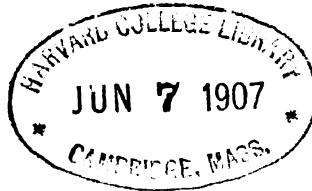
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FRANK L. ...
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ON THE GEOMETRICAL MEAN DISTANCES OF RECTANGULAR AREAS AND THE CALCULATION OF SELF-INDUCTANCE.

By Edward B. Rosa.

1. THE FORMULÆ.

The formula for the self-inductance of a multiple layer coil of rectangular section is obtained by integrating an expression for mutual inductance twice over the area of the section of the coil. If $dx dy$ is an infinitesimal element of the section ABCD at P and $dx' dy'$ is a second element at Q, the mutual inductance of the two circles of which these elementary areas are sections is given by the following formula of Maxwell:¹

$$M = 4\pi a \left\{ \log \frac{8a}{r} \left(1 + \frac{y'}{2a} + \frac{3x^2 + y^2}{16a^2} - \dots \right) - \left(2 + \frac{y}{2a} + \frac{x^2 - 3y^2}{16a^2} - \dots \right) \right\} \quad (I)$$

where a is the radius of the smaller circle, $a + y$ is the radius of the larger circle, x is the distance between their planes and r , the distance between the two infinitesimal sections, is $\sqrt{x^2 + y^2}$.

If we integrate equation (I) with respect to $dx' dy'$ over the entire rectangle ABCD and divide by the area of this rectangle, we shall have the mutual inductance of the circle whose section is $dx dy$ and the large ring whose section is ABCD. If we then integrate this expression with respect to $dx dy$ again over the area ABCD (and divide again by the area), we shall have the mutual inductance of the ring whose section is ABCD on itself, which is of course its self-inductance. The current is sup-

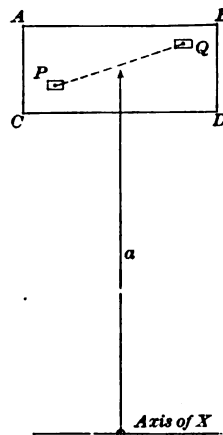


Fig. 1

¹ Electricity and Magnetism, II, § 705.

posed to be uniformly distributed over the entire cross section of the ring.

If the area be subdivided into n equal squares, it may be taken to represent the section of a circular coil wound with square wire, the latter being insulated by a covering of infinitesimal thickness. The self-inductance of such a coil is n^2 times as great as that of a single ring occupying the same space. Hence by inserting n^2 as a factor in the expression previously derived the formula for the self-inductance of a coil of wire of n turns, mean radius a , axial width b , and radial depth c was derived by Weinstein.²

Stefan³ put the formula into more convenient form for calculation by combining several terms which depend only on the ratio of b to c into two terms, y_1 and y_2 , the values of which could be taken from tables prepared by him for the purpose. Stefan's formula is as follows:

$$L = 4\pi n^2 a \left\{ \log \frac{8a}{\sqrt{b^2 + c^2}} \left(1 + \frac{3b^2 + c^2}{96a^2} \right) - y_1 + \frac{b^2}{16a^2} y_2 \right\} \quad (2)$$

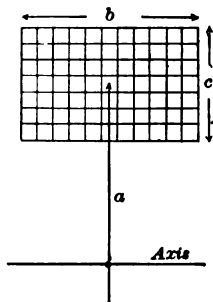


Fig. 2

where b and c are the breadth and depth of the coil as stated above, a is the mean radius, n is the number of turns, and y_1 and y_2 are taken from the table, using $x = \frac{b}{c}$ or $\frac{c}{b}$ as arguments, x being never greater than unity. This table is given in the appendix to this article.

When the section of the coil is square Weinstein's formula is much simplified as follows, where c is the length of one side of the square section:

$$L = 4\pi n^2 a \left\{ \log \frac{8a}{c} \left(1 + \frac{c^2}{24a^2} \right) + .03657 \frac{c^2}{a^2} - 1.194913 \right\} \quad (3)$$

Formula (2) may be used where b and c have any ratio; formula (3) applies only where $b=c$. Neither is accurate unless b and c are much less than the radius a .

² Wied. Annalen, **21**, pp. 353-354; 1884; Maxwell, *Elect. & Mag.*, 3d ed., II, p. 350.

³ Wied. Annalen, **22**, p. 107; 1884; and Sitzungsberichte Kais. Akad. der Wiss., Wien, **88** (2), p. 1201; 1883.

2. THE CORRECTION TERMS FOR INSULATED ROUND WIRE.

As stated above, these formulæ apply only to the case of a coil of square wire insulated by a covering of infinitesimal thickness, since the current is supposed to be distributed uniformly over the entire cross section of the coil. This is a condition impossible to realize in practice. When the coil is made up of round wire insulated by a covering of appreciable thickness, the self-inductance is different, as Maxwell pointed out,⁴ for three distinct reasons:

(1) The self-inductance of each turn of round wire is greater than that of a square wire when the diameter of the round wire is equal to that of the square, since the geometric mean distance R of a square from itself is greater than that of the inscribed circle, and of course (2) it is still greater than that of a smaller circle.

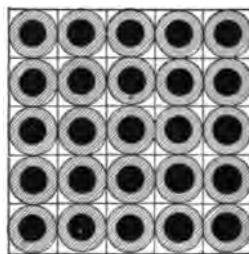


Fig. 8

For the square, $\log R_s = \log D + \frac{1}{3} \log 2 + \frac{\pi}{3} - \frac{25}{12}$. For the smaller circle $\log R_c = \log d - \log 2 - \frac{1}{4}$ where D is the side of the square and d is the diameter of the circle. The difference is

$$\begin{aligned} \log \frac{R_s}{R_c} &= \log \frac{D}{d} + \frac{4}{3} \log 2 + \frac{\pi}{3} - \frac{11}{6} \\ &= \log \frac{D}{d} + 0.1380605 \end{aligned} \quad (4)$$

Hence, the excess of the self-inductance of the coil of round wires due to the greater self-inductance of each of the n turns is

$$\Delta_1 L = 4\pi n a \left\{ \log_e \frac{D}{d} + 0.1380605 \right\} \quad (5)$$

(3) But in addition to this, the mutual inductances of the round wires on each other are different from the mutual inductances of the square wires on one another. Maxwell states⁵ that "the induc-

⁴ Electricity and Magnetism, II, § 693.

⁵ Electricity and Magnetism, II, § 693.

tive action of the eight nearest round wires on the wire A under consideration is less than that of the corresponding eight square wires on the square wire in the middle by 2×0.01971 " per unit of length, Fig. 4. "The correction for the wires at greater distance may be neglected," says Maxwell, and hence the total correction may be written, subtracting 0.01971 from 0.13806,

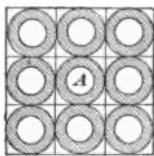


Fig. 4

$$\Delta L = 4\pi na \left\{ \log \frac{D}{a} + 0.11835 \right\} \quad (6)$$

This value of ΔL is to be added to L_u computed by Weinstein's or Stefan's formula to give L the corrected self-inductance of the coil.

Stefan's paper giving his modification of Weinstein's formula was published in 1884, and in this paper he stated that Maxwell's value of the absolute term in the correction ΔL (0.11835) was not right,⁶ but that it ought to be 0.15494. He did not give his derivation of this constant, but the context suggests that it is obtained by the use of the summation formula for self-inductance, and not as Maxwell had done by means of the geometrical mean distance.

In the third edition of Maxwell the value of the correction due to the difference in the mutual inductances of round wires and of square wires is recomputed by Mr. Chree, using the values of the geometrical mean distances given by Maxwell in his paper "On the Geometrical Mean Distances of Two Figures in a Plane."⁷ He obtains substantially the value given previously, namely 0.118425. The correction for reducing from a square to a circle (0.1380605) is right, and Stefan undoubtedly so understood it. The difference between Maxwell and Stefan was as to the correction for the mutual inductances ΔM , where

$$\Delta M = 4\pi anE \quad (7)$$

According to Maxwell, $E = -0.01971$

According to Stefan, $E = +0.01688$

That is, the numerical values of the correction are of opposite sign, although of the same order of magnitude numerically. As Stefan did not show how he obtained his value of the correction, and Maxwell

⁶ Wied. Annalen, 22, p. 116; 1884.

⁷ Transactions of the Royal Society of Edinburgh, 26, p. 729; 1872.

did not show how he obtained the values of the geometrical mean distances from which his value is computed, the discrepancy has not been explained. Accordingly, some authors give Maxwell's value and some give Stefan's.

I shall, in what follows, derive the correction first by Maxwell's method, using the geometrical mean distances, and then verify it by means of formulæ of self-inductance. I shall show that Maxwell's values of the geometrical mean distances of neighboring squares are wrong, and hence that his value for E (equation 7) is wrong; that Stefan's value is right as to sign, but only a first approximation as to magnitude; and that the value of this correction term given by Stefan as constant for all cases is not a constant, but is a function of the number of turns in the coil, and the shape of its cross section.

3. THE GEOMETRIC MEAN DISTANCE OF ADJACENT SQUARES.

Maxwell states⁷ that the geometrical mean distance of two squares side by side is .99401 times the distance of their centers of gravity, and that the g. m. d. of two squares corner to corner is 1.0011 times the distance of their centers of gravity. He does not, however, give the general expression for these distances.

Gray⁸ gives an excellent treatment of the subject of geometrical mean distances, but there are a number of misprints in equations 104, 109, 111, and 113 which it is necessary to correct before using them in numerical computations.⁹

Equation (8) below is Gray's equation (113) corrected, with c in place of b' and d in place of a' .

$$\alpha = \frac{1}{2} \left(d - a \right)$$

$$\beta = \frac{1}{2} \left(d + a \right)$$

⁷ Absolute Measurements, Vol. II, Part I, pp. 288-306.

⁹ The sign of the first term of equation 104 should be +. The signs before p^2 in the coefficients of the log in the first four terms of equation 109 should all be minus; thus $\frac{1}{4}(\beta^2 - p^2)$, $-\frac{1}{4}(\alpha^2 - p^2)$, $-\frac{1}{4}[(a - \beta)^2 - p^2]$, $+\frac{1}{4}[(a - \alpha)^2 - p^2]$. Similarly in equation 111 the coefficients of the first two terms should be $\frac{1}{2}(\beta^2 - p^2)$ and $-\frac{1}{2}(\alpha^2 - p^2)$. In equation 113 the coefficient of β^4 in each of the first four terms should be $\frac{1}{2}$ instead of $\frac{1}{4}$ and the first term should have $\log[(p + b + b')^2 + \beta^2]$ instead of $\log[(p + b + b')^2 - \beta^2]$.

p = distance between the rectangles, which are symmetrically placed, as shown in Fig. 5. R is the geometrical mean distance between the rectangles.

$$\begin{aligned}
 abcd \log R = & \frac{1}{4} \left[(p+b+c)^2 \left\{ \beta^2 - \frac{(p+b+c)^2}{6} \right\} - \frac{\beta^4}{6} \right] \log \left((p+b+c)^2 + \beta^2 \right) \\
 & - \frac{1}{4} \left[(p+b)^2 \left\{ \beta^2 - \frac{(p+b)^2}{6} \right\} - \frac{\beta^4}{6} \right] \log \left((p+b)^2 + \beta^2 \right) \\
 & - \frac{1}{4} \left[(p+c)^2 \left\{ \beta^2 - \frac{(p+c)^2}{6} \right\} - \frac{\beta^4}{6} \right] \log \left((p+c)^2 + \beta^2 \right) \\
 & + \frac{1}{4} \left[p^2 \left\{ \beta^2 - \frac{p^2}{6} \right\} - \frac{\beta^4}{6} \right] \log (p^2 + \beta^2) \\
 & - (\text{the same series of terms with } \beta \text{ replaced by } a) \\
 & + \frac{\beta}{3} (p+b+c)^3 \tan^{-1} \frac{\beta}{p+b+c} + \frac{\beta^3}{3} (p+b+c) \tan^{-1} \frac{p+b+c}{\beta} \\
 & - \frac{\beta}{3} (p+b)^3 \tan^{-1} \frac{\beta}{p+b} - \frac{\beta^3}{3} (p+b) \tan^{-1} \frac{p+b}{\beta} \\
 & - \frac{\beta}{3} (p+c)^3 \tan^{-1} \frac{\beta}{p+c} - \frac{\beta^3}{3} (p+c) \tan^{-1} \frac{p+c}{\beta} \\
 & + \frac{\beta}{3} p^3 \tan^{-1} \frac{\beta}{p} + \frac{\beta^3}{3} p \tan^{-1} \frac{p}{\beta} \\
 & - (\text{the same series of terms with } \beta \text{ replaced by } a) \\
 & - \frac{(\beta^2 - a^2)}{8} \left\{ (p+b+c)^2 - (p+b)^2 - (p+c)^2 + p^2 \right\} - \frac{11}{6} abcd.
 \end{aligned}
 \tag{8}$$

Equation (8a) below gives the g. m. d. for two *adjacent* symmetrically placed rectangles. The sides of these rectangles are a , b , and c , d , respectively, as before, the distance p apart being reduced to zero, Fig. 5a.

$$a = \frac{1}{2} (d - a)$$

$$\beta = \frac{1}{2} (d + a)$$

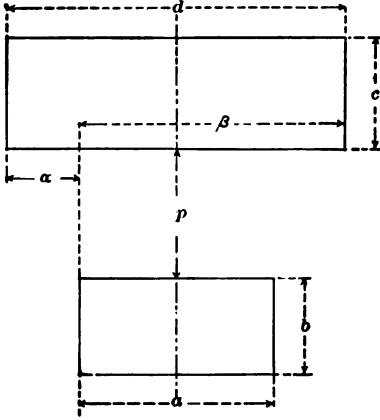


Fig. 5

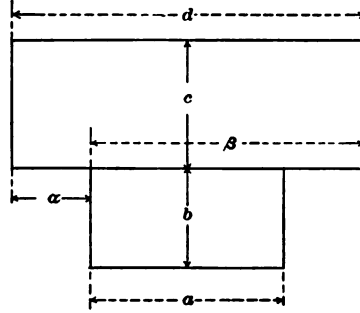


Fig. 5a

$$\begin{aligned}
 abcd \log R = & \frac{1}{4} \left[(b+c)^3 \left\{ \beta^2 - \frac{(b+c)^2}{6} \right\} - \frac{\beta^3}{6} \right] \log \left\{ (b+c)^2 + \beta^2 \right\} \\
 & - \frac{1}{4} \left[b^3 \left\{ \beta^2 - \frac{b^2}{6} \right\} - \frac{\beta^3}{6} \right] \log (b^2 + \beta^2) \\
 & - \frac{1}{4} \left[c^3 \left\{ \beta^2 - \frac{c^2}{6} \right\} - \frac{\beta^3}{6} \right] \log (c^2 + \beta^2) - \frac{\beta^4}{24} \log \beta^2 \\
 & - (\text{the same series of terms with } \beta \text{ replaced by } a) \\
 & + \frac{\beta}{3} (b+c)^3 \tan^{-1} \frac{\beta}{b+c} + \frac{\beta^3}{3} (b+c) \tan^{-1} \frac{b+c}{\beta} \\
 & - \frac{\beta b^3}{3} \tan^{-1} \frac{\beta}{b} - \frac{\beta^3 b}{3} \tan^{-1} \frac{b}{\beta} \\
 & - \frac{\beta c^3}{3} \tan^{-1} \frac{\beta}{c} - \frac{\beta^3 c}{3} \tan^{-1} \frac{c}{\beta} \\
 & - (\text{the same series of trigonometrical terms with } \beta \\
 & \quad \text{replaced by } a) \\
 & - \frac{1}{8} (\beta^2 - a^2) \left\{ (b+c)^2 - b^2 - c^2 \right\} - \frac{11}{6} abcd \quad (8a)
 \end{aligned}$$

For two adjacent squares of side a , $a=b=c=d$, $\alpha=0$, $\beta=a$, and equation (8a) reduces to the following:

$$\begin{aligned} a^4 \log R_1 = & \frac{7a^4}{24} \log 5a^2 - \frac{a^4}{6} \log 2a^2 - \frac{a^4}{6} \log 2a^2 - \frac{a^4}{24} \log a^2 + \frac{2a^4}{3} \log 4a^2 \\ & - \frac{2a^4}{24} \log a^2 + \frac{8a^4}{3} \tan^{-1} \frac{1}{2} + \frac{2a^4}{3} \tan^{-1} 2 - \frac{4a^4}{3} \tan^{-1} 1 - \frac{25}{12} a^4 \\ \therefore \log R_1 = & \log a + \frac{7}{24} \log 5 + \log 2 + 2 \tan^{-1} \frac{1}{2} - \frac{25}{12} \end{aligned} \quad (9)$$

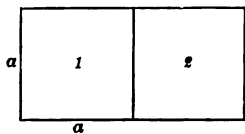


Fig. 6

$$\frac{7}{24} \log 5 = 0.4694194$$

$$\log 2 = 0.6931472$$

$$2 \tan^{-1} \frac{1}{2} = 0.9272952$$

$$2.0898618$$

$$-\frac{25}{12} = \frac{2.0833333}{0.0065285} = a_1 = \log \frac{R_1}{a}$$

$$\therefore \log R_1 = \log a + .0065285 \quad (10)$$

$$= \log (1.0065498 a)$$

$$\therefore R_1 = 1.0065498 a, \text{ (instead of } .99401 a \text{ as given by Maxwell)}$$

Therefore, the *g. m. d.* of two adjacent squares is 1.0065498 times the distance between their centers of gravity.

It is to be noticed that formula (9) is an exact one, and any required degree of accuracy can be attained in the value of R by carrying out the values of the four terms to the required number of decimal places. The degree of accuracy above is greater than is ordinarily necessary, but not greater than is desirable for the present purpose.

4. GEOMETRIC MEAN DISTANCE OF OTHER SQUARES IN THE SAME ROW OR COLUMN.

For two squares, as 1 and 3, whose centers are distant $2a$, the geometric mean distance is given by the following expression, obtained from equation (8):

$$\log R_s = \log a - \frac{7}{4} \log 5 + \frac{27}{4} \log 3 - \frac{11}{3} \log 2 + 8 \tan^{-1} \frac{1}{3} - 4 \tan^{-1} \frac{1}{2} - \frac{25}{12} \quad (11)$$

Substituting numerical values as before,

$$\begin{aligned} \log R_s &= \log a + .6936576 \\ &= \log (2.0010212a) \\ \therefore R_s &= 2.0010212a \quad (12) \\ R_s &= 1.0005106 \times 2a \end{aligned}$$

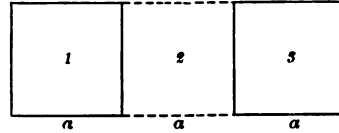


Fig. 7

Therefore, the g. m. d. of the first and third squares is 1.0005106 times the distance between their centers of gravity.

Since the g. m. d. between the circles inscribed in these two squares is the distance between their centers, the excess of the $\log R_s$ over $\log R'_s$ (which is $\log 2$) is $0.6936576 - .6931472 = .0005104 = a_3$. Thus a_3 is the naperian logarithm of $1.0005106 = R_s \div 2a$.

We may obtain a check on the values of a_2 and a_3 by the following method. We first obtain the geometric mean distance of square 1 (Fig. 7) from the rectangle made up of squares 2 and 3, by means of equation (8a). The equation will be,

$$\begin{aligned} \log R_s &= \log a + 4 \tan^{-1} \frac{1}{3} - \tan^{-1} \frac{1}{2} - \frac{3}{4} \log 2 - \frac{7}{48} \log 5 \\ &\quad + \frac{27}{8} \log 3 - \frac{7}{12} \log 10 - \frac{25}{12} \\ &= 0.35009307 + \log a \\ \therefore 2 \log R_s &= 0.70018606 + 2 \log a \end{aligned}$$

The geometric mean distance between the circles inscribed in these squares being the distances of their centers, we have, when $a = 1$,

$$\begin{aligned} 2 \log R_r &= \log 1 + \log 2 = .69314718 \\ \text{Thus, } 2 \log R_s - 2 \log R_r &= a_2 + a_3 = .0070389 \\ \text{Above we found } a_2 &= .0065285 \\ a_3 &= .0005104 \\ \text{Sum } a_2 + a_3 &= .0070389 \end{aligned}$$

This exact agreement between the results of two independent calculations assures the correctness of a_2 and a_3 .

The g. m. d. of the first and fourth squares we will obtain by the second of the above methods. In equation (8a) substitute $b=a$, $c=3a$, $d=a$, and we find the g. m. d. of the square ABEF from the rectangle BCDE. In this way we find

$$3 \log R_{2,3,4} = 3 \log a + \frac{131}{3} \log 2 - \frac{27}{4} \log 3 + \frac{7}{6} \log 5 - \frac{161}{24} \log 17 \\ + 20 \tan^{-1} \frac{1}{4} - 8 \tan^{-1} \frac{1}{3} - \frac{25}{4} \quad (13)$$

$$= 3 \log a + 1.7989008 \quad (14)$$

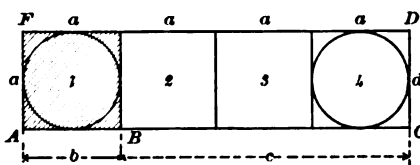


Fig. 9

By the theory of the geometrical mean distance the logarithms of the g. m. d. of the rectangle BCDE from the square is one-third the sum of the logarithms of the g. m. d.'s of each of the squares 2, 3, and 4, which make up the rectangle, from the first square. Therefore, since

$$\log R_2 = \log a + .0065285$$

$$\log R_3 = \log a + .6936576$$

and $\log R_4 = 3 \log R_{2,3,4} - (\log R_2 + \log R_3)$, we have

$$\log R_4 = \log a + 1.0987148$$

$$= \log (3.000307a),$$

$$\therefore R_4 = 1.000102 \times a$$

The g. m. d. of the circle inscribed in square 4 from the circle inscribed in square 1 is $3a$, the distance between their centers; therefore

$$\log R'_4 = \log 3a = \log a + 1.0986123$$

$$\log R_4 - \log R'_4 = .0001025 = a_4$$

Proceeding in the same way for the g. m. d. of squares 5 and 6, with respect to square 1, we find

$$\log R_5 - \log R'_5 = .0000325 = a_5$$

$$\log R_6 - \log R'_6 = .0000136 = a_6$$

In the calculation of the corrections due to the differences in the mutual inductances for round wires from their values for square wires the differences in logarithms of the g. m. d.'s are used, and not the g. m. d.'s themselves. Hence the differences a_2, a_3 , etc., are the important quantities to determine, rather than R_1, R_2 , etc. All the differences for parallel squares in the same row we have found to be positive, which shows that the mutual inductances of straight round wires at a given distance apart are greater than for square wires at the same distances. The same would be true for the squares in the same column with square number 1. Thus, we have found all the corrections to be applied to the middle wire for the 20 wires in the same row and column for a group of 121 making up a square 11 by 11. These corrections decrease rapidly as we go away from the wire in question, the first correction term being 10 times the sum of the next four.

						α_6					
						α_5					
						α_4					
						α_3					
						α_2					
α_8	α_7	a_4	α_3	α_2	α_1	α_2	α_3	α_4	α_5	α_6	
						α_7					
						α_6					
						α_5					
						α_4					
						α_3					
						α_2					

Fig. 9

5. CORRECTIONS FOR THE SECOND ROW.

If we find the g. m. d. of square 1 to the various squares in the second row, we can get the differences in the logarithms of the g. m. d.'s for squares and for circles; that is, the corrections $\beta_1, \beta_2, \beta_3$,

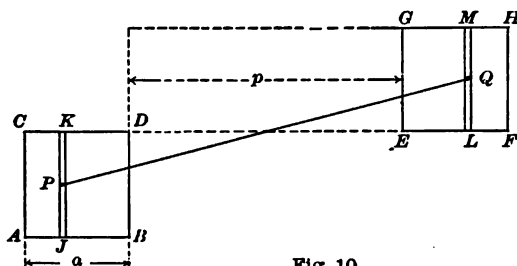


Fig. 10

etc., as we have found a_2, a_3, a_4 , etc., for the first row. Taking two squares, as shown in Fig. 10, we express the logarithm of the distance PQ in terms of the coordinates of P and Q. First, we inte-

grate with respect to P along the line JK, and find the g. m. d. from Q to the line JK. Second, integrate with respect to Q along LM, and so find the g. m. d. of LM from JK. Third, integrate with respect to JK along AB and find the g. m. d. of LM from the square ABCD. Fourth, integrate along EF and so find the g. m. d. of one square from the other.

The formula for the g. m. d. of two squares as shown in Fig. 10, their horizontal distance apart being p , determined as above is as follows:

$$\begin{aligned}
 \log R_{\beta} = & \left[\frac{(p+2a)^2}{2a^2} - \frac{(p+2a)^4}{48a^4} - \frac{1}{3} \right] \log \left(4a^2 + (p+2a)^2 \right) \\
 & - \left[\frac{(p+a)^2}{a^2} - \frac{(p+a)^4}{24a^4} - \frac{2}{3} \right] \log \left(4a^2 + (p+a)^2 \right) \\
 & + \left[\frac{p^2}{2a^2} - \frac{p^4}{48a^4} - \frac{1}{3} \right] \log (4a^2 + p^2) \\
 & - \left[\frac{(p+2a)^2}{4a^2} - \frac{(p+2a)^4}{24a^4} - \frac{1}{24} \right] \log \left(a^2 + (p+2a)^2 \right) \\
 & + \left[\frac{(p+a)^2}{2a^2} - \frac{(p+a)^4}{12a^4} - \frac{1}{12} \right] \log \left(a^2 + (p+a)^2 \right) \\
 & - \left[\frac{p^2}{4a^2} - \frac{p^4}{24a^4} - \frac{1}{24} \right] \log (a^2 + p^2) \\
 & - \frac{(p+2a)^4}{24a^4} \log (p+2a) + \frac{(p+2a)^4}{12a^4} \log (p+a) - \frac{p^4}{24a^4} \log p \\
 & + \frac{1}{3} \left[\frac{4p}{a} + 8 - \frac{(p+2a)^3}{a^3} \right] \tan^{-1} \frac{p+2a}{2a} - \frac{2}{3} \left[\frac{4p}{a} + 4 - \frac{(p+a)^3}{a^3} \right] \tan^{-1} \frac{p+a}{2a} \\
 & - \frac{1}{3} \left[\frac{p}{a} + 2 - \frac{(p+2a)^3}{a^3} \right] \tan^{-1} \frac{p+2a}{a} + \frac{2}{3} \left[\frac{p}{a} + 1 - \frac{(p+a)^3}{a^3} \right] \tan^{-1} \frac{p+a}{a} \\
 & + \frac{1}{3} \left[\frac{4p}{a} - \frac{p^3}{a^3} \right] \tan^{-1} \frac{p}{2a} - \frac{1}{3} \left[\frac{p}{a} - \frac{p^3}{a^3} \right] \tan^{-1} \frac{p}{a} \\
 & - \frac{1}{8} \left[\frac{(p+2a)^2}{a^2} - 2 \frac{(p+a)^2}{a^2} + \frac{p^2}{a^2} \right] - \frac{11}{6}
 \end{aligned} \tag{17}$$

For two squares corner to corner the distance p is zero and equation 17 reduces to the following:

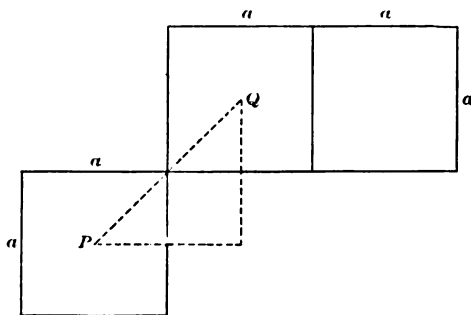


Fig. 11

$$\begin{aligned}\log R_{\beta_1} &= \log a + 3 \log 2 - \frac{7}{12} \log 5 + \pi - 4 \tan^{-1} \frac{1}{2} - \frac{25}{12} \\ &= \log a + 0.34427164 \\ &= \log (1.410962 a)\end{aligned}\quad (18)$$

The distance between centers of these two squares is $a\sqrt{2}$. Therefore,

$$\begin{aligned}\log R'_{\beta_1} &= \log a + \frac{1}{2} \log 2 = \log a + .3465736 \\ \log R_{\beta_1} - \log R'_{\beta_1} &= -.0023019 = \beta_1\end{aligned}$$

Also, $R_{\beta_1} = 0.997701 \times a\sqrt{2}$ (instead of $1.0011 \times a\sqrt{2}$ as given by Maxwell). That is, the g. m. d. of two squares, corner to corner, is 0.997701 times the distance between their centers.

In a similar manner, putting $p=a$ in equation (17) we find the g. m. d. for the next square in the second row, and also the correction, β_2 . Thus,

$$\begin{aligned}\log R_{\beta_2} &= \log a - \frac{17}{3} \log 2 - \frac{27}{8} \log 3 + \frac{91}{48} \log 5 + \frac{119}{48} \log 13 \\ &\quad - \frac{\pi}{2} - 8 \tan^{-1} \frac{1}{3} + 5 \tan^{-1} \frac{1}{2} + 5 \tan^{-1} \frac{2}{3} - \frac{25}{12} \\ &= \log a + .8046295 \\ \log R'_{\beta_2} &= \log a + \frac{1}{2} \log 5 \\ &= \log a + .8047190 \\ \therefore \log R_{\beta_2} - \log R'_{\beta_2} &= -.0000895 = \beta_2\end{aligned}\quad (19)$$

In a similar manner we find, putting $p=2a$ in equation (17)

$$\begin{aligned}\log R_{\beta_3} &= \log a - \frac{47}{3} \log 2 + \frac{27}{4} \log 3 - \frac{7}{24} \log 5 \\ &\quad - \frac{119}{24} \log 13 + \frac{161}{24} \log 17 \\ &\quad - 10 \tan^{-1} \frac{2}{3} - 20 \tan^{-1} \frac{1}{4} + 14 \tan^{-1} \frac{1}{2} + 16 \tan^{-1} \frac{1}{3} - \frac{25}{12} \quad (20) \\ &= \log a + 1.1513161 \\ \log R'_{\beta_3} &= \log a + \frac{1}{2} \log 10 = \log a + 1.1512925 \\ \therefore \log R_{\beta_3} - \log R'_{\beta_3} &= +.00002355 = \beta_3\end{aligned}$$

It is to be noticed that the correction β_3 is positive, whereas β_1 and β_2 have been negative.

6. CHECK ON VALUES OF $\beta_1, \beta_2, \beta_3$.

We may obtain a check on the values of $\beta_1, \beta_2, \beta_3$ just calculated by formula (17), by means of formula (8). The logarithm of the

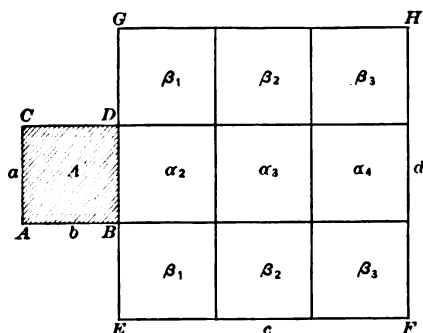


Fig. 12

g. m. d. of the square ABCD from the square EFGH (Fig. 12) is equal to the sum of the logarithms of the g. m. d.'s of A for the nine squares making up EFGH. Since a, a_2, a_4 are the differences of the logarithms of the corresponding squares and the inscribed circles, we may compute the sum of these differences by means of equation

(8a) and compare with the values found. This will give a check on the values of the β s.

Substituting in equation (8a) $a=1, \beta=2, b=1, c=3, a=1, d=3$ we obtain the following equation for the value of the $\log R$ for the adjacent squares ABCD and EFGH, Fig. 12:

$$\begin{aligned}9 \log R &= 9 \log a + 7 \log 2 + \frac{77}{24} \log 5 - \frac{119}{24} \log 13 + \frac{161}{24} \log 17 \\ &\quad + 30 \tan^{-1} \frac{1}{2} + 8 \tan^{-1} \frac{1}{3} - 20 \tan^{-1} \frac{1}{4} - 10 \tan^{-1} \frac{2}{3} + \pi - \frac{75}{4} \\ &= 9 \log a + 6.3993349 \quad (21)\end{aligned}$$

For the corresponding inscribed circles we have

$$\begin{aligned} 9 \log R' &= 9 \log a + 3 \log 2 + \log 3 + 2 \log 5 \\ &= 9 \log a + 6.3969297 \end{aligned} \quad (22)$$

$$\therefore 9 \log R - 9 \log R' = .0024056 = a_1 + a_2 + a_3 + 2(\beta_1 + \beta_2 + \beta_3) \quad (23)$$

This is the algebraic sum of the a s and β s for the nine squares as given by means of equation (8). We have already checked the values of the a s; if the a s and β s sum up to the above value we shall have also a check on the β s.

$$\begin{aligned} \text{Thus} \quad a_2 &= .0065285 \\ a_3 &= .0005104 \\ a_4 &= .0001024 \\ 2\beta_3 &= .0000471 \\ &\quad + .0071884 \\ 2\beta_1 &= -.0046038 \\ 2\beta_2 &= -.0001790 = -.0047828 \\ \hline \text{Sum} &= +.0024056 = a_1 + a_2 + a_3 + 2(\beta_1 + \beta_2 + \beta_3) \end{aligned} \quad (24)$$

This complete agreement with the value above (23) checks the correctness of the calculations of the β s made from formula (17).

Instead of finding β_4 by equation (17) we may use equation (8a),

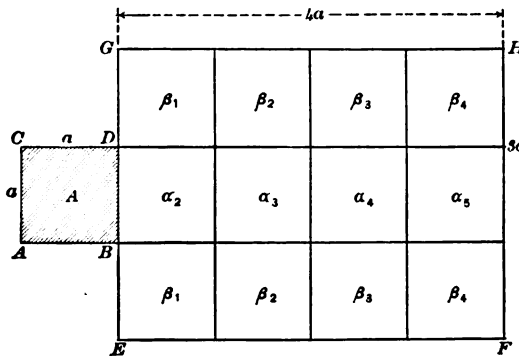


Fig. 13

thus: Find $\log R$ for square A with respect to square EFGH, and $\log R'$ for circle A with respect to the 12 circles inscribed in the 12 squares of EFGH. The difference gives the sum of the a s and β s.

Since all are separately known except β_4 , the latter then becomes known. Thus, by means of equation (8a) we have

$$\begin{aligned} 12 \log R &= 12 \log a + \frac{28}{3} \log 2 - \frac{119}{24} \log 5 + \frac{119}{6} \log 13 \\ &\quad - \frac{161}{24} \log 17 - \frac{41}{24} \log 29 + 20 \tan^{-1} \frac{1}{4} - 40 \tan^{-1} \frac{1}{5} \\ &\quad + 70 \tan^{-1} \frac{2}{5} - 34 \tan^{-1} \frac{1}{2} + \pi - 25 \\ &= 12 \log a + 10.6189076 \end{aligned} \quad (25)$$

For circle A and the 12 circles of EFGH we have

$$\begin{aligned} 12 \log R' &= 12 \log a + 5 \log 2 + \log 3 + 2 \log 5 + \log 17 \\ &= 12 \log a + 10.6164374 \\ \therefore 12 \log R - 12 \log R' &= .0024702 = a_2 + a_3 + a_4 + a_5 + 2(\beta_1 + \beta_2 + \beta_3 + \beta_4) \\ \text{By (23), } .0024056 &= a_2 + a_3 + a_4 + 2(\beta_1 + \beta_2 + \beta_3) \\ \therefore .0000646 &= a_5 + 2\beta_4 \\ \text{But } .0000325 &= a_5 \\ \therefore .0000321 &= 2\beta_4 \\ .0000160 &= \beta_4 \end{aligned}$$

7. CORRECTION FOR THE FIRST TWO LAYERS OF WIRE.

Considering the square ABCD made up of 25 smaller squares,

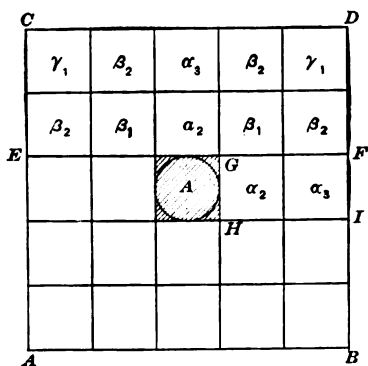


Fig. 14

the geometric mean distances of the middle square A from the rectangle CDEF and from the rectangle FGHI are given by formula (8a). If R_{10} is the geometric mean distance of A from the rectangle CDEF (made up of 10 small squares) and R_2 is the g. m. d. of A from the rectangle FGHI (consisting of 2 small squares), we have from formula (8a) when $a = 1$,

$$\begin{aligned} 10 \log R_{10} &= 6.68642698 \\ 2 \log R_2 &= 0.70018606 \end{aligned} \quad (26)$$

Adding these quantities and dividing by 12 we get the log of the g. m. d. of A from the whole space outside A inclosed within the larger square, that is, of A from the 24 small squares outside A. Calling this latter g. m. d. R_{24} we have

$$\log R_{24} = .61555109.$$

The g. m. d. of the circle inscribed in A from a circle inscribed in any of the other squares is of course the distance between their centers. The log of the g. m. d. of the circle in A from the 24 circles in the other squares is found from the following expression:

$$\begin{aligned} 24 \log R_{24} &= 4 \log 1 + 12 \log 2 + 4 \log 5 \\ &= 14.75551782 \\ \therefore \log R'_{24} &= .61481324. \end{aligned} \quad (27)$$

The mean correction for the mutual inductance of the 24 wires on the middle wire is the difference of these logarithms. Thus

$$\log R_{24} - \log R'_{24} = .00073785.$$

The total correction for the 24 wires making up the two layers surrounding wire A is 24 times as great, namely,

$$\Delta M_2 = .017708$$

The correction for the eight wires constituting the first layer surrounding A is $4(a_2 + \beta_1) = 4(.0065285 - .0023019)$. Thus

$$\Delta M_1 = .016906$$

Thus the difference, representing the correction due to the 16 wires in the second layer is about 5 per cent of the effect of the first 8 wires.

8. THE CORRECTIONS, $\gamma_1, \gamma_2, \gamma_3, \delta_1$.

We have found the corrections for all the wires separately except the corner ones, the correction for which is γ_1 . The sum of the corrections for the six wires of one quadrant is one-fourth of ΔM_2 , or .00442707. This is equal to

$$\begin{aligned} a_2 + a_3 + \beta_1 + 2\beta_2 + \gamma_1 \\ a_2 + a_3 &= .0070389 \\ \beta_1 + 2\beta_2 &= -.0024810 \\ \therefore a_2 + a_3 + \beta_1 + 2\beta_2 &= .0045579 \\ \text{Sum of 6} &= .0044271 \\ \therefore \gamma_1 &= -.0001308 \end{aligned}$$

In a similar manner, calculating the log of the g. m. d. of square A from the 15 squares making up the rectangle ABCD, and get-

ting the excess over the log for the corresponding inscribed circles we find the sum of the corrections for the 15 squares. All of them

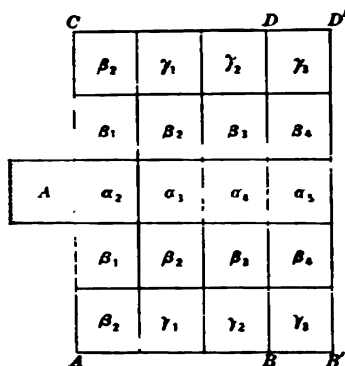


Fig. 15

having been previously calculated except γ_3 , the latter becomes known. Thus,

$$\gamma_3 = -.0000347$$

Instead of doing similarly with the rectangle AB'CD' to get γ_3 , I have computed the g. m. d. of the square A with respect to the rectangle BB'DD', consisting of 5 squares. The not very simple expression is as follows:

$$\begin{aligned} 5 \log R &= 59 \log 2 + 27 \log 3 - \frac{471}{6} \log 5 - \frac{119}{24} \log 13 \\ &\quad + \frac{161}{6} \log 17 + \frac{41}{24} \log 29 - 80 \tan^{-1} \frac{3}{5} - 64 \tan^{-1} \frac{1}{2} \\ &\quad - 56 \tan^{-1} \frac{3}{4} - 70 \tan^{-1} \frac{2}{5} - 10 \tan^{-1} \frac{2}{3} - \frac{125}{12} \\ &= 7.2152930 \end{aligned} \quad (28)$$

$5 \log R' = 7.2152400$, for the corresponding inscribed circles.

Difference = $0.0000530 = \alpha_5 + 2\beta_4 + 2\gamma_3$

$$\alpha_5 = .0000325$$

$$2\beta_4 = .0000321$$

$$\therefore \alpha_5 + 2\beta_4 = .0000646$$

$$\text{Sum of } 5 = .0000530$$

$$\therefore 2\gamma_3 = -.0000116$$

$$\gamma_3 = -.0000058$$

To get these small corrections accurately by this method of difference requires extending the values of the logarithms and inverse tangents to seven or eight decimal places. The above expression for $5 \log R$ gives that quantity as the difference between large positive and negative quantities ($225.24224423 - 218.02695118$), and then the total correction for the 5 squares is the very small difference between two large quantities ($7.2152930 - 7.2152400$). This small

difference is then the sum of 5 corrections, of which 3 are known. That these final small differences which are the corrections sought are accurately determined is proved by the exact agreement of their values as checked by different formulæ.

The value of δ_1 is $-.0000256$. Mr. F. W. Grover has kindly calculated for me the values of $\gamma_4, \delta_2, \delta_3, \epsilon_1, \epsilon_2, \zeta_1$, and verified the calculation of all the others. Figure (16) gives the 30 corrections for one quadrant; the other quadrants would of course have the same corrections for squares in corresponding positions. It will be noted that the α corrections at the right and left and above and below the square A are positive, whereas the corrections for squares at 45° with the principal axes as $\beta_1, \gamma_1, \delta_1, \epsilon_1, \zeta_1$, are negative. On either side of the 45° lines the corrections are also negative, as $\beta_2, \gamma_2, \gamma_3$, etc. Nearer the axes the corrections become positive, as $\beta_3, \beta_4, \beta_5, \gamma_4$. The correction δ_1 is four times γ_3 , although the squares are almost equally distant from A, because γ_3 lies at a lower angle, somewhat off the diagonal.

ζ_1	ϵ_2	δ_3	γ_4	β_5	α_6	+8.4 β_5	+0.7 γ_4	-4.0 δ_3	-4.5 ϵ_2	-3.1 ζ_1
ϵ_2	ϵ_1	δ_2	γ_3	β_4	α_5	+16.0 β_4	-5.8 γ_3	-11.3 δ_2	-8.1 ϵ_1	-4.5 ϵ_2
δ_3	δ_2	δ_1	γ_2	β_3	α_4	+23.6 β_3	-34.8 γ_2	-26.6 δ_1	-11.3 δ_2	-4.0 δ_3
γ_4	γ_3	γ_2	γ_1	β_2	α_3	-89.5 β_2	-130.8 γ_1	-34.8 γ_2	-5.8 γ_3	+0.7 γ_4
β_5	β_4	β_3	β_2	β_1	α_2	-2301.9 β_1	-89.5 β_2	+23.6 β_3	+16.0 β_4	+8.4 β_5
α_6	α_5	α_4	α_3	α_2	A	+6528.5 α_2	+510.4 α_3	+102.5 α_4	+32.5 α_5	+13.6 α_6
β_5	β_4	β_3	β_2	β_1	α_2	β_1	β_2	β_3	β_4	β_5
γ_4	γ_3	γ_2	γ_1	β_2	α_3	β_2	γ_1	γ_2	γ_3	γ_4
δ_3	δ_2	δ_1	γ_2	β_3	α_4	β_3	γ_2	δ_1	δ_2	δ_3
ϵ_2	ϵ_1	δ_2	γ_3	β_4	α_5	β_4	γ_3	δ_2	ϵ_1	ϵ_2
ζ_1	ϵ_2	δ_3	γ_4	β_5	α_6	β_5	γ_4	δ_3	ϵ_2	ζ_1

Fig. 16.

9. THE TOTAL CORRECTION FOR 3 LAYERS AND 5 LAYERS OF WIRES.

We have already found the correction for the first layer of 8 wires and the second layer of 16 wires. The correction for the third layer of 24 wires may now easily be calculated. It is

$$\begin{aligned}\Delta M_3 - \Delta M_2 &= 4(a_1 + 2\beta_2 + 2\gamma_3 + \delta_1) \\ &= 4(102.5 + 47.2 - 69.6 - 25.6) \times 10^{-6} \\ &= 0.000218 \\ \therefore \Delta M_3 &= 0.017708 + 0.000218 \\ &= 0.017926\end{aligned}\quad (29)$$

ΔM_3 is the correction for 3 layers of 48 wires.

Fig. 16 gives the correction terms for each of 120 squares on the central square A, in a larger square of 11 x 11 unit squares. The four quadrants are symmetrical, and in each quadrant values are symmetrical about the diagonal.

1170	1589	1614	1619	1621	1622	1622		184	184	185	187	192	217	636
1589	1777	1793	1800	1804	1804.5			1.4	1.5	2	6	13	29	217
1614	1793	1796	1800	1803	1804.8			1.2	1.2	3	6	10	13	192
1619	1800	1800	1801	1803	1804.7				1.3	3	5	6	6	187
1621	1804	1803	1803	1804	1805				1	2	3	3	2	185
1622	1804.5	1804.8	1804.7	1805	1806				0	1	1.3	1.2	1.5	184
1622	1804.5	1804.8	1804.7	1805	1806				0	1	1.3	1.2	1.5	184
1621	1804	1803	1803	1804	1805				1	2	3	3	2	185
1619	1800	1800	1801	1803	1804.7				1.3	3	5	6	6	187
1614	1793	1796	1800	1803	1804.8			1.2	1.2	3	6	10	13	192
1589	1777	1793	1800	1804	1804.5			1.4	1.5	2	6	13	29	217
1170	1589	1614	1619	1621	1622	1622		184	184	185	187	192	217	636

Fig. 17.

To find the correction for 120 wires constituting five layers we find the logarithm of the g. m. d. of the square A from the space outside of it contained within the larger square, the latter consisting

of 120 small squares. Then find the g. m. d. of the circle inscribed in the square A from the 120 circles inscribed in the 120 squares. The result is as follows:

$$\log R_{120} = 1.35674951, \text{ for the square}$$

$$\log R'_{120} = 1.35659900, \text{ for the circles}$$

$$\text{Difference} = .00015051$$

Multiplying by 120, $\Delta M_6 = .018061 = \text{total correction for 120 squares.}$

SUMMARY OF CORRECTIONS.

For 1st layer of 8 = .016906 $\therefore \Delta M_1 = .016906$ for 1 layer.
 For 2d layer of 16 = .000802 $\Delta M_2 = .017708$ for 2 layers.
 For 3d layer of 24 = .000218 $\Delta M_3 = .017926$ for 3 layers.
 For 4th & 5th layers of 72 = .000135 $\Delta M_5 = .018061$ for 5 layers.

These are the values of the correction for mutual induction for a single turn of wire in a coil surrounded by other wires, assuming the wires are of small section and the curvature small. Such wires as lie on or near the surface of the coil will evidently have smaller values for ΔM .

CORRECTION FOR MUTUAL INDUCTANCE NEAR SURFACE OF A COIL.

To find the correction to be made for a particular wire in a coil of rectangular section add together the corrections due to the neighboring wires. Thus, let NOP be a portion of the boundary of the section of a coil, one turn of wire occupying each square. Let us find the correction for the corner wire A_1 . The space to the right of N_1O_1P is a complete quadrant, and the correction due to it is one-fourth of .018061, or .004515. The wires in the column above A_1 give corrections a_2, a_3, a_4 , etc. $= \Sigma a = .007188$. The sum of these is .01170 which is the total correction for mutual inductance for the corner wire. The corrections for B_1 , Fig. 19, are as follows:

$$\text{For quadrant } N_1O_1P_1, .018061 \div 4 = .004515$$

$$\text{For } C_1, D_1, E_1, \text{ etc.} = \Sigma a = .007188$$

$$\text{For } A_1, a_2 = .006528$$

$$\text{Sum} = .018231$$

$$\text{For } A_2, A_3, A_4, \text{ etc., } \Sigma \beta = -.002343$$

$$\text{Total correction} = .015888$$

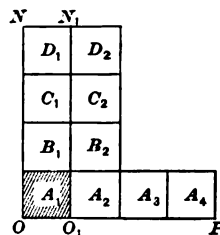


Fig. 18

In a similar manner the correction can be found for B_n , Fig. 20, or for any wire in a coil.

In Fig. 17 the values of ΔM are plotted on the left for the different wires, and the differences between the respective values and .01806, the maximum value, are plotted on the right, the numbers on the right being parts in 100,000. The differences are practically confined to the outer layer and a few wires in each corner. The average value of the differences for a coil of 10 x 10 turns is .00093, making the average correction +.01806-.00093 = +.01713.

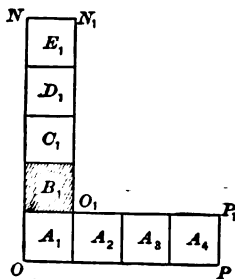


Fig. 19

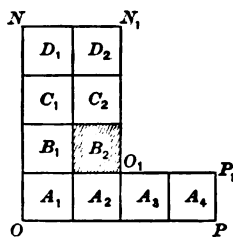


Fig. 20

$$\therefore \Delta M = 4\pi na \times .01713. \quad (30)$$

10. INDEPENDENT DETERMINATION OF GEOMETRIC MEAN DISTANCE OF ADJACENT SQUARES.

The geometric mean distance of a square from itself is given by the expression:

$$\log R = \log a + \frac{1}{3} \log 2 + \frac{\pi}{3} - \frac{25}{12} \quad (31)$$

$$\therefore R = .447049 \times a$$

where R is the geometric mean distance and a is the side of the square.

The geometric mean distance of a rectangle from itself is¹⁰

$$\begin{aligned} \log R = & \log \sqrt{a^2 + b^2} - \frac{1}{6} \frac{a^2}{b^2} \log \sqrt{1 + \frac{b^2}{a^2}} - \frac{1}{6} \frac{b^2}{a^2} \log \sqrt{1 + \frac{a^2}{b^2}} \\ & + \frac{2}{3} \frac{a}{b} \tan^{-1} \frac{b}{a} + \frac{2}{3} \frac{b}{a} \tan^{-1} \frac{a}{b} - \frac{25}{12} \end{aligned} \quad (32)$$

where a and b are the breadth and length of the rectangle respectively.

When $b = 2a$,

$$R = 0.670803 \times a$$

¹⁰ Maxwell, II, § 692.

Suppose ABCD, Fig. 21, represents a rectangle in which the length is two and the breadth one, and EF divides it into two unit squares. Let R be the g. m. d. of the entire rectangle from itself, R_1 the g. m. d. of the square from itself, and R_2 the g. m. d. of one square from the other. Then $R = .670803$, $R_1 = .447049$, and R_2 is to be determined.

By the principle of the geometric mean distance

$$2 \log R = \log R_1 + \log R_2 \quad (33)$$

$$\text{or } R^2 = R_1 R_2$$

$$2 \log_{10} R = \bar{1}.6531900$$

$$\log_{10} R_1 = \bar{1}.6503551$$

$$\therefore \log_{10} R_2 = 0.0028349$$

$$R_2 = 1.006549$$

$$\log_e R_2 = .006528 = a_2$$

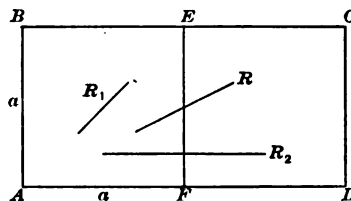


Fig. 21

This is the value of the geometric mean distance of adjacent squares already found by means of formula (8a), differing by less than one part in a million, this difference occurring because the logarithms are not carried out as far in this case as before.

11. SQUARES CORNER TO CORNER.

Let ABCD be a square of length 2 units, divided into four unit squares. Let R be the g. m. d. of the large square from itself $= .894098$; R_1 be the g. m. d. of each small square from itself $= .447049$; R_2 the g. m. d. of any small square from an adjacent one $= 1.006549$ (see above) and R_3 the g. m. d. of two squares corner to corner, as 1 from 4 or 2 from 3. R_3 is to be determined. By the principle of the geometrical mean distance,

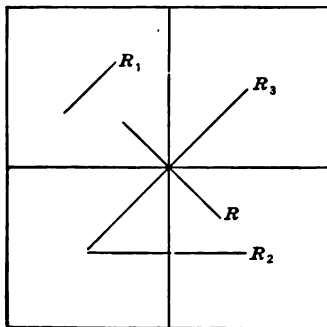


Fig. 22

$$4 \log R = \log R_1 + 2 \log R_2 + \log R_3 \quad (35)$$

$$\text{or, } R^4 = R_1 R_2^2 R_3$$

$$\log_{10} R_1 = \bar{1}.6503551$$

$$2 \log_{10} R_2 = 0.0056698$$

$$\text{Sum} = \bar{1}.6560249$$

$$4 \log R = \bar{1}.8055404$$

$$\log R_3 = 0.1495155$$

$$\frac{1}{2} \log 2 = 0.1505150$$

$$\therefore \log \frac{R_3}{\sqrt{2}} = \bar{1}.9990005$$

$$\therefore R_3 = 0.997701 \times \sqrt{2}$$

That is, the g. m. d. of two squares, corner to corner, is 0.997701 times the distance between their centers. This is the result previously found from equation (17).

12. SQUARES NOT ADJACENT; R_3 .

Let the rectangle of Fig. 23 consist of three unit squares. Let R = g. m. d. of the rectangle from itself, which formula (32) shows to be 0.894657; R_1 = g. m. d. of a single square from itself = 0.447049, R_2 = g. m. d. of one square on adjacent one = 1.006549, R_3 = g. m. d. of square 1 on square 3, which is to be determined. By the principle of the geometrical mean distance

$$9 \log R = 3 \log R_1 + 4 \log R_2 + 2 \log R_3 \quad (36)$$

$$\text{or, } R^9 = R_1^3 R_2^4 R_3^2$$

$$\text{and } R_3 = \sqrt{\frac{R^9}{R_1^3 R_2^4}} = 2.001022$$

$$= 1.000511 \times 2a$$

for squares of side a .

$$\therefore \frac{R_3}{R_2} = 1.000511 \text{ agreeing closely with the value found p. 9.}$$

$$\therefore a_3 = .000511$$

We have thus derived from the comparatively simple formulæ (31 and 32) the same values of a_2 and a_3 that were obtained from equations (8a) and (8).

13. SELF-INDUCTANCE OF A CIRCLE OF SQUARE SECTION.

THREE FORMULÆ COMPARED.

Problem 1.—Circle of square section.

As a first illustration of the use of the geometric mean distance in calculating inductances let us take the case of a single circular turn of wire of square section. The formulæ are more exact as the

ratio of the radius to the side of the cross section is greater; hence, we may take $a=25$ cm and $b=c=0.1$ cm as a case favorable to high accuracy. The three formulæ to be compared will be Weinstein's, Stefan's, and Maxwell's, in the latter using the geometric mean distance of the square section. The formulæ are as follows for the case of a square section:

Weinstein's:

$$L = 4\pi a \left\{ \log \frac{8a}{b} \cdot \left(1 + \frac{b^2}{24a^2} \right) + .03657 \frac{b^2}{a^2} - 1.194913 \right\} \quad (37)$$

Stefan's:

$$L = 4\pi a \left\{ \log \frac{8a}{b\sqrt{2}} \cdot \left(1 + \frac{b^2}{24a^2} \right) - y_1 + \frac{b^2}{16a^2} y_2 \right\} \quad (38)$$

where $y_1=0.848340$ and $y_2=0.8162$ for a square.

Maxwell's:

$$L = 4\pi a \left\{ \log \frac{8a}{R} \cdot \left(1 + \frac{3R^2}{16a^2} \right) - 2 - \frac{R^2}{16a^2} \right\} \quad (39)$$

where R is the geometric mean distance of the square section from itself = .0447049 cm in this case.

Substituting $a=25$ cm and $b=0.1$ cm we find the following values for the self-inductance:

By Weinstein's:

$$\frac{8a}{b} = 2000. \quad \log_e 2000 = 7.600902$$

$$\frac{1}{24} \frac{b^2}{a^2} \cdot \log \frac{8a}{b} = .000005$$

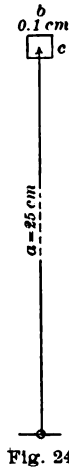
$$.03657 \frac{b^2}{a^2} = .000001$$

$$7.600908$$

$$- 1.194913$$

$$6.405995$$

$$4a = 100 \therefore \frac{L}{\pi} = 640.5995 \text{ cm.}$$



By Stefan's:

$$\begin{aligned}
 \log_e 2000 &= 7.600902 \\
 -\frac{1}{2} \log 2 &= -\frac{0.346573}{7.254329} \\
 \frac{b^2}{24a^2} \log \frac{8a}{b\sqrt{2}} &= .000005 \\
 \frac{b^2}{16a^2} \gamma_1 &= \frac{.000001}{7.254335} \\
 -\gamma_1 &= -\frac{0.848340}{6.405995} \\
 \therefore \frac{L}{\pi} &= 640.5995
 \end{aligned}$$

By Maxwell's:

$$\begin{aligned}
 \frac{8a}{R} = \frac{200}{.447049} &= \frac{2000}{.447049} & \log_e 2000 &= 7.600902 \\
 -\log .447049 &= +\frac{.805087}{8.405989} \\
 \frac{3R^2}{16a^2} \times \log \frac{8a}{R} &= \frac{0.000005}{8.405994} \\
 -\left(2 + \frac{R^2}{16a^2}\right) &= -\frac{2.000000}{6.405994} \\
 \therefore \frac{L}{\pi} &= 640.5994
 \end{aligned}$$

These three values of L are practically identical. Stefan's formula is derived from Weinstein's, but Maxwell's is an independent one. This confirms the value of R for the square.

Problem 2.—Circle of Larger Section.

As already stated, the geometric mean distance is calculated on the basis of a straight conductor, which is equivalent to a circle of infinite radius. We see by the above example that it must be substantially the same for a circle where the radius is 250 times the side of the square as it is for a circle of infinite radius. For a square of 1.0 cm side, the radius of the circle being 25 cm, and hence the radius only 25 times the side of the square, the value of R would not be as exact, and hence the agreement among the three formulæ would

not be as good as the foregoing. Using the above formulæ for this case, Fig. 25, we find the following results:

By Weinstein's and Stefan's formulæ, $L = 410.3816$

By Maxwell's, using the g. m. d. . . $L = 410.3750$

Difference = .0066

The difference is one part in 60,000; the second value being smaller shows the g. m. d. is a little too great when applied to a circle where the curvature is considerable.

Problem 3.—Coil of Two Turns.

Calculating the self-inductance of a coil of two turns of round wire by the method of summation, and also by Stefan's formula, we can obtain the correction for mutual inductance independently of the formulæ for geometric mean distance. The self-inductance of the coil of two turns is

$$L = 2L_1 + 2M_{12}$$

Suppose $a = 99.85$ cm and the diameter of the section of the wire is 0.1 cm, Fig. 26.

By Wien's formula $L_1 = 791.694990 \times 4\pi$

By Maxwell's formula $M_{12} = 697.521875 \times 4\pi$

$$\therefore \frac{L_\sigma}{4\pi} = 2978.43373 \text{ cm.}$$

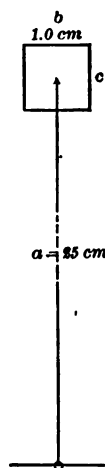


Fig. 25

The self-inductance of a coil of two turns of *square* section by Stefan's formula, the section of the coil being 1×2 mm and the radius the same as before is

$$\frac{L_u}{4\pi} = 2949.55944$$

The correction to reduce to a round section is

$$na (.1380605), na = 199.7, \therefore \frac{A_1 L}{4\pi} = 27.57068$$

$$\therefore L \div 4\pi = \text{sum} = 2977.13012$$

$$\text{Difference from } L_\sigma \div 4\pi = 1.30361$$

If there were no correction for mutual induction these two values should agree. Their difference is the correction in question. That is

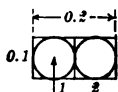


Fig. 26

$$\begin{aligned} \Delta_s L &= naa_s = 1.30361 \\ na &= 199.7 \therefore a_s = \frac{1.30361}{199.7} = .0065278 \end{aligned}$$

This is almost identically the value found for a_s by the method of geometrical mean distances. If we had used .0065285 and applied the correction naa_s based upon it we should have found $\Delta_s L/4\pi = 1.30374$ and $L/4\pi = 2978.43386$. This differs from the value $L_\sigma/4\pi$ by the method of summation given above by less than one in 20,000,000. This is a very interesting confirmation of the result obtained by the method of geometrical mean distances, and shows that when the radius of the coil is large in comparison with the dimensions of the cross section that the formulæ are very accurate.

Problem 4.—Coil of Four Turns.

We will now find β_1 , the correction for diagonal squares, by the method just employed to find a_s . The self-inductance of this coil by the method of summation is

$$L = 2L_1 + 2L_2 + 2M_{12} + 2M_{34} + 4M_{13} + 4M_{14} \quad (40)$$

As before, L_1 and L_2 are calculated by Wien's formula for circles and the M s by Maxwell's formula for mutual inductances. The results are as follows:

$$\begin{aligned} 2L_1 &= 1583.3900 \\ 2L_2 &= 1585.1758 \\ 2M_{12} &= 1395.0438 \\ 2M_{34} &= 1396.6410 \\ 4M_{13} &= 2791.6843 \\ 4M_{14} &= 2653.1941 \\ \hline \therefore L_\sigma/4\pi &= 11405.1290 \end{aligned}$$



By Stefan's and Weinstein's formula $L_u/4\pi$ is found on the assumption of a uniform distribution of current over the cross section of the coil.

$$\frac{L_u}{4\pi} = 11345.6632$$

Correction to reduce from square to round

$$\text{wire} = na(.1380605) = 399.6 \times .1380605 = \frac{55.1690}{}$$

$$\text{Sum} = \frac{L'}{4\pi} = 11400.8322$$

$$\therefore \frac{A_2 L}{4\pi} = \frac{L_\sigma - L'}{4\pi} = 4.2968 = \text{the correction for mutual induction.}$$

$$= na(2a_2 + \beta_1)$$

$$\therefore 2a_2 + \beta_1 = \frac{4.2968}{399.6} = .010753$$

$$2a_2 = .013057$$

$$\therefore \beta_1 = -.002304$$

By the method of geometric mean distances we found β_1 to be $-.002302$. This slight difference amounts to only one part in 15,000,000 of the whole value of the self-inductance of the coil. Thus we see that the two largest correction terms a_2 and β_1 derived by means of formulæ of self and mutual inductance agree with the values found by the method of geometric mean distances.

It is probable that Stefan derived his value for the correction in this way. As we have already seen, the effect of the eight nearest wires on any given wire, Fig. 28, is

$$4a_2 + 4\beta_1 = .026114 - .009204 \\ = .01691$$

β_1	α_2	β_1
α_2		α_2
β_1	α_2	β_1

Fig. 28

and this is practically the value of E (page 4), which results from the correction given by Stefan. This neglects all the other wires and assumes the effect equal on all the wires of the section. This is indeed a very good first approximation to the correction, and amply accurate for most purposes.

Problem 5.—Coil of Four Turns in one Layer.

The self-inductance of a coil of four turns of wire in one layer by the method of summation is

$$L = 4L_1 + 6M_{12} + 4M_{13} + 2M_{14} \quad (41)$$

The L s and M s computed as in the last article are as follows:

$$4 L_1 = 12688.549\pi$$

$$6 M_{12} = 16769.275\pi$$

$$4 M_{13} = 10070.481\pi$$

$$2 M_{14} = 4710.869\pi$$

$$\therefore L_\sigma = 44239.174\pi$$

By Stefan's formula, the self-inductance of a coil of four turns of square wire filling the entire section is, for the above dimensions:

$$L_u = 44001.729\pi$$

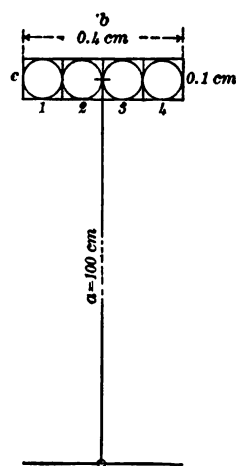


Fig. 29

We must add to this the correction to reduce to a round section and the correction for mutual induction. The latter involves a_2 , a_3 , a_4 . Taking each wire separately it will be seen that the sum of these corrections is $6a_1 + 4a_2 + 2a_4$

$$6a_1 = .0065285 \times 6 = .0391710$$

$$4a_2 = .0005104 \times 4 = .0020416$$

$$2a_4 = .0001025 \times 2 = .0002050$$

$$.0414176$$

$$E = \text{average for each wire} = .0103544$$

$$\text{Correction } C = .1380605$$

$$C + E = .148415$$

$$4\pi an = 1600\pi$$

$$\therefore \Delta L = .148415 \times 1600\pi = 237.464\pi$$

$$L_u = 44001.729\pi$$

$$\therefore L = 44239.193\pi$$

$$\text{Difference from } L_\sigma = .019\pi$$

This difference is less than one part in two million. Without the correction E for mutual induction (depending on a_1, a_2, a_4) the discrepancy would be 1,000 times as large.

Problem 6.—Coil of Ten Turns.

Let us take a coil of ten turns of round covered wire, the radius being 25 cm, diameter of bare wire being 0.8 mm, of covered wire 1.0 mm, the whole length of coil being 1 cm. The self-inductance of this coil has been calculated in a previous paper¹¹ to be 47385.82π cm using the method of summation.

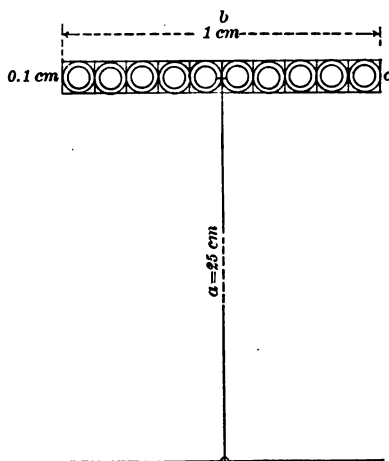


Fig. 80

By Stefan's formula (substituting $a = 25$, $n = 10$, $b = 1.0$, $c = 0.1$) we find

$$L_u = 47011.93\pi \text{ cm.}$$

This assumes a uniform distribution of current over the rectangular section, as though the coil were wound with square wire having insulation of infinitesimal thickness. The correction to apply to reduce it to the actual winding of round insulated wire is as follows:

$$\Delta L = 4\pi an \left[\log_e \frac{D}{d} + .1380605 + \frac{1}{n} \Sigma a \right]$$

¹¹ Calculation of Self-Inductances of Single Layer Coils, Bulletin of Bureau of Standards, 2, p. 165; 1906.

Σa for a single layer coil of n turns is as follows:

$$\Sigma a = 2 \left[(n-1)a_1 + (n-2)a_2 + (n-3)a_3 + (n-4)a_4 + (n-5)a_5 + \dots \right] \quad (42)$$

For a coil of ten turns we have

$$.006528 \times 9 = .058756$$

$$.000511 \times 8 = .004083$$

$$.000102 \times 7 = .000717$$

$$.000033 \times 6 = .000195$$

$$.000013 \times 5 = .000068$$

$$\text{Sum} = .063819$$

$$\therefore \Sigma a = .127638$$

$$E = \frac{1}{n} \Sigma a = .012764$$

$$C = .138060$$

$$F = \log_e \frac{D}{d} = .223144$$

$$\text{Sum} = .37397$$

$$\Delta L = 4\pi a n \times 0.37397$$

$$\text{or } \Delta L = 383.97\pi \text{ cm}$$

$$L_n = 47011.93\pi \text{ cm}$$

$$\therefore L = 47385.90\pi \text{ cm}$$

$$\text{whereas } L\sigma = 47385.82\pi \text{ cm} \quad (\text{See p. 31.})$$

$$\text{Difference} = .08\pi \text{ cm}$$

That is, the difference between the self-inductance by the very accurate formulæ employed in the method of summation, and the value by Stefan's formula for a uniform distribution of current plus the corrections to reduce to the actual winding amounts to one part in 600,000.

Problem 7.—Coil of Twenty Turns.

$$a = 25 \quad b = 2 \text{ cm} \quad c = 0.1 \text{ cm} \quad n = 20$$

Diameter of bare wire 0.6 mm, of covered wire 1.0 mm.

In the last case we obtained the self-inductance of the coil by two distinct methods, the first being the method of summation, the sec-

ond by assuming the current uniformly distributed over the section and then applying the three corrections C, F, E . In this problem we may calculate L , first, by use of the current sheet formula and apply the corrections A and B to reduce to round wires, and then, second, by Stefan's formula for uniform distribution and apply the three corrections C, F, E and reduce to round wires.

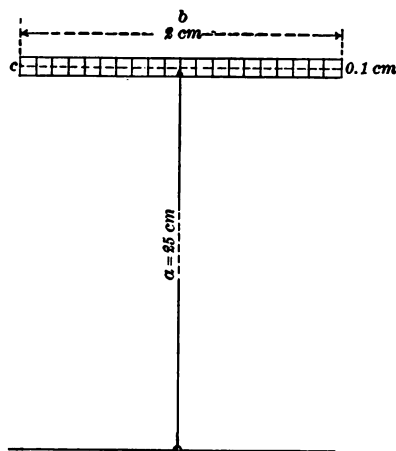


Fig. 81

Rayleigh's formula for a narrow current sheet is as follows:

$$L = 4\pi an^2 \left\{ \log \frac{8a}{b} - 0.5 + \frac{b^2}{32a^2} \left(\log \frac{8a}{b} + \frac{1}{4} \right) \right\}$$

Here $\frac{8a}{b} = 100$ $\log_e 100 = 4.605170$

$$\frac{4}{20,000} \left(\log \frac{8a}{b} + \frac{1}{4} \right) = \frac{.000971}{4.606141} - 0.500000$$

$$\frac{4.106141}{4.106141}$$

$$4\pi an^2 = 40,000\pi \therefore L_s = 164,245.64\pi \text{ cm}$$

This is the self-inductance of a winding of 20 turns of infinitely thin tape, each turn being 1 mm wide, with edges touching without making electrical contact, which arrangement fulfills the conditions

of a current sheet. To reduce this to the case of round wires we must apply the corrections A and B for self and mutual induction.¹²

$$\text{By Table VII, for } \frac{d}{D} = 0.6, \quad A = .0460$$

$$\text{By Table VIII, for } n = 20, \quad B = .2974$$

$$A + B = .3434$$

$$4\pi an = 2000\pi$$

$$\therefore \Delta L = 4\pi an (A + B) = 686.8\pi \text{ cm}$$

$$\therefore L = L_u - \Delta L = 163,558.84\pi \text{ cm}$$

By Stefan's formula we find, substituting the above values of a , n , b , c , and taking $y_1 = .548990$ and $y_2 = .1269$

$$L_u = 162,234.60\pi \text{ cm}$$

The correction $E = \frac{1}{n} \Sigma a$ is found by substituting 20 for n in equation (42). This gives $E = .01357$. The three corrections are then as follows:

$$C = .13806$$

$$F = .51082$$

$$E = .01357$$

$$.66245$$

$$\therefore \Delta L = 4\pi an(C + F + E) = 1324.90\pi \text{ cm}$$

$$\therefore L = L_u + \Delta L = 163,559.50\pi \text{ cm}$$

This value of L is greater than the value found by the other method by only four parts in a million. Thus we see that the method of calculating L_u by Stefan's or Weinstein's formula and applying the corrections C, F, E gives practically identical results with the method of summation and also with the current sheet method *for short coils*. When, however, the coils are longer the agreement is not so good, for the reason that the formula of Weinstein (and Stefan's derived from it) are not as accurate when the section of the coil is greater. Thus if the coil in the above problem had been 5 cm long and 2.5 mm deep, and wound with 20 turns of heavier wire the difference would have been 1 part in 25,000 (still very good agreement), and if it were

¹² Rosa, Bulletin of Bureau of Standards, 2, p. 161; 1906.

10 cm long and 0.5 cm deep (the radius being 25 cm) it would have been 1 part in 2,200. For most experimental work, therefore, the formula is amply accurate.

Problem 8.—Coil of Sixteen Turns.

$$a = 100. \quad b = c = 4 \text{ mm.}$$

The coil is wound with round 1 mm wire. It is a somewhat tedious operation to calculate the self-inductance of a coil of 16 turns by the method of summation, but it seemed worth while to test the correction terms $\alpha, \beta, \gamma, \delta$, in this way. By taking a large radius and a small section the work is diminished without sacrificing the accuracy. The result is

$$L_{\sigma} = 656,954.28\pi \text{ cm}$$

By Weinstein's formula, substituting $a = 100$, $b = 0.4$,
 $n = 16$,

$$L_u = 655,973.87\pi \text{ cm}$$

To find the correction for mutual induction we have to consider the effect on each wire of the sixteen of all the other wires, and take the mean.

The correction α_s applies twice to each of the four corner wires, three times to each of the 8 wires numbered 2, 3, 5, 9, 14, 15, 8, 12; and four times to each of the four wires 6, 7, 10, 11, Fig. 33. Thus

$$(2 \times 4 + 3 \times 8 + 4 \times 4) \div 16 = 3$$

That is, α_s applies three times to each wire *on the average*. In the same way we find that α_r applies twice to each wire on the average and α_l once. Considering all the correction terms in this way we find

$$\frac{1}{n} \Sigma(\alpha, \beta, \gamma, \delta) = E = 3\alpha_s + 2\alpha_r + \alpha_l + \frac{3}{2}\beta_s + \frac{9}{4}\beta_l + 3\beta_r + \gamma_1 + \gamma_s + \frac{\delta_1}{4} \quad (43)$$



The sum of the coefficients is $15 = n - 1$

$\frac{13}{\alpha_4}$	$\frac{14}{\beta_3}$	$\frac{15}{\gamma_2}$	$\frac{16}{\delta_1}$
$\frac{9}{\alpha_3}$	$\frac{10}{\beta_2}$	$\frac{11}{\gamma_1}$	$\frac{12}{\gamma_2}$
$\frac{5}{\alpha_2}$	$\frac{6}{\beta_1}$	$\frac{7}{\beta_2}$	$\frac{8}{\beta_3}$
$\frac{1}{\alpha_1}$	$\frac{2}{\alpha_2}$	$\frac{3}{\alpha_3}$	$\frac{4}{\alpha_4}$

Fig. 88

$$3a_4 = .019586$$

$$2a_3 = .001021$$

$$a_4 = .000102$$

$$\frac{3}{2}\beta_3 = .000035 = .020744$$

$$\frac{9}{4}\beta_1 = -.005179$$

$$3\beta_2 = -.000269$$

$$\gamma_1 + \gamma_2 = -.000166$$

$$\frac{\delta_1}{4} = -.000006 = -.005620$$

$$\therefore E = +.015124$$

The correction F is zero, since we have assumed the diameter of the bare wire to be 1 mm, and hence $D = d$.

$$C = .13806$$

$$E = .01512$$

$$\text{Sum} = .15318$$

$$4\pi na = 6400\pi$$

$$\Delta L = 4\pi na(C + E) = 980.35\pi \text{ cm}$$

From above,

$$L_u = 655,973.87\pi \text{ cm}$$

$$\therefore L = L_u + \Delta L = 656,954.22\pi \text{ cm}$$

This value of L agrees with the value L_σ obtained by the method of summation within one part in 10,000,000. In the calculations of L_σ and L_u the values were carried out two more decimal places than are given above, so that the close agreement is not merely accidental. With the large radius and small section chosen the formulæ are very exact, and this gives an excellent test of the corrections, α , β , γ , δ , for mutual induction.

The relative importance of the corrections C , F , and E decreases with the number of turns, for ΔL is proportional to n whereas L is proportional to n^2 . Thus for a coil of a large number of turns it is quite unnecessary to know E accurately, so far as knowing the self-inductance for experimental purposes. If we take $C = .138$, $E = .018$, or $C + E = 0.156$, we shall be amply accurate, when n is large, for the most refined experimental work. For a coil of few turns E

is smaller and relatively more important, and a value can be chosen from the above list that will be nearly right, or it can be calculated with great precision from the values of the corrections, α , β , γ , δ , etc., given in Fig. 16. These corrections of course are carried out much further than is necessary for any requirements of experimental purposes. Measurements can not be made of the section of the coil and of the diameter of the wire accurately enough to justify such refinements. It is better, however, that the formulæ should be more accurate than necessary rather than less so, and for the purpose of testing other formulæ it is desirable to be able to calculate self-inductance by this method with the highest precision. If further justification for these corrections is needed it must be found in the interest for its own sake of the subject of geometrical mean distance.

Summary of the values of E found for the various cases considered:

2 turns	$E =$	.006528	(Problem 3)
3 " (one layer)	$E =$	.009045	
4 " (two layers)	$E =$	.01691	(" 4)
4 " (one layer)	$E =$	.01035	(" 5)
8 " (two layers)	$E =$	.01335	
10 " (one layer)	$E =$	.01276	(" 6)
20 " (" ")	$E =$	.01357	(" 7)
16 " (four layers)	$E =$	.01512	(" 8)
100 " (ten layers)	$E =$	.01713	
400 " (20 $\times$ 20)	$E =$	.01764	
1000 " (50 $\times$ 20)	$E =$	.01778	
Infinite turns	$E =$	.01806	

14. CORRECTION FOR CURVATURE.

We have derived the α , β , γ , δ , ϵ corrections on the hypothesis of linear conductors, or of circular conductors in which the ratio of the radius to the side of the section is very great. When the section is increased the values of the equivalent distances between two coils is reduced; that is, the distance found by the method of geometrical mean distances is too great.

We can find the magnitude of this difference by computing the mutual inductance of two circular coils of square section by formulæ

which I have given elsewhere,¹³ and finding the distance apart of the equivalent circles. This distance is greater than the mean distance between centers of the coils when they are near and less when

they are further apart, and it is always less than the geometrical mean distances for straight conductors when the radii are equal and the section considerable.



Fig. 84

The two straight conductors whose square sections are shown in Fig. 34 may be replaced by

two single conductors O_1 , O_2 near their centers, the slight displacement from the center being such as to be equivalent to the correction for section. The distance apart of O_1 , O_2 would of course be the geo-

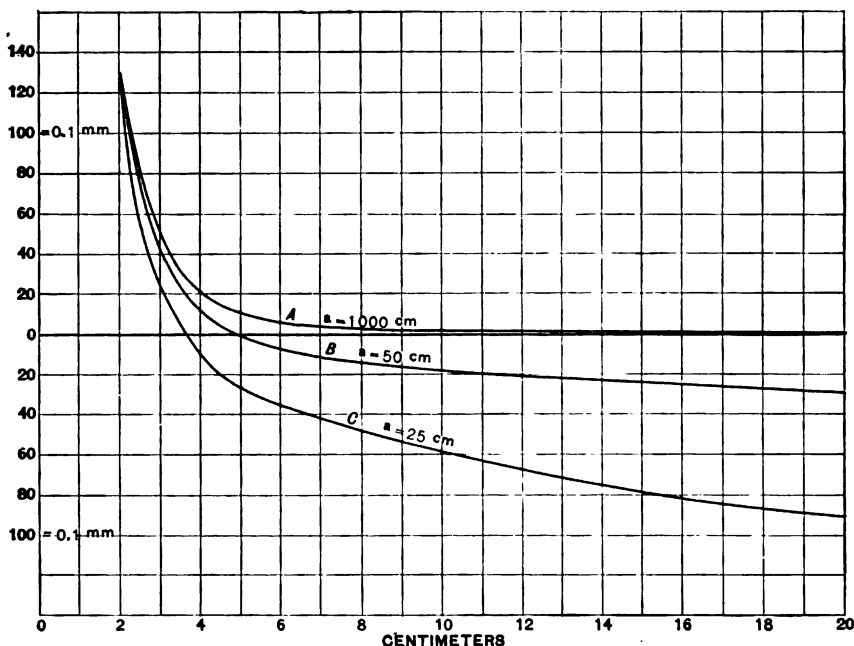


Fig. 35.

metrical mean distance of A and B. This is a little greater than the distance between centers, as we have seen for parallel squares in the position of A and B, Fig. 34. The curve A (Fig. 35) shows

¹³ E. B. Rosa, Bulletin of Bureau of Standards, 2, p. 359; 1906.

how much the distance $O_1 O_2$ exceeds the distance between centers for distances between 2 cm (when A and B are in contact) and 20 cm. Each square represents 1 cm horizontally and .02 mm vertically. Curve A shows how rapidly the displacement Δd falls off as the conductors are further apart, Δd being only .001 mm at $d = 10$ cm.

If, however, A B represent the sections of circular conductors the corrections by the method of geometrical mean distances as we have seen is only reliable when the section is very small in comparison with the radius. I have calculated the correction for section for the case of three circular coils, all of square section 2×2 cm, with radii 25, 50, and 1,000 cm; this correction is ΔM .¹⁴ I have found the displacement Δd which gives the same value of ΔM by the following formula:

$$\Delta M = 4\pi a \cdot \Delta d \left\{ \log \frac{8a}{d} \left(\frac{3d}{8a^3} - \frac{15d^3}{256a^4} + \frac{105d^5}{64 \cdot 128a^6} \right) - \left(\frac{1}{d} + \frac{5d}{16a^3} - \frac{77d^3}{1024a^4} + \frac{282d^5}{(128)^2 a^6} \right) \right\} \quad (44)$$

These displacements are plotted in the curves A, B, C, Fig. 35. Curve A represents the coil of 1,000 cm radius, which is 500 times the side of the section, and approximates closely to the case of a straight conductor, having sensibly the same correction for section (represented by the displacement) as the straight conductors. The correction for section is negative (represented by a positive value for Δd) in all cases when the conductors are near each other, but quickly becomes positive for the circular conductors of radius 25 and 50 cm as the distance increases. In the case of the coils of smaller radius the negative displacement Δd is relatively great at the larger distances, although it is actually very small; the greatest value for $d = 20$ for the smallest circle is less than 0.1 mm.

These curves show very forcibly how appreciable is the correction of the geometrical mean distance for curvature; or, in other words, that the geometrical mean distances calculated for straight conductors can be applied to circular conductors only when the distances between conductors is very small in comparison with the radius.

¹⁴ Bulletin of Bureau of Standards, 2, p. 349; 1906, equation (41).

APPENDIX.

Table of Inverse Tangents to nine decimal places.

$\tan^{-1} \frac{1}{12}$	0.083 141 232	$\tan^{-1} \frac{1}{3}$	0.321 750 554
$\tan^{-1} \frac{1}{11}$	.090 659 887	$\tan^{-1} \frac{1}{2}$	.463 647 609
$\tan^{-1} \frac{1}{10}$	.099 668 652	$\tan^{-1} \frac{2}{11}$	.179 853 506
$\tan^{-1} \frac{1}{9}$	.110 657 221	$\tan^{-1} \frac{2}{5}$	.380 506 377
$\tan^{-1} \frac{1}{8}$	.124 354 994	$\tan^{-1} \frac{2}{3}$	.588 002 603
$\tan^{-1} \frac{1}{7}$	.141 897 055	$\tan^{-1} \frac{3}{4}$	.643 501 109
$\tan^{-1} \frac{1}{6}$	.165 148 681	$\tan^{-1} \frac{3}{5}$	.540 419 500
$\tan^{-1} \frac{1}{5}$	.197 395 560	$\tan^{-1} \frac{4}{5}$	.674 740 942
$\tan^{-1} \frac{1}{4}$	.244 978 663	$\tan^{-1} \frac{5}{6}$	.694 738 276

$$\tan^{-1} 3 = \frac{\pi}{2} - \tan^{-1} \frac{1}{3}$$

$$\tan^{-1} 2 = \frac{\pi}{2} - \tan^{-1} \frac{1}{2} \text{ etc.}$$

Table of Constants for Stefan's Equation

b/c or c/b	y_1	y_2	b'/c or c'/b	y_1	y_2
0.00	0.50000	0.1250	0.55	0.80815	0.3437
0.05	.54899	.1269	0.60	.81823	.3839
0.10	.59243	.1325	0.65	.82648	.4274
0.15	.63102	.1418	0.70	.83311	.4739
0.20	.66520	.1548	0.75	.83831	.5234
0.25	.69532	.1714	0.80	.84225	.5760
0.30	.72172	.1916	0.85	.84509	.6317
0.35	.74469	.2152	0.90	.84697	.6902
0.40	.76454	.2423	0.95	.84801	.7518
0.45	.78154	.2728	1.00	.84834	.8162
0.50	.79600	.3066			

Table of Napierian Logarithms to nine decimal places for Numbers from 1 to 100.

1	0.000 000 000	51	3.931 825 633
2	0.693 147 181	52	3.951 243 719
3	1.098 612 289	53	3.970 291 914
4	1.386 294 361	54	3.988 984 047
5	1.609 437 912	55	4.007 333 185
6	1.791 759 469	56	4.025 351 691
7	1.945 910 149	57	4.043 051 268
8	2.079 441 542	58	4.060 443 011
9	2.197 224 577	59	4.077 537 444
10	2.302 585 093	60	4.094 344 562
11	2.397 895 273	61	4.110 873 864
12	2.484 906 650	62	4.127 134 385
13	2.564 949 357	63	4.143 134 726
14	2.639 057 330	64	4.158 883 083
15	2.708 050 201	65	4.174 387 270
16	2.772 588 722	66	4.189 654 742
17	2.833 213 344	67	4.204 692 619
18	2.890 371 758	68	4.219 507 705
19	2.944 438 979	69	4.234 106 505
20	2.995 732 274	70	4.248 495 242
21	3.044 522 438	71	4.262 679 877
22	3.091 042 453	72	4.276 666 119
23	3.135 494 216	73	4.290 459 441
24	3.178 053 830	74	4.304 065 093
25	3.218 875 825	75	4.317 488 114
26	3.258 096 538	76	4.330 733 340
27	3.295 836 866	77	4.343 805 422
28	3.332 204 510	78	4.356 708 827
29	3.367 295 830	79	4.369 447 852
30	3.401 197 382	80	4.382 026 635
31	3.433 987 204	81	4.394 339 155
32	3.465 735 903	82	4.406 719 247
33	3.496 507 561	83	4.418 840 608
34	3.526 360 525	84	4.430 816 799
35	3.555 348 061	85	4.442 651 256
36	3.583 518 938	86	4.454 347 296
37	3.610 917 913	87	4.465 908 119
38	3.637 586 160	88	4.477 336 814
39	3.663 561 646	89	4.488 636 370
40	3.688 879 454	90	4.499 809 670
41	3.713 572 067	91	4.510 859 507
42	3.737 669 618	92	4.521 788 577
43	3.761 200 116	93	4.532 599 493
44	3.784 189 634	94	4.543 294 782
45	3.806 662 490	95	4.553 876 892
46	3.828 641 396	96	4.564 348 191
47	3.850 147 602	97	4.574 710 979
48	3.871 201 011	98	4.584 967 479
49	3.891 820 298	99	4.595 119 850
50	3.912 023 005	100	4.605 170 186

THE COMPENSATED TWO-CIRCUIT ELECTRODYNAMOMETER.

By Edward B. Rosa.

1. ALTERNATING CURRENT AMMETERS FOR PRECISION MEASUREMENTS.

The Kelvin balance is a standard instrument for the measurement of alternating currents, but is a slow instrument in practice and is difficult to use for currents not entirely steady. Moreover, in the instruments for heavy currents there are eddy currents in the metal parts of the balance. This causes an appreciable error, at least with currents of higher than the ordinary frequency, and with currents of distorted wave shape.

The new Leeds and Northrup comparator is an excellent instrument for precision measurements of large or small alternating currents, and we have found it very useful at the Bureau of Standards. It is a differential hot wire instrument and is used in connection with a shunt for large currents. It is essential in this instrument also, for precision measurements, that the current to be measured shall remain very steady; and at least for currents of frequency higher than the ordinary, the self-inductances of the shunts used with the instrument should be known.

The use of an electrometer in connection with a standardized shunt for measuring large alternating currents is subject to the same condition, that the inductance of the shunt must be known; and for shunts of very small resistances on relatively high frequencies, or for distorted wave shapes, the inductance must be very small indeed to be neglected. Thus, with a shunt of .001 ohm (for measuring say 200 to 500 amperes) and a frequency of 180 cycles an inductance of only 100 cm, or 0.1 microhenry, would make the impedance of the shunt greater than its resistance by nearly one per cent, and hence introduce an error of this magnitude in the result, unless the

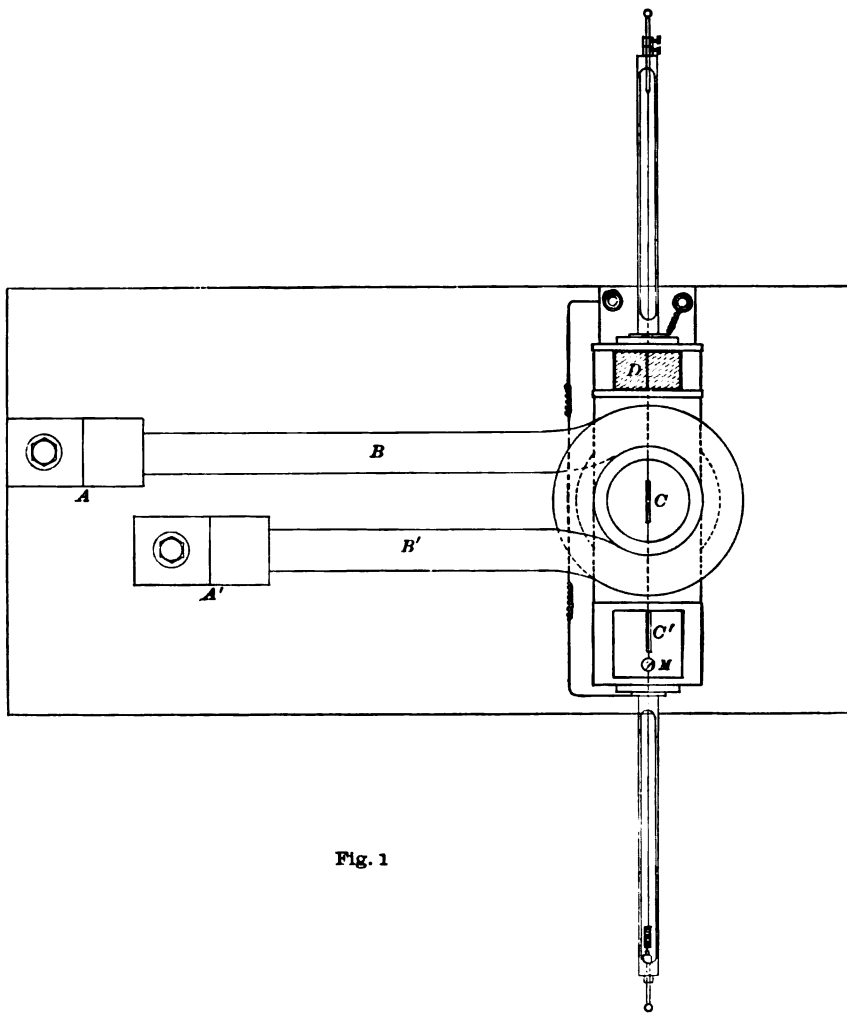


Fig. 1

Fig. 1.—Drawing of Heavy Current Ammeter.

Two-circuit electro-dynamometer, with current shunts, to measure alternating currents from 100 to 1,000 amperes. The field coil consists of two turns of heavy stranded cable B, B'; the two moving coils *c*, *c'* give an astatic system unaffected by distant stray fields, or by the earth's field when the instrument is calibrated by direct currents. The deflections are read on a curved scale by means of a telescope. A lever on the bottom of the lower suspension gives a ready means of changing the zero position of the moving system, required in testing the compensation for the self-inductance of the potential circuit, and in testing for the presence of eddy currents.

inductance were known and a correction applied. If, however, one is measuring currents not of sine wave form the correction can not be calculated unless the wave form is known. It is obviously better, if possible, in precision measurements, to employ an instrument that is free from error arising from the small residual inductance which can not be entirely eliminated even from a "non-inductive" shunt, and that is correct for all frequencies and all wave shapes.

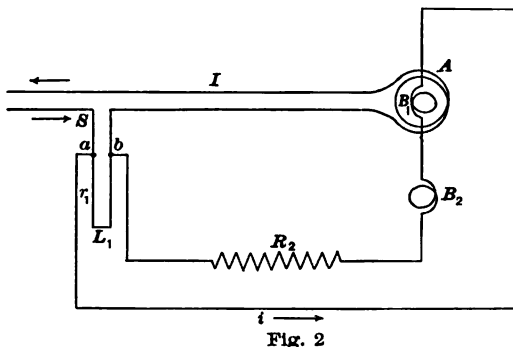
2. THE TWO-CIRCUIT ELECTRODYNAMOMETER.

A fourth form of instrument which we have been using for some time at the Bureau of Standards, and which seems to meet these conditions, is a two-circuit electrodynamicometer, with standardized shunt. Several of these

instruments, for measuring alternating currents of different ranges, were built in the instrument shop of the Bureau about two years ago and have recently been carefully investigated. They are quick in action and convenient to operate, and

when compensated for the resultant inductance and capacity of the moving coil circuit, they have the same constant for alternating current as for direct. They may, therefore, be calibrated by direct currents, measured by means of standard shunts, a potentiometer and standard cell, the moving coil circuit of the electrodynamicometer being connected to the terminals of a shunt designed to carry alternating current, the larger shunts being submerged in an oil bath which is water cooled and stirred. It is, of course, desirable that the inductance of the shunt be kept small, but no error arises from this cause, either in the ideal simple instrument or in the actual compensated instrument, the theory of which will be given presently.

That the reading of the ideal simple instrument is independent of the self-inductance of the shunt may easily be shown. By ideal simple instrument is meant one in which the self-inductance of the



moving coil circuit and the mutual inductance of the two circuits are both zero, while eddy currents in the region of the moving coils and capacity in the resistance R_s are supposed absent. Fig. 2 shows the circuits of the instrument. The main current I to be measured flows in series through the shunt S and the fixed coil A of the electro-dynamometer. From the potential points a, b , of the shunt the potential current i flows through the resistance R_s and the moving coils $B_1 B_2$ of the instrument. This current is proportional to the difference of potential between the points a, b , and therefore to the product of the current into the impedance of the shunt. Thus,

$$i = \frac{(I-i)\sqrt{r_1^2 + p^2 L_1^2}}{R_s} \quad (1)$$

where r_1 is the resistance and L_1 the inductance of the shunt, p being $2\pi n$. This current i lags behind the main current I by the angle ϕ , where

$$\cos \phi = \frac{r_1}{\sqrt{r_1^2 + p^2 L_1^2}}$$

Hence the torque of the instrument, which is proportional to I, i and $\cos \phi$ is

$$\begin{aligned} T &= K I i \cos \phi \\ &= \frac{K I (I-i) r_1}{R_s} \end{aligned} \quad (2)$$

That is, the deflection is unaffected by the self-inductance of the shunt. The constant K is determined by passing a direct current through the instrument, the potential current in that case being,

$$i = \frac{(I-i)r_1}{R_s}$$

The deflection is the same when the direct current is equal to the square root of the mean square of the alternating current. The deflection being independent of the frequency is also independent of the wave form.

3. THE COMPENSATED INSTRUMENT.

In the above simple theory of the two circuit electro-dynamometer it was assumed that L_2 and M , the self-inductance of the potential circuit and the mutual inductance between the two circuits, were zero. As this is not realized in the actual instrument it is necessary to calculate the effect upon the deflection of these quantities, as well as the effect of eddy currents in the fixed winding or other parts of the instrument.

In the potential circuit there are two electromotive forces acting, e_1 due to the current $I - i_1$ in the shunt (where I is the total current flowing, which passes through the fixed coil, and i_1 is the component of the current in the potential circuit in phase with I), and e_2 due to the mutual inductance M between the two coils. The first is equal to the current times the impedance, the second is proportional to M , I and the frequency. Thus, using complex quantities,

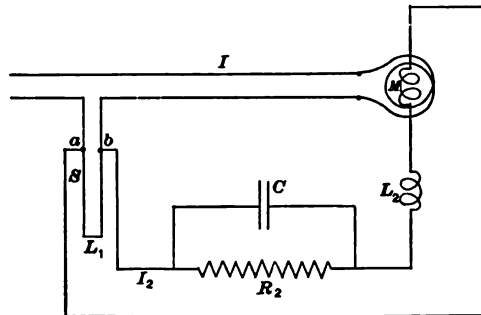


Fig. 8

$$\left. \begin{aligned} e_1 &= (I - i_1)(r + j\omega L_1) \\ e_2 &= -j\omega MI \\ e &= e_1 + e_2 \end{aligned} \right\} \quad (3)$$

The current I_2 in the potential circuit is equal to the electromotive force e divided by the impedance of the potential circuit, namely, $R_2 + j\omega L_2$. Thus,

$$\begin{aligned} I_2 &= \frac{(I - i_1)(r + j\omega L_1) - j\omega MI}{R_2 + j\omega L_2} \\ &= \frac{(I - i_1)(R_2 r + \omega^2 L_1 L_2) - \omega^2 M L_2 I - j\omega [(I - i_1)(r L_2 - R_2 L_1) + M R_2 I]}{R_2^2 + \omega^2 L_2^2} \end{aligned} \quad (4)$$

This current I_2 is not in phase with the main current, but lags by an angle ϕ such that

$$\tan \phi = \frac{p[(I-i_1)(L_2r - R_2L_1) + MR_2I]}{(I-i_1)(R_2r + p^2L_1L_2) - p^2ML_2I} \quad (5)$$

Putting i_1 and i_2 for the two components in phase and 90° out of phase with I , so that

$$I_2 = i_1 + ji_2$$

we have for the component in phase with the main current I ,

$$i_1 = \frac{(I-i_1)(R_2r + p^2L_1L_2) - p^2ML_2I}{R_2^2 + p^2L_2^2} \quad (6)$$

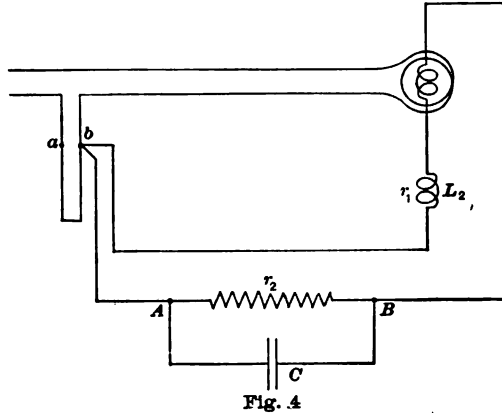
The component i_2 , 90° out of phase with I , contributes nothing to the torque on the moving coils and may be neglected. The torque due to the component i_1 is

$$\begin{aligned} T &= KIi_1 \\ &= \frac{KI(I-i_1)r}{R_2} \left[\frac{1 + \frac{p^2L_1L_2}{R_2r} - \frac{p^2ML_2}{R_2r} \frac{I}{I-i_1}}{1 + \frac{p^2L_2^2}{R_2^2}} \right] \quad (7) \end{aligned}$$

If L_2 , the self-inductance of the moving coil were zero, this expression for the torque would reduce to equation (2), which is the value for direct current. If L_2 is not zero, the expression in the bracket gives the correction to the torque, which depends not only on the constants L_1 and L_2 , but also on the variable mutual inductance M and the frequency of the current. If R_2 is large and the frequency is low, the correction can be neglected. But for frequencies as large as 100 to 180, and such resistances as we can use without reducing the deflection too much the correction may be important. I have therefore arranged to compensate the self-inductance of the potential circuit by a capacity, so as to make the potential current have the same phase as the impressed electromotive force in the potential circuit. This is equivalent to making $L_2=0$, and so making the correction in brackets in (7) equal to zero.

Fig. 4 represents the potential circuit of the dynamometer with

an outside resistance r_2 in parallel with which is a condenser C . Its terminals have been removed from the shunt and joined together, or what amounts to the same thing the two terminals are joined to the same point on the shunt, so that no electromotive force is impressed upon the potential circuit except that due to mutual inductance.



The conductance from A to B is $\frac{I}{r_2} + j\phi C$

The impedance Z_2 is therefore

$$Z_2 = \frac{I}{\frac{I}{r_2} + j\phi C} = \frac{r_2}{1 + j\phi C r_2} = \frac{r_2 - j\phi C r_2^2}{1 + \phi^2 C^2 r_2^2}$$

The impedance of the remainder of the circuit, Z_1 , is

$$Z_1 = r_1 + j\phi L_2$$

The total impedance of the potential circuit is therefore

$$Z = r_1 + j\phi L_2 + \frac{r_2 - j\phi C r_2^2}{1 + \phi^2 C^2 r_2^2}$$

If $\phi^2 C^2 r_2^2$ can be neglected in comparison with unity, this reduces to

$$Z = r_1 + r_2 - j\phi (C r_2^2 - L_2) \quad (8)$$

and if r_2 is so adjusted that $Cr_2^2 = L_2$, the impedance reduces to the

$$Z = r_1 + r_2 = R_2$$

That is, the impedance is equal to the ohmic resistance.

The compensation for the inductance in this case is true at all frequencies, and hence the correction factor in brackets in equation 7 reduces to unity.

In a particular case L_2 was 0.328 milli-henry, C was 1 microfarad, and hence r_2 was 18.1 ohms; $p^2 C^2 r_2^2$ was, therefore, for 180 cycles about .0004, and for 900 cycles, .01. The compensation is thus seen to be independent of frequency for a considerable range, and hence independent of the wave form.

4. APPLYING THE COMPENSATION.

To test the need of compensation join up the dynamometer as in Fig. 4 without the condenser or the auxiliary resistance r_2 and pass the full load current of the instrument through the fixed coil. If R_2 is the resistance of the moving coil circuit and L_2 its inductance, the current I_2 in the moving coil circuit will be,

$$I_2 = \frac{-j\dot{p}MI}{R_2 + j\dot{p}L_2} = \frac{-(R_2 - j\dot{p}L_2)j\dot{p}MI}{R_2^2 + \dot{p}^2 L_2^2} = \frac{-\dot{p}^2 L_2 MI - j\dot{p}MR_2 I}{R_2^2 + \dot{p}^2 L_2^2} \quad (9)$$

$$= i_1 + j\dot{i}_2$$

The component $j\dot{i}_2$ produces no torque; the component i_1 gives a torque

$$T = KIi_1$$

$$= -\frac{K\dot{p}^2 L_2 MI^2}{R_2^2 + \dot{p}^2 L_2^2} \quad (10)$$

$$= -\frac{K\dot{p}^2 L_2 MI^2}{R_2^2}$$

when L_2 is small, as is usually the case, and $\dot{p}^2 L_2^2$ is negligible in comparison with R_2^2 .

If the moving coils are so placed that the mutual inductance M changes sign as the coils are deflected over the range of the scale, this torque will reverse sign accordingly. In an instrument with a torsion head where the moving coil always occupies the same

position, the position can be so chosen that M , and hence the torque due to mutual inductance, shall be zero. In a deflection instrument this is not practicable, but, as we have seen above, the torque is made zero by compensating with capacity so as to have the same effect as reducing L_1 to zero.

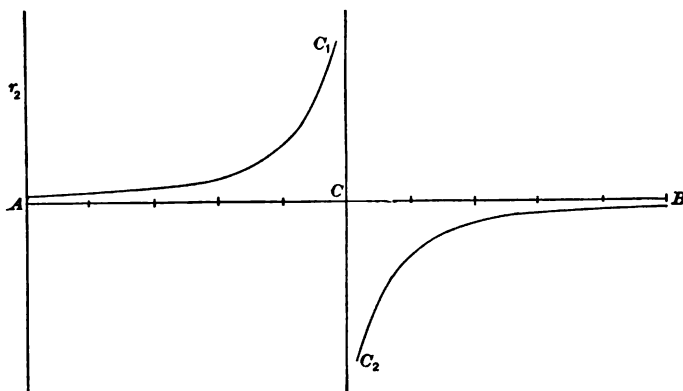


Fig. 5

By introducing a condenser and resistance r_2 into the circuit, varying the latter as necessary, the deflection can be reduced to zero for all positions of the moving coils, *provided there are no eddy currents in the fixed coil*. If there are eddy currents, which produce a torque in addition to that expressed by (10), the resistance r_2 in parallel with the capacity C required to give zero deflection will vary over the

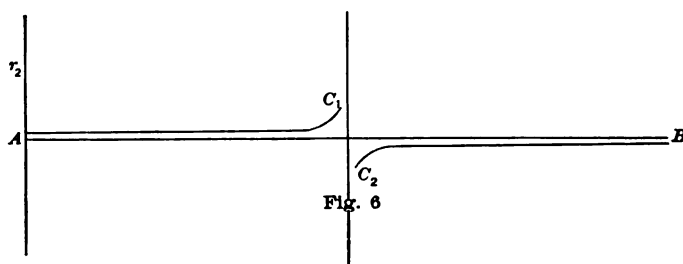


Fig. 6

scale as shown in Fig. 5. AB represents a scale 1 meter long. The curve AC_1 shows how the compensating resistance r_2 varies over the first half of the scale, and C_2B represents it over the second part. The reading C corresponds to the position of the coil when the mutual inductance M is zero. Fig. 6 shows the curve when the eddy cur-

rent effect is very small, and the compensating resistance is constant except for a short distance in the region of zero mutual inductance, which may of course be displaced considerably from the center.

5. TORQUE DUE TO EDDY CURRENTS.

To investigate the effect of eddy currents in the fixed coil or in any masses of metal near the moving coils, assume a third circuit C (Fig. 7) inductively connected with the fixed coil A and the moving coil circuit B.

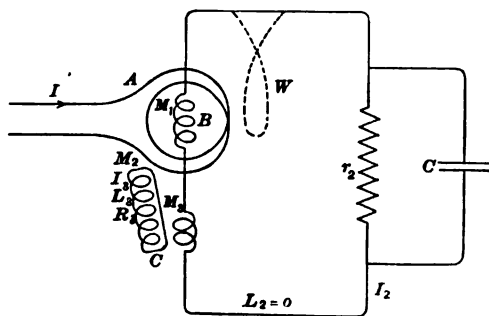


Fig. 7

Let M_1 be the mutual inductance of the fixed and moving coil circuits.

Let M_2 be the mutual inductance of the fixed and eddy current circuits.

Let M_3 be the mutual inductance of the moving coil and eddy current circuits.

Let the moving coil circuit be compensated for self-inductance so that we take L_2 as zero; L_3 is the self-inductance of the eddy current circuit. R_2 and R_3 are the resistances of the moving coil and eddy current circuits respectively.

Let I be the current in the fixed coil, its phase being taken as zero.

$I_2 = i'_2 + ji''_2$ is the current in the moving coil circuit.

$I_3 = i'_3 + ji''_3$ is the current in the eddy current circuit.

The electromotive force E_3 in the eddy current circuit is then

$$E_3 = -j\dot{p}M_3I \quad (11)$$

neglecting the insignificant effect of the small current I_2 .

Dividing E_s by the impedance of the eddy current circuit,

$$I_s = \frac{-j\dot{p}M_s I}{R_s + j\dot{p}L_s} = -\frac{\dot{p}^2 M_s L_s I + j\dot{p}M_s R_s I}{R_s^2 + \dot{p}^2 L_s^2} \quad (12)$$

$$\begin{aligned} \therefore i'_s &= -\frac{\dot{p}^2 M_s L_s I}{R_s^2 + \dot{p}^2 L_s^2} \quad (2) \\ i''_s &= -\frac{j\dot{p}M_s R_s I}{R_s^2 + \dot{p}^2 L_s^2} \quad (4) \end{aligned} \quad \left. \vphantom{\begin{aligned} \therefore i'_s &= -\frac{\dot{p}^2 M_s L_s I}{R_s^2 + \dot{p}^2 L_s^2} \quad (2) \\ i''_s &= -\frac{j\dot{p}M_s R_s I}{R_s^2 + \dot{p}^2 L_s^2} \quad (4) \end{aligned}} \right\} (13)$$

The electromotive force E_s acting in the moving coil circuit is due to the mutual inductance of the other two circuits. Thus:

$$\begin{aligned} E_s &= -j\dot{p}M_1 I - j\dot{p}M_s I_s = -j\dot{p}M_1 I - \frac{\dot{p}^3 M_s M_s R_s I - j\dot{p}^3 M_s M_s L_s I}{R_s^2 + \dot{p}^2 L_s^2} \\ I_s &= \frac{E_s}{R_s} = -\frac{\dot{p}^3 M_s M_s R_s I}{R_s(R_s^2 + \dot{p}^2 L_s^2)} - \frac{j\dot{p}I}{R_s} \left\{ M_1 - \frac{\dot{p}^3 M_s M_s L_s}{R_s^2 + \dot{p}^2 L_s^2} \right\} \quad (14) \\ &= i'_s + j i''_s, \end{aligned}$$

$$\begin{aligned} \therefore i'_s &= -\frac{\dot{p}^3 M_s M_s R_s I}{R_s(R_s^2 + \dot{p}^2 L_s^2)} \quad (3) \\ i''_s &= -\frac{j\dot{p}I}{R_s} \left\{ M_1 - \frac{\dot{p}^3 M_s M_s L_s}{R_s^2 + \dot{p}^2 L_s^2} \right\} \quad (5) \end{aligned} \quad \left. \vphantom{\begin{aligned} \therefore i'_s &= -\frac{\dot{p}^3 M_s M_s R_s I}{R_s(R_s^2 + \dot{p}^2 L_s^2)} \quad (3) \\ i''_s &= -\frac{j\dot{p}I}{R_s} \left\{ M_1 - \frac{\dot{p}^3 M_s M_s L_s}{R_s^2 + \dot{p}^2 L_s^2} \right\} \quad (5) \end{aligned}} \right\} (15)$$

The resultant torque on the moving coil system is due to the current I_s in the fields due to I and I_s . The

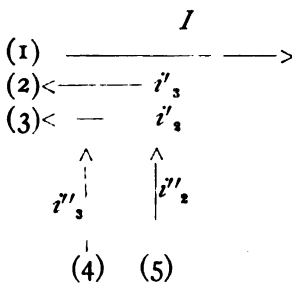


Fig. 8.

rectangular components of these currents, which in general have different phases, are shown graphically in Fig. 8. The current i'_s produces a torque with respect to I and i'_s , while i''_s produces a torque with respect to i''_s . The whole torque T may be regarded as made up of two parts, T_1 and T_2 , where T_1 is due to the currents represented by the horizontal arrows and T_2

is due to the currents represented by the vertical arrows. If K_1 is the constant of the instrument for the field winding through which the current I flows, and K_s the constant for the eddy current circuits, we shall have

$$T_1 = K_1 I i'_1 + K_3 i'_1 i'_3 = \frac{\rho^3 M_1 R_3 I^2}{R_3(R_3^2 + \rho^2 L_3^2)} \left\{ -K_1 M_3 + \frac{K_3 \rho^3 M_2 M_3 L_3}{R_3^2 + \rho^2 L_3^2} \right\}$$

$$T_2 = K_3 i''_3 i''_1 = \frac{\rho^3 M_2 R_3 I^2}{R_3(R_3^2 + \rho^2 L_3^2)} \left\{ +K_3 M_1 - \frac{K_3 \rho^3 M_2 M_3 L_3}{R_3^2 + \rho^2 L_3^2} \right\}$$

The total torque is therefore

$$T = T_1 + T_2 = - \frac{\rho^3 M_2 R_3 I^2}{R_3(R_3^2 + \rho^2 L_3^2)} \left\{ K_3 M_1 - K_1 M_3 \right\} \quad (16)$$

In order to make this torque zero the quantity $K_3 M_1 - K_1 M_3$ must be made equal to zero, as none of the other quantities can be made zero except perhaps M_2 , and that only by rebuilding the instrument. The mutual inductances M_1 and M_3 each consist of two parts and we may write:

$$M_1 = M'_1 + M''_1 \quad \text{where } M'_1 = \text{constant}$$

$$M''_1 \propto K_1 = a K_1$$

$$M_3 = M'_3 + M''_3 \quad \text{where } M'_3 = \text{constant}$$

$$M''_3 \propto K_3 = a K_3$$

The constant parts of these mutual inductances, M'_1 and M'_3 , are due to the inductances of the fixed and eddy current circuits upon all the moving coil circuit exclusive of the moving coils themselves. The variable parts are the inductances upon the moving coils, which vary as the coils are deflected, and reverse sign as they pass through the position of no mutual inductance. The mutual inductance M''_1 is proportional to the strength of magnetic field at the coil due to unit current in the fixed coil, as is also the constant K_1 ; hence M''_1 is proportional to K_1 . Thus we have

$$K_3 M_1 - K_1 M_3 = K_3 M'_1 + a K_1 K_3 - K_1 M'_3 - a K_3 K_1$$

$$= K_3 M'_1 - K_1 M'_3 = A, \text{ a constant.} \quad (17)$$

Thus if we adjust M'_1 so that $K_3 M'_1 = K_1 M'_3$, the constant A will be zero and the torque due to the eddy currents will be zero. This we can do by inserting a coil of a few turns in series with the moving coil circuit, *W* Fig. 7, and placing it in such a position with respect to the fixed coil that the mutual inductance M'_1 is given the desired value. This adjustment holds for all

positions of the moving coils and for all values and frequencies of the current.

Writing A for the bracket in equation (16) we have for the torque due to eddy currents and mutual inductance without the compensating inductance,

$$T = -\frac{p^2 M_1 R_2 I^2 A}{R_2 (R_2^2 + p^2 L_2^2)} \quad (18)$$

This equation I have verified experimentally¹ by placing coils of known resistance and inductance near the instrument and varying the several quantities separately. When L_2 and p are small the torque is proportional to p^2 , that is, to the *square of the frequency*; when pL_2 is large relatively to R_2 the torque is *independent of the frequency*. For the case of masses of metal the torque falls between these extremes and is approximately proportional to the frequency.

The torque is constant and in the same direction over the entire scale; hence it is easy to distinguish between the torque due to eddy currents and that due to L_2 , the self-inductance of the moving coil circuit. The latter is variable and changes sign where M_1 is zero. Hence in making the double compensation one should change r_2 until the deflection is constant over the scale, then adjust the compensating inductance until it is reduced to zero. This compensation should be made with a heavy current in the field and as high a frequency as possible, and R_2 relatively small. At lower frequency the torque is smaller, and will be negligible if compensated as nearly as possible at the higher frequency.

6. VARYING THE RANGE OF THE INSTRUMENT.

To vary the constant of the instrument, the resistance R_2 may be increased by varying amounts above its minimum. The resistance of the moving coils and connections being r_1 and the compensating resistance being r_2 , the minimum value of R_2 is $r_1 + r_2$. Additional resistance r_3 may be added to any extent without disturbing the compensation, provided this resistance is free from inductance or

¹ In this verification as well as in the other experimental work with these instruments I have received valuable assistance from Messrs. M. G. Lloyd, P. G. Agnew, T. T. Fitch, and F. W. Grover.

capacity. This will vary K over a wide range and so enable one to measure a considerable range of current.

If r_s does possess appreciable inductance or capacity it will slightly alter the compensation, but usually not appreciably. For the inductance or capacity of r_s is not likely to be appreciable in comparison with L_s unless r_s is large; and if r_s and hence R_s is large, the torque due to L_s for which compensation is made will be very small, being inversely proportional to R_s . However, as a test for compensation can quickly be made at any time, there is no necessity for the compensation of the instrument to be wrong, even though the potential resistance R_s is frequently altered.

One of the chief advantages of this form of alternating current ammeter is the ease with which the constant can be altered in a known ratio. The formula for the torque given by equation (2), which is (7) with the correction factor reduced to unity, may be written,

$$D = K_1 \frac{I^2 r_1}{R_s}$$

for heavy currents, where i is inappreciable in comparison with I and D is the deflection. K_1 will not of course be quite constant over the scale. If R_s is increased in the ratio of 4, 9, 16, etc., the value of I for the same deflection will be increased 2, 3, 4, etc., times. If r_1 is decreased 4 times (by using a shunt of larger capacity and lower resistance), a given deflection will correspond to a current twice as large. Thus the heaviest current the instrument can carry (suppose 100 amperes) will be measured with the smallest value of r_1 and the largest value of R_s , and this will give a full scale deflection. For the lightest currents to be measured, suppose r is 10 times larger and R_s 10 times smaller, then 10 amperes will give full scale deflection, and 5 amperes can be measured with good accuracy. If the instrument has a double winding, a still wider range of current can be measured with satisfactory accuracy.

For the heavy current ammeter shown in Fig. 1 there are two manganin shunts provided, arranged as shown in Fig. 9. Either one can be shortcircuited by the copper block a , b being an insulating block; a and b can be interchanged in position quickly when the screw S is released.

The heavy bar conductors are not disturbed in this change. The shunts are submerged in oil, which is cooled by a circulation of water. Thus the temperature of the shunts will be raised very little by the full load currents for which they are designed, namely, 250 and 1,000 amperes.

7. WATTMETERS.

Two-circuit electrodynamicometers are very commonly used as wattmeters, with a high resistance in the potential circuit. For measuring the power in circuits of relatively high electromotive force and of large power factor, the effect of the usually small inductance of the coils of the potential circuit and of any slight eddy currents that may be present is negligible. But when the wattmeter is used on circuits of very low electromotive force, where the resistance R_s must be made small, or when the power factor is very small, these sources of error may be very important. It is therefore desirable to use the same compensation of resistance in parallel with capacity in precision wattmeters as is used in two-circuit ammeters, and also to test for the presence of eddy currents. If these eddy currents are in the metallic parts of the instrument (the field conductors being so thoroughly stranded and insulated as to obviate them in the field windings), they may be fully compensated for by a loop in the potential circuit, placed inductively with respect to the field, as described above. If, however, they are in the field coil themselves, this compensation for their torque still leaves a possible source of error—namely, in the different distribution of the alternating current in the field coils from that of the direct current employed in calibrating the instrument. The field coils should therefore be carefully protected from eddy currents by making them of insulated strands.

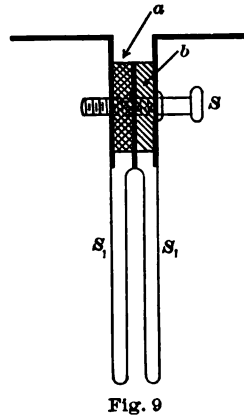


Fig. 9

A wattmeter compensated in this manner may be employed to measure the power expended on condensers or on a cable on open circuit. Or it may be used to test other wattmeters, having the

potential current displaced 90° from the field current, either by the generating machine or by other means; or it may be used to show when a given current and a given electromotive force differ by 90° , or any other desired angle, and so employed to calibrate phase meters.

When a wattmeter or alternating ammeter is to be made very sensitive, that is to measure small power or small current, it is necessary to make the constant K large. If the instrument is compensated for the self-inductance of the moving coils one can put a larger number of turns on these coils than otherwise, and allow L , to have a relatively large value. This will give a greater sensibility without danger of error arising from self-inductance.¹

¹Since the above was written I have noticed that M. H. Abraham has used a capacity and a resistance to compensate in a similar manner an alternating moving coil galvanometer.

THE COMPLETE FORM OF FECHNER'S LAW.

By P. G. Nutting.

Underlying vision, audition, and other sense perception is a fundamental quantitative relation between stimulus and sensation, between the objective and the subjective. From this fundamental relation, once established, may be derived another relation giving the least perceptible increment to the stimulus in terms of the whole. Fechner's law is such a relation derived from experimental data. It states that the least perceptible increment is proportional to the whole stimulus over quite a wide range of moderate intensities. That is, the ratio of least perceptible increment to total stimulus is a constant. This constant experiment shows to be of the order of about two per cent.

Now, just at the threshold value of a stimulus, evidently the least perceptible increment is the whole, so that, calling S this stimulus and δS the least perceptible increment, $\delta S:S=1$, while according to Fechner's law $\delta S:S=\text{constant}=\text{about } 0.02$, hence Fechner's law does not and can not hold in this form at low intensities for any sense organ. The general law must be such that $\delta S:S=1$ for $S=S_0$, the threshold value, while for large values of S , $\delta S:S=\text{a small constant}$. From the complete form of Fechner's law may be obtained the general relation between sensibility and intensity, and from the sensibility the desired general relation between stimulus and sensation. If this general relation is expressed in the proper mathematical form, it may be expected to hold with different constants for all sense organs.

Of the various sense perceptions, vision permits of far the most accurate quantitative determination. In this field, König and Brodhun¹ have published some excellent data bearing on Fechner's

¹A. König and E. Brodhun. Sitzungsberichte der Berliner Akad. 1888, pp. 917-932.

law. They determined the least perceptible increment for light ranging in intensity from just above the threshold value to about 100,000 meter-candles and in color from violet to deep red. The data relating to König's own eye—a normal trichromat—are reproduced in the accompanying table and figure.

TABLE I.

$\lambda =$	670	605	575	505	470	430
$L_0 =$	0.060	0.0056	0.0029	0.00017	0.00012	0.00012
L			$\delta L : L$			
200,000		0.0425				
100,000		0.0241	0.0325			
50,000	0.0210	0.0255	0.0260			
20,000	0.0160	0.0183	0.0205	0.0195		
10,000	0.0156	0.0163	0.0179	0.0181		
5,000	0.0176	0.0158	0.0166	0.0160		
2,000	0.0165	0.0180	0.0180	0.0175	0.0180	
1,000	0.0169	0.0198	0.0185	0.0184	0.0167	0.0178
500	0.0202	0.0235	0.0180	0.0194	0.0184	0.0214
200	0.0220	0.0225	0.0225	0.0220	0.0215	0.0245
100	0.0292	0.0278	0.0269	0.0244	0.0225	0.0246
50	0.0376	0.0378	0.0320	0.0252	0.0250	0.0272
20	0.0445	0.0460	0.0385	0.0295	0.0320	0.0345
10	0.0655	0.0610	0.0582	0.0362	0.0372	0.0396
5	0.0918	0.103	0.0888	0.0488	0.0464	0.0494
2	0.1710	0.167	0.136	0.0655	0.0715	0.0600
1	0.258	0.212	0.170	0.0804	0.0881	0.0740
0.5	0.376	0.276	0.208	0.0910	0.096	0.0966
0.2		0.332	0.268	0.110	0.127	0.116
0.10			0.396	0.133	0.138	0.137
0.05				0.183	0.185	0.154
0.02				0.251	0.209	0.223
0.01				0.271	0.189	0.249
0.005				0.325	0.300	0.312
0.002						0.369

König and Brodhun did not include the increment (δL) in the total light (L) in calculating the values of $\delta L:L$. Logically the increment should be included, otherwise the ratio $\delta L:L$ becomes meaningless as the threshold value is reached. The values quoted have been recalculated with increment included. In the figure $\delta L:L$ is plotted as a function of the natural logarithm of L , the curves being extended from the data of König and Brodhun to the point $\delta L:L=1$ for $L=L_0$ the threshold value, shown as a dotted ordinate in the figure.

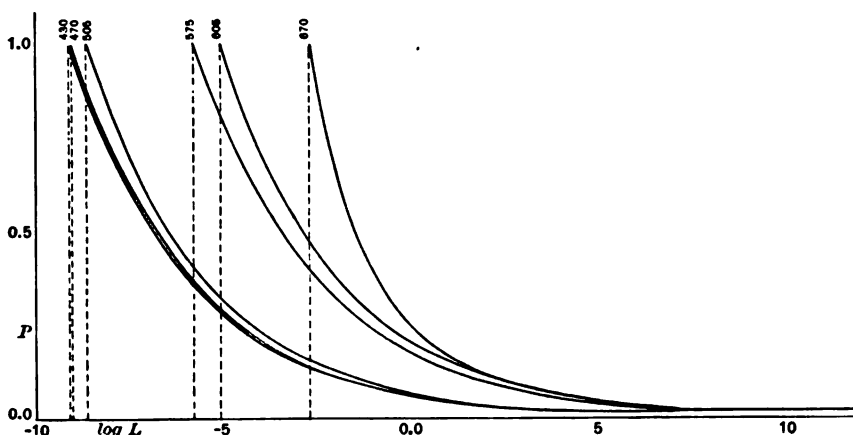


Fig. 1.—Least Perceptible Increment. $P(L)$

We have now to determine the mathematical form of these curves by the methods of synthetic function theory. For brevity, call $\delta L:L=P(L)$ or simply P , the photometric function. Denote the abscissa $\log L$ by x and the minimum value of P (about 0.016) by P_m .

All the curves have evidently the threshold value—the abscissa of the dotted ordinate—as natural origin of coordinates. That is, they would be strictly comparable if plotted as functions of $x-x_0$ or $\log(L:L_0)$ instead of x or $\log L$. Further, they are all of the general class of functions included in

$$P=a+b e^{-z}$$

when a and b are constants (independent of x) and z is a function of x . As a special case we may consider the particular function $z(x)=c(x-x_0)$ just as we should choose a particular integral of a differential equation as the practical working solution.

Taking then

$$P = a + b e^{-c(x-x_0)}$$

the parameter a is identified with P_m , being the value of P for x very large, b is $1 - P_m$ (since $a + b = 1$), c is from the plot (since $b c$ is the slope of the tangent at $x = x_0$) a number equal to about $\frac{1}{4}$ to $\frac{1}{2}$. Since $x - x_0$ is $\log L - \log L_0$ or $\log (L : L_0)$, P as a function of L is, according to the above,

$$P = P_m + (1 - P_m) (L_0 : L)^c$$

At very high intensities, instead of remaining constant, the values of $\delta L : L$ show a tendency to increase again. To represent this would require an additional term in the photometric function P . But at these high intensities the light sensation becomes painful and we have every reason to believe that the mechanism of vision is altered if there is not an actual destruction of tissue. It would serve no useful purpose to extend the function to cover such a case of heterogeneity.

From the photometric function $P(L)$ may be derived the sensibility of the eye as a function of intensity. The sensibility of the eye as a physical instrument we know decreases steadily to accommodate itself to increasing intensity, otherwise its range would be very small. It can easily be used with illuminations varying by a factor of a million and that within a very few minutes. We have to determine the law of variation of sensibility $\sigma(L)$ with intensity L .

Consider any physical instrument—a galvanometer for instance—capable of indicating on a scale the amount of a stimulus affecting it. The scale reading will be some function $\rho(S)$ of this stimulus. The sensibility of the instrument is also in general a function of the stimulus. Call this function $\sigma(S)$. This sensibility function is the rate of change of scale reading with the intensity of the stimulus.

$$\sigma = \frac{\partial \rho}{\partial S}, \quad \rho = \int \sigma ds.$$

Now, in the case of vision, the sensibility to variations of intensity varies inversely as the least perceptible increment, the δL above. But $\delta L : L$ has been expressed as a function $P(L)$ of intensity, and δL is $\delta L : L$ multiplied by L , or $\delta L = LP$. Sensibility as a function $\sigma(L)$ of intensity is then

$$\sigma = 1/LP = \frac{1}{L(P_m + (1-P_m)(L_o/L)^c)}$$

For low intensities σ is approximately proportional to L^{c-1} , for high intensities $\sigma = 1/P_m L$; the sensibility varies inversely as the intensity at high intensities exactly at low intensities σ varies from L^{-1} to L^{-1} .

Having the sensibility function, the scale reading is at once obtained by direct integration. In the visual case, the scale reading is the sensation of visual brightness. This is to be expressed as a function of the stimulus light or luminosity of radiation. For the brightness $B(L)$ we have

$$\begin{aligned} B &= \int \sigma dL = \int \frac{dL}{L(P_m + (1-P_m)(L_o/L)^c)} \\ &= \frac{1}{P_m c} \log \left(1 + P_m (L^c L_o^{-c} - 1) \right) + \text{constant.} \end{aligned}$$

The integration constant is zero, since for $L = L_o$ at the threshold value the visual sensation must be zero. The coefficient $1/CP_m$ may be included in a more general one which we may call B_o . Thus the general relation between visual sensation and the stimulus is

$$B = B_o \log \left(1 + P_m (L^c L_o^{-c} - 1) \right)$$

It is of interest here to note what was accomplished by Fechner and subsequent workers in this field. Fechner himself set up the form (using the notation here employed)

$$\frac{\delta L}{L + L_o}$$

to represent his own observations. He proceeded further to treat δL as a true differential, placing the above expression equal to a differential brightness dB and integrating, thus obtaining

$$B = K \log (L + L_o)$$

But δL and dB are fixed finite quantities and by no means to be regarded as differentials vanishingly small. At low intensities for instance, δL is as large as L itself. Nor does there appear to be any possible interpretation of the integration performed.

SUMMARY OF CONCLUSIONS.

1. The least perceptible increment δS of a stimulus S is not strictly proportional to that stimulus ($\delta S : S = \text{constant}$) but a function of the form

$$\frac{\delta S}{S} = a + b e^{-T}$$

in which a and b are constants and T is a function of S .

2. A special form of the above function closely representing visual sensation is

$$\frac{\delta L}{L} = P = P_m + (1 - P_m) (L_0 : L)^c$$

when P_m is a number equal to about 0.016 and c varies between one-fourth and one-half.

3. Sensibility as a function of stimulus S has been derived in the form $\sigma :: 1 : \delta S$, or

$$\sigma = \frac{1}{S(S_m + (1 - S_m)(S_0 : S_0)^c)}$$

4. Taking sensibility, by analogy with physical instruments, as the derivative of sensation (scale reading) with respect to stimulus, the integration of the above expression for sensibility gives for sensation in terms of stimulus the relation

$$B = B_0 \log \left(1 + S_m (S^c S_0^{-c} - 1) \right).$$

While the derivation of these relations is such that they are not to be regarded as more than particular solutions of the general problem, yet they are put forward with some confidence that more extensive experimental investigation will more fully substantiate them.

A COMPARISON OF THE UNIT OF LUMINOUS INTENSITY OF THE UNITED STATES WITH THOSE OF GERMANY, ENGLAND, AND FRANCE.

By Edward P. Hyde.

In inaugurating the work in photometry at the Bureau of Standards several years ago it was decided to adopt at once a working standard which could be maintained uniform, and in terms of which lamps submitted for verification could be standardized, and to defer the adoption of a primary standard until a thorough investigation of various primary photometric standards could be made. A comparison of the standards used by a large number of incandescent lamp manufacturers in the United States showed differences of 10 per cent or more and emphasized the need of authoritative photometric standards, copies of which could be supplied to lamp manufacturers and others making photometric measurements.

Obviously it was desirable to have the secondary standard which was to be maintained at the Bureau in as close agreement as possible with the standards employed in industrial practice in the United States. In the gas industry the British parliamentary candle is the recognized primary standard and has been used quite generally as the working standard, although in recent years it has been supplanted to some extent by the Harcourt (10-cp) pentane lamp. In the photometry of electric lamps, following the recommendation of the American Institute of Electrical Engineers, the Hefner lamp has been recognized as the primary standard, but the luminous intensity of a source has been expressed in British parliamentary candles, using the ratio 1 Hefner = 0.88 British candle.

The British candle is thus universally the unit of luminous intensity in the United States, and it was therefore adopted as the unit in terms of which the standard of the Bureau should be expressed.

But the British candle as a standard is unreliable and may have values differing by 5 per cent or more, so that the value of a candle unit obtained from a standard sperm candle would not be satisfactory. It was therefore decided to obtain the candle unit in the way recommended by the American Institute of Electrical Engineers, with one essential difference. Since Hefner lamps which do not differ in intensity from the standards of the Reichsanstalt by more than 2 per cent, are certified by that institution as correct, and since, owing to the difference in color between the Hefner lamp and ordinary illuminants, comparisons between the Hefner and other sources depend somewhat on the observer, two observers using different certified Hefner lamps may obtain results differing by several per cent. Therefore, instead of measuring at the Bureau a number of incandescent lamps in terms of several Hefner lamps in our possession, the Hefner unit was obtained directly from the Reichsanstalt by means of incandescent lamps verified at that laboratory in terms of their Hefner lamps. In this way it was hoped to obtain a standard expressed in terms of the Hefner unit of the Reichsanstalt.

The unit of the Bureau was obtained from these lamps by the use of the ratio, 1 Hefner = 0.88 candle. This unit was found to be in approximate agreement with the mean value of the candle in use by many lamp factories and testing laboratories, and was therefore in no sense an innovation. Since the establishment of the unit, all the principal lamp factories and testing laboratories have obtained copies of it directly or indirectly, so that this unit has now become in practice as well as in theory the standard of the United States in the photometry of electric lamps.

Since obtaining the original lamps, several comparisons of the standard of the Bureau with that of the Reichsanstalt have been made by means of seasoned incandescent lamps sent to Germany, each comparison indicating that the ratio of our unit to that of the Reichsanstalt had remained unchanged to within the errors of observation. Moreover, measurements made both at the Reichsanstalt and at the Bureau of Standards on incandescent lamps belonging to the Electrical Testing Laboratories of New York ¹ showed the Bureau unit to equal very closely $\frac{100}{88}$ of the Reichsanstalt unit.

¹ Sharp: Trans. International Electrical Congress, St. Louis, I, p. 492; 1904.

In order to obtain an accurate comparison of the unit of the United States with those of England and France, and to obtain a further comparison with the unit of Germany, the writer carried a number of seasoned lamps abroad in the spring of 1906, and had them compared with the recognized standards of these countries. Eighteen lamps were taken, nine of which were 50-volt, 64-watt, horseshoe filament lamps which had been standardized for intensity in a specific direction defined by lines etched on the bulb. The other nine lamps were 110-volt, 64-watt, oval anchored filament lamps which had been standardized for mean horizontal intensity (the mean intensity perpendicular to the axis of the lamp) by the method of rotating the lamps at 180 revolutions per minute. Three lamps of each type were left at the National Physical Laboratory at London as a check on the others which were taken to Paris and Berlin.

The lamps were hermetically sealed in copper boxes and were carried entirely by hand in order to avoid excessive jarring. They were taken in the order named to the National Physical Laboratory, the Laboratoire Central d'Électricité at Paris, the Physikalisch-Technische Reichsanstalt at Berlin, and then again to the National Physical Laboratory. They were intercompared and measured in terms of the standards of the Bureau immediately before leaving and as soon after returning as practicable.

At each laboratory, except the Reichsanstalt, the method employed for determining the mean horizontal intensity of the six 110-volt lamps was the same as that used at the Bureau. At the Reichsanstalt the method of rotating a pair of mirrors about the lamp was employed. Moreover, the distance at which both the stationary and rotating lamps were measured was approximately 125 cm at each laboratory except the Reichsanstalt, where the 50-volt stationary lamps were compared at a distance of 300 cm, and the other six lamps at a distance of 450 cm. The difference in distance was occasioned by the use of the *two* mirrors in measuring the mean horizontal intensity of the 110-volt lamps.

An investigation was subsequently undertaken at the Bureau to determine what differences in the values of mean horizontal intensity might result from the two different methods of making the measurements. The results of this investigation, which was extended to a study of all the possible errors incident to the deter-

mination of mean horizontal intensity by the rotating-lamp method, are given in a separate paper in the last volume² of this Bulletin. For oval anchored filament lamps the values of mean horizontal intensity determined in the two ways would probably not differ appreciably, although it is impossible to conclude definitely from the results obtained by one observer what the results would be for a second observer. The flicker, which is plainly perceptible even with lamps of the oval anchored filament type rotating at 180 revolutions per minute, is more annoying and more productive of error with some observers than with others. Two observers at the Bureau, however, obtained correct values of mean horizontal intensity for these lamps to within negligible differences.

The possible errors due to the difference in distance at which the measurements were made were also investigated, by comparing the intensities of the lamps at a long distance with their intensities at a much shorter distance, using the rotating sector disk. In the case of the stationary lamps a bad centering of the filament produces different apparent intensities at different distances. With the rotating lamps images are cast by the bulb in different directions and at various distances from the lamp, and the light radiating from these images has of course a different effective center from that radiating directly from the filament. In the orientation of stationary lamps care is always exercised to avoid a direction in which an image is thrown, but in rotating a lamp the light in every direction in the horizontal plane falls on the screen so that it must be shown by experiment what differences in the apparent intensities obtained at different distances may result from these images. Although the error due to such an effect would no doubt be quite appreciable if measurements were made with the lamp in such a stationary position that an image would fall directly on the screen, in rotating the lamp the effect of such images is distributed over the entire horizontal zone, so that the resultant error would probably be small. A comparison of the lamps at 300 cm and at 125 cm revealed no appreciable differences, so that the measurements made at the Reichsanstalt are comparable with those made in England and France and at this Bureau.

² Vol. 2, p. 415.

The results of the determinations made in England were obtained by comparison with incandescent lamps, which in turn had been compared with the standard Harcourt pentane lamp of the National Physical Laboratory. This lamp is supposed to have an intensity of 10 British parliamentary candles, and therefore the intensities of the incandescent lamps verified for the Bureau are expressed in British candles. In France the comparisons were made against incandescent lamps which had been measured at the Laboratoire Central in terms of the Carcel lamp. By assigning to the Carcel lamp an intensity of 9.6 bougies décimales, a value obtained by Violle from a direct comparison with the platinum standard, the results are expressed in bougies décimales.

In Germany the comparisons were made against incandescent lamps which had in turn been compared against the Hefner amyl acetate lamp. The results are expressed in Hefner candles. Since at the Reichsanstalt the electromotive force of a Clark cell at 15° C is taken as 1.4328 volts, instead of 1.4340 volts (the value used in England, France, and the United States), the voltages and currents of the incandescent lamps were modified accordingly, but in the following tables, for purposes of comparison, all voltages and currents are expressed in terms of the same units.

In Table I are given the complete results obtained for the individual lamps at the different laboratories, in the order in which the laboratories were visited. The first six lamps, B. S. Nos. 44, 45, 46, 49, 50, and 64, are the 50-volt stationary lamps, the remaining six, B. S. Nos. 4, 7, 8, 13, 30, and 40, are the 110-volt lamps that were standardized for mean horizontal intensity and are hereafter designated as the *rotating* lamps. The six lamps, three stationary and three rotating, which were left at the National Physical Laboratory, are not included in this table. They were used merely as a check on the constancy of the other twelve lamps.

In the first column are given the numbers of the lamps, in the second column the voltages at which the lamps were burned, and in the remaining columns the values of candlepower and current found at the respective laboratories. At the Bureau of Standards and at the National Physical Laboratory they were measured twice. The values given in the fifth and sixth columns are those obtained at the first visit to the National Physical Laboratory, but since for

TABLE I.
Values of Candlepower and Current Obtained at the Different Laboratories.

Lamps	Voltage	Bureau of Standards		National Physical Laboratory		Laboratoire Central		Reichsanstalt		National Physical Laboratory		Bureau of Standards	
		Candle-power	Current	Candle-power	Current	Candle-power in bougies décimales	Current	Candle-power in Hefner's	Current	Candle-power	Current	Candle-power	Current
B. S. No. 44.	49.0	17.4 ₁	1.3326	17.6 ₈	1.3330	17.7 ₀	1.3325	19.9	1.333	17.7 ₂	1.3330	17.3 ₃	1.3325
B. S. No. 45.	49.0	17.3 ₁	1.3304	17.6 ₈	1.3306	17.5 ₃	1.330	19.6	1.331	17.4 ₃	1.3306	17.2 ₄	1.3302
B. S. No. 46.	49.5	17.3 ₁	1.3225	17.6 ₈	1.3229	17.5 ₃	1.323	19.5	1.323	17.6 ₀	1.3230	17.2 ₃	1.3225
B. S. No. 49.	49.5	17.5 ₃	1.3301	17.7 ₀	1.3302	17.7 ₃	1.3295	19.9	1.330	17.7 ₄	1.3305	17.5 ₃	1.3301
B. S. No. 50.	49.5	17.3 ₃	1.3283	17.5 ₃	1.3284	17.5 ₀	1.328	19.6	1.328	17.5 ₀	1.3288	17.3 ₀	1.3282
B. S. No. 64.	50.5	17.7 ₀	1.3328	17.9 ₃	1.3325	18.0 ₀	1.333	20.1	1.333	18.0 ₀	1.3331	17.6 ₃	1.3328
Means		17.4 ₃	1.3294	17.7 ₁	1.3296	17.6 ₃	1.3293	19.7 ₁	1.3297	17.6 ₃	1.3298	17.4 ₀	1.3294
B. S. No. 4.	109.0	16.2 ₃	0.5973	16.5 ₀	0.5975	16.5 ₃	0.5965	18.5	0.597	16.5 ₁	0.5975	16.1 ₃	0.5974
B. S. No. 7.	108.0	15.9 ₀	0.5947	16.1 ₁	0.5944	16.0 ₃	0.5937	18.0	0.594	16.2 ₃	0.5946	15.8 ₃	0.5944
B. S. No. 8.	109.0	16.0 ₃	0.5851	16.3 ₀	0.5852	16.4 ₀	0.5850	18.3	0.584	16.3 ₀	0.5852	16.0 ₃	0.5850
B. S. No. 13.	109.0	15.7 ₃	0.5936	16.0 ₃	0.5938	16.4 ₀	0.5932	18.1	0.593	16.0 ₃	0.5939	15.6 ₁	0.5937
B. S. No. 30.	109.0	15.7 ₃	0.5852	16.0 ₃	0.5855	16.0 ₃	0.5848	18.0	0.585	16.0 ₃	0.5855	15.7 ₃	0.5854
B. S. No. 40.	109.0	16.2 ₃	0.5984	16.7 ₃	0.5986	16.7 ₀	0.5970	18.6	0.598	16.6 ₃	0.5986	16.3 ₁	0.5984
Means		16.0 ₃	0.5924	16.2 ₃	0.5925	16.3 ₃	0.5917	18.2 ₃	0.5923*	16.3 ₁	0.5926	15.9 ₃	0.5924

* The current values in this column were only given to three decimal places by the Reichsanstalt, but since the corrections for difference in value of the standard cell amounts to 5 in the next place for the rotating lamps, and is the same for all the lamps, the mean is increased by 5 in the fourth decimal place.

lack of time only a single set of readings was made, the results are not guaranteed by the National Physical Laboratory to a high accuracy. They are therefore omitted in the final comparisons of the units.

Table I reveals several interesting and important facts in regard to the present stage of development of the science of photometry. A comparison of the mean values of the stationary and rotating lamps before being carried abroad and after their return gives some idea of the constancy and portability of seasoned incandescent lamps as secondary photometric standards. The lamps were carried about for two months, were packed and unpacked five times, were burned in all about two or three hours, and yet the mean value of the six stationary lamps was found to have changed by less than 3 parts in 1000, while the mean value of the six rotating lamps apparently changed less than 1 part in 1000. It is not to be inferred from this that the lamps necessarily remained as constant as the results would indicate, since it is impossible to measure with certainty even the mean value of the lamps with an accuracy better than several parts in 1000. The constancy of the resistance of the lamp filaments as shown by the agreement between the mean current values is another indication of the constancy of the lamps and emphasizes the importance of accurate electrical measurements in the photometry of incandescent lamp standards.

A second interesting fact, on which some light is thrown by Table I, is the ultimate accuracy of photometric comparisons. Of course this accuracy depends largely on the nature of the sources to be compared, their color, relative intensity, constancy, etc. Moreover, any single observer with a particular set of apparatus may at any one time, or even at different times, obtain relative values between the two different sources to a remarkably high degree of accuracy. A very important question, however, is the accuracy with which different experienced observers using different apparatus can obtain the relative values of two light sources under favorable circumstances.

This question is partially answered by Table I. The conditions of comparison—at least for the stationary lamps—were as nearly ideal as could be realized with any known sources. The lamps were all of the same color (the voltages having been chosen initially to accomplish this); they were approximately of the same intensity,

and those of each type burned at approximately the same voltage. The observers were experienced in photometry, and the measurements were made in laboratories of international reputation, but the apparatus and the methods of measurement employed were somewhat different.

In order to show the agreement among the different measurements the results are expressed in a more convenient way in Table II. Since the units of the different countries are not the same the values of the individual lamps are expressed in terms of the mean of the six lamps taken as unity.

TABLE II.
Relative Values of Individual Lamps in Terms of Mean.

Lamps	Bureau of Standards	National Physical Laboratory	Laboratoire Central	Reichsanstalt	National Physical Laboratory	Bureau of Standards	Mean Relative Values
B. S. No. 44	0.998	0.998	1.001	1.007	1.003	0.999	1.001
B. S. No. 45	0.992	0.995	0.993	0.991	0.986	0.991	0.991
B. S. No. 46	0.992	0.995	0.994	0.986	0.997	0.993	0.993
B. S. No. 49	1.008	1.005	1.004	1.007	1.005	1.007	1.006
B. S. No. 50	0.996	0.993	0.990	0.991	0.991	0.994	0.992
B. S. No. 64	1.014	1.014	1.018	1.017	1.019	1.016	1.016
Means	1.000	1.000	1.000	1.000	1.000	1.000	1.000
B. S. No. 4	1.016	1.013	1.012	1.014	1.012	1.010	1.013
B. S. No. 7	0.994	0.989	0.981	0.986	0.996	0.993	0.990
B. S. No. 8	1.001	1.001	1.002	1.003	0.999	1.004	1.002
B. S. No. 13	0.986	0.986	1.002	0.992	0.986	0.989	0.990
B. S. No. 30	0.986	0.995	0.981	0.986	0.986	0.984	0.985
B. S. No. 40	1.016	1.027	1.021	1.019	1.020	1.020	1.020
Means	1.000	1.000	1.000	1.000	1.000	1.000	1.000

In the first column are given the numbers of the lamps, the first six being the stationary lamps and the last six the rotating lamps. The next six columns contain the relative values of the lamps as determined at the various laboratories indicated at the head of the respective columns. The last column contains the mean relative values of the individual lamps, being the means of the values in the preceding six columns. If these mean relative values are taken as the true values of the lamps, by comparing them with the values in the other columns, an idea is obtained of the accuracy with which

the individual lamps were measured in terms of the six standards. It is seen that the greatest deviation from the true value is only 0.7 per cent for the stationary lamps and 1.2 per cent for the rotating lamps, whereas the average deviation for the measurements at any one laboratory is in no case as large as 0.3 per cent for the stationary lamps, or 0.5 per cent for the rotating lamps. The accuracy of reading is naturally somewhat less with the rotating lamps than with the stationary lamps, as the flicker for oval anchored filaments rotating at 180 revolutions per minute is quite perceptible and annoying, even to observers accustomed to it.

In comparing the relative values found at the different laboratories with the mean relative values it should be noted that the Reichsanstalt only certified the lamps to one-tenth of a Hefner candle, and therefore the relative values calculated from the Hefner candle values given by that laboratory are only significant to within several units in the last decimal place in Table II. Moreover, as stated above, the first values given by the National Physical Laboratory represent only a single set of readings, and consequently the agreement between the two columns of values given by that institution should not be compared with the agreement between the two columns of values found at this Bureau, and which represent the mean of a number of determinations.

In order to compare the unit of the Bureau with those of the other laboratories, the values given in Table I are grouped in a more convenient form in Table III. In this table the unit of each laboratory is expressed in terms of the unit of the Bureau of Standards, except that the Reichsanstalt unit is expressed in terms of 0.88 times the Bureau unit. The ratios are given as obtained both from the stationary and from the rotating lamps, and it is interesting to note the differences between the two values. In every case the ratio obtained from the rotating lamps is lower than that obtained from the stationary lamps. This difference, which amounts to 0.7 per cent at one of the laboratories, may be due in part to causes discussed in the paper referred to above on the errors incident to the rotating lamp method of measuring mean horizontal intensity. It may also be due in part to small errors in electrical measurements. Although all of the stationary lamps were of one type and all of the rotating lamps were of one type, these two types were not the same. The

stationary lamps were 50-volt lamps, taking a current of 1.3 amperes, whereas the rotating lamps were 110-volt lamps, with currents of 0.6 amperes, so that the electrical conditions in the two cases were quite different.

TABLE III.

Values of Units of Different Laboratories in Terms of the Unit of the Bureau of Standards.

Units of Various Laboratories	Values in Terms of B. S. Unit:		
	Stationary Lamps	Rotating Lamps	Means
Bureau of Standards unit	1.000	1.000	1.000
Reichsanstalt unit	1.001×0.88	0.996×0.88	0.998×0.88
National Physical Laboratory unit	0.987	0.981	0.984
Laboratoire Central unit	0.985	0.978	0.982

It is seen from Table III that the unit of the Bureau is equal to $\frac{1.00}{0.88}$ of the Reichsanstalt unit to within the errors of measurement. Since the unit was originally obtained in this way nearly four years ago it indicates that the Bureau has maintained through incandescent lamp secondary standards, a unit which, compared with the unit of the Reichsanstalt, has not changed appreciably in that time. But though the Bureau has been able to maintain for several years a constant unit by means of incandescent lamp secondary standards, and could probably continue to maintain it in this way for an indefinite period, nevertheless the need of a suitable primary photometric standard is fully recognized and an investigation of primary standards will be undertaken in the near future.

It is evident from Table III that the value of the British candle obtained from the Hefner by the use of the fraction 0.88, which is supposed to represent the mean of the best determinations of the ratio, is not the same as the British candle defined in terms of the Harcourt (10-cp) pentane lamp at the National Physical Laboratory. The difference is about 1.5 per cent, according to the values given in Table III; but, as will be seen later, there is an uncertainty in this ratio amounting to several per cent. Before passing to a brief résumé of the recent determinations of the ratios of the three lamps

recognized as standards in England, France, and Germany, it is desirable to call attention to the approximate agreement between the bougie décimale, the British candle, and the candle in use in the United States in the testing of electric lamps. The difference of about 2 per cent is small compared with the accuracy usually obtained in industrial photometry, and no serious results would follow a compromise by which the three countries would agree upon a unit to be the same for all. This highly desirable end will become more difficult of accomplishment each year, as a higher accuracy of photometric measurement is demanded, and even now a change in unit of over 1 per cent would be felt in the United States. Unfortunately the unit of Germany is so different from all the others that it would probably be impossible to agree upon a unit common to the four countries, but the ratio of the Hefner candle to the candle of France, England, and the United States could be accurately fixed.

The question of a uniform unit is to be distinguished from that of a uniform standard lamp in terms of which the unit is expressed. Thus at present the Carcel lamp in France has an intensity of 9.6 bougies décimales, and the Harcourt pentane lamp in England has an intensity of 10 British candles. Only in Germany has the standard an intensity of one unit, and even here in stating the intensity the word *candle* is affixed (Hefnerkerze). Of course it would be very desirable to adopt a common standard as well as a common unit, but the adoption of the unit is the more important of the two. As soon as an entirely satisfactory standard is presented, its general adoption will probably follow, and this is much more likely to result if the adoption of a new standard does not entail the adoption of a new unit. The Hefner lamp was recommended by the Geneva Congress as a secondary standard, but owing partly to the fact that the Hefner was not found as satisfactory in other laboratories as it had been found at the Reichsanstalt, and partly, no doubt, because of the different value of the unit, its general adoption did not follow. The Carcel continues to be used in France, and the Harcourt pentane lamp is coming into general favor in England.

In 1903 the commission which was appointed by the Paris International Congress of Gas Engineers to fix rules for the photometry

of gas burners decided, upon hearing the report of Doctor Bunte, that the ratios of the three standards, the Hefner, the Carcel, and the Harcourt (10-cp) lamp, were not sufficiently well determined, and recommended that determinations of these ratios be made at competent laboratories in Germany, France, and England. The most probable values of these ratios, as given by Bunte,¹ are reproduced in part in Table IV.

TABLE IV.

Most Probable Ratios of the Hefner, Carcel, and Harcourt (10-cp) Lamps as Given by Bunte.

	Hefner	Harcourt (10-cp)	Carcel
Hefner.....	1.0	0.088	0.092
Harcourt (10-cp).....	11.4	1.0	1.05
Carcel.....	10.87	.95	1.0

In response to requests from the commission new determinations of the ratios of the three lamps have been undertaken in laboratories of England, France, and Germany, and the results of some of these investigations have already been published. The values found at the Reichsanstalt² are given in Table V.

TABLE V.

P. T. R. Values of Harcourt (10-cp) and Carcel Lamps in Terms of Hefner Candles.

Intensity of Harcourt (10-cp) lamp	=	10.9 HK
Intensity of Carcel lamp	=	10.7 HK

The value of the Harcourt lamp is given for the atmospheric conditions adopted as standard at the National Physical Laboratory, which are a barometric height of 760 mm of mercury, and a humidity of 10 liters of water vapor in one cubic meter of pure, dry air. The value given for the Carcel lamp was 10.8 Hefner candles at a

² Recueil des Travaux et Compte rendu des Séances de la Commission internationale de Photométrie. Première Session, Zurich, 1903, p. 45.

¹ Zs. für Instrumentenkunde, 26, p. 186; 1906: also Liebethal, Jour. für Gasbeleuchtung, 49, p. 559; 1906.

humidity of 8.8 liters of water vapor in one cubic meter of pure, dry air; but since the atmospheric conditions adopted as normal in Paris³ are the same as those normal to England, the value was corrected accordingly in order to make the results comparable in the various tables.

In France comparisons were undertaken both at the Laboratoire Central d'Électricité and at the Laboratoire d'Essais. The results of the comparisons at the former laboratory have been published in full⁴, and in an appendix the results obtained at the Laboratoire d'Essais are summarized. At the Laboratoire Central the comparisons were made principally by measuring each lamp against incandescent lamps, but a few experiments were made in which the lamps were compared directly one against another. The results of these last experiments are combined with the results obtained at the Laboratoire d'Essais where but one method was employed, viz, that of comparing the lamps directly one against another. The values which will be submitted by the French laboratories to the International Commission are given in Tables VI and VII, in which the intensity of each lamp is reduced to the value it would have under the atmospheric conditions adopted as normal to the respective country.

In Table VI are given the results obtained at the Laboratoire

TABLE VI.

Ratios Determined at Laboratoire Central by Indirect Comparison Through Incandescent Lamps.

	Carcel	Hefner	Harcourt (10-cp)
Carcel	1.0	10.76	1.0
Hefner	0.0929	1.0	0.0929
Harcourt (10-cp)	1.0	10.76	1.0

Central by indirect comparison through incandescent lamps. In Table VII are given the means of two series of measurements made

³ Laporte et Jouaust, Bulletin de la Société Internationale des Electriciens, Series 2, 6, p. 375; 1906.

⁴ Laporte et Jouaust, loc. cit.

at the Laboratoire d'Essais and one series made at the Laboratoire Central, all by direct comparison.

TABLE VII.

Ratios Determined at Laboratoire d'Essais and Laboratoire Central by Direct Comparison.

	Carcel	Hefner	Harcourt (10-cp)
Carcel	1.0	10.73	0.991
Hefner	0.0932	1.0	0.0933
Harcourt (10-cp)	1.009	10.72	1.0

The results of the comparisons made at the National Physical Laboratory have not as yet been published in full, but in the summary of the work of the Reichsanstalt for the last year,⁵ where the results of the measurements made at the Reichsanstalt are given, the values given in Table VIII are recorded as having been determined at the National Physical Laboratory. The value of the Carcel lamp is corrected to the values it would have under normal atmospheric conditions.

TABLE VIII.

N. P. L. Values of Harcourt (10-cp) and Carcel Lamps in Terms of Hefner Candles.

Intensity of Harcourt (10-cp) lamp	=	10.95 HK
Intensity of Carcel lamp	=	10.68 HK

In the determinations of the ratios made at the various laboratories, of necessity different lamps were employed. At each laboratory the respective lamp or lamps recognized as the standard of the laboratory were probably used, but at the Laboratoire Central, for example, the Hefner and Harcourt (10-cp) lamps that were used were not the particular lamps that are taken as the standards of the Reichsanstalt and the National Physical Laboratory, respectively, so that any lack of reproducibility of any of the standards might cause differences in the ratios determined at the different laboratories. It is

⁵ Loc. cit.

interesting therefore to compare these ratios with those calculated from the values assigned by the various laboratories to the twelve incandescent lamps of the Bureau of Standards. The ratios calculated from the incandescent lamp measurements are given in Table IX. In making the computations the intensity of the Carcel lamp

TABLE IX.

Ratios Calculated from Intensities Assigned to B. S. Incandescent Lamps.

	Harcourt (10-cp)	Hefner	Carcel
Harcourt (10-cp)	1.0	11.19	1.043
Hefner	0.0893	1.0	0.0932
Carcel	0.959	10.73	1.0

was taken as 9.6 bougies décimales, the Harcourt (10-cp) lamp as equal to 10 British candles, and the Hefner lamp as equal to 1 Hefner candle.

It would seem that the values obtained through the incandescent lamps are in some ways the more accurate ratios of the *units* of the various countries, because the primary standard of each country is operated under the usual conditions by the most experienced observers

TABLE X.

Summary of Ratios of Hefner, Harcourt (10-cp), and Carcel Lamps.

	Harcourt (10-cp)	Carcel	Harcourt (10-cp)
	Hefner	Hefner	Carcel
Reichsanstalt (using an electric comparison lamp) ..	10.9	10.7	1.02
Laboratoire Central (using an electric comparison lamp)	10.76	10.76	1.00
Laboratoire Central and Laboratoire d'Essais (direct comparison)	10.72	10.73	1.009
National Physical Laboratory	10.95	10.69	1.024
Ratios computed from Bureau of Standards incandescent lamps	11.19	10.73	1.043
Best previous values (Bunte)	11.4	10.87	1.05

with the respective lamp, whereas in the direct comparisons different lamps are used, and the conditions of operation in the different countries may be slightly different from the conditions normal to the laboratory in which the respective lamp is a standard.

The ratios obtained at the various laboratories together with the values obtained through the Bureau of Standards incandescent lamps and the best previous values, as given by Bunte, are summarized in Table X. It is seen from this table that the best agreement is obtained in the ratio of the Carcel lamp to the Hefner lamp, whereas discrepancies amounting to 4 or 5 per cent occur in the other ratios. The most important result to engineers of the United States is the discrepancy in the ratio of the Hefner candle to the British candle as given by the Harcourt (10-cp) lamp. The value given by Bunte as the best previous ratio is 0.88, and so is the same as that recommended by the American Institute of Electrical Engineers and adopted by the Bureau. This value, however, is 2 per cent different from that obtained through the Bureau of Standards incandescent lamps, and 5 or 6 per cent different from the values found at the various laboratories. If, therefore, the British candle be taken as one-tenth the intensity of the Harcourt (10-cp) pentane lamp at the National Physical Laboratory, according to the values assigned the Bureau of Standards incandescent lamps, it is 2 per cent different from the candle in this country obtained from the Hefner lamp by using the ratio 0.88. If the British candle be taken as one-tenth the intensity of any Harcourt (10-cp) lamp, it would seem from Table X to be different from the candle in this country by 5 or 6 per cent. Whatever the amount of the difference, it would seem that the candle in use in the photometry of electric lamps in the United States is sufficiently different from the British candle, as obtained through Harcourt (10-cp) pentane lamps, to justify defining the unit obtained through the Hefner by use of the ratio 0.88 as the *candle* instead of the *British candle*. It is hoped, however, that the suggestion made in a previous paragraph in regard to an international *unit* may be carried out, and that by a constant intercomparison of the standards of the different countries by the interchange of electric incandescent lamps the unit may be maintained uniform.

GEOMETRICAL THEORY OF RADIATING SURFACES WITH DISCUSSION OF LIGHT TUBES.

By Edward P. Hyde.

Theoretical photometry assumes two general laws of radiation. (1) The law of variation of the intensity of illumination of a surface in inverse proportion to the square of the distance of the surface from the luminous source is merely a statement of a geometrical property, if the rectilinear propagation of light is assumed. (2) Lambert's law of variation of the intensity of a luminous surface in direct proportion to the cosine of the angle of emission is an empirical law based primarily on the observation that a uniformly bright sphere, when viewed at a distance, appears as a uniformly bright disk. It would seem to follow from Kirchhoff's law that Lambert's cosine law can be true only for a black body, but no satisfactory experiments have been made, so far as the writer knows, to test the law in its application to glowing surfaces. Numerous investigations of the cosine law as applied to diffusing screens have been undertaken, but the results of these are at considerable variance, due principally to the difficulty of obtaining perfect mat surfaces.

These two laws, the inverse square law and Lambert's cosine law, which are assumed as the basis of theoretical photometry, are applicable to infinitesimal sources. The inverse square law follows rigorously for a point source only, and the cosine law is always assumed as applicable primarily to the infinitesimal elements of a surface, since this is the inference from the observation that a uniformly bright sphere, when viewed at a distance, appears as a uniformly bright disk. These two laws may be stated mathematically as follows:

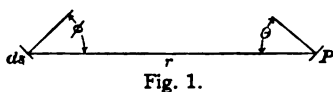


Fig. 1.

$$J = \frac{i dS \cos \phi \cos \theta}{r^2} \quad (1)$$

where

J = intensity of illumination of the screen at P, i. e. the quantity of light per second falling normally on a unit area of the screen at P.

i = specific light intensity of the radiating element of surface dS , i. e. the quantity of light per second emitted normally by dS .

ϕ = angle of emission from dS .

θ = angle of incidence on the screen at P.

r = distance between dS and P.

Since these two laws are deduced for sources of infinitesimal dimensions, errors of considerable magnitude may result in applying them to extended sources. Particularly is this so in the case of the inverse square law. This law underlies the great majority of practical photometric measurements, and its applicability is seldom questioned. Indeed, when it is questioned, it is usually on the ground that since the source is extended it is impossible to determine the effective center of radiation. The source is considered as an aggregate of point sources, rather than as a continuous surface, to each element of which both the inverse square law and the cosine law apply, and which, therefore, considered as a whole, will have a complex law of its own, essentially different from the simple inverse square law.

In a previous paper¹ the writer solved the case for a finite cylinder in connection with a study of Talbot's law as applied to the rotating sector disk. Since this case serves as an excellent illustration of the difference between a point source and a radiating surface, showing the errors resulting from an assumption of the inverse square law for such a surface, its solution will be repeated here with some additions.

Assuming Lambert's law and the inverse square law for infinitesimal elements of surface, as combined in equation (1), if we let ϕ (Fig. 2) be the angle of emission from any element of surface dS of the radiating cylinder; θ the angle of incidence of the ray from dS on a screen at P placed at right angles to OP, where OP lies in

¹ This Bulletin, 2, p. 1; 1906.

the plane perpendicular to the axis of the cylinder at its middle point, O ; r the distance from the element dS to the screen at P ; and i the specific light intensity, supposed to be constant over the surface of the cylinder: then the intensity of illumination of the screen at P is

$$J = \iint \frac{i \cos \phi \cos \theta}{r^2} dS$$

taken over that part of the curved surface of the cylinder convex toward P ,

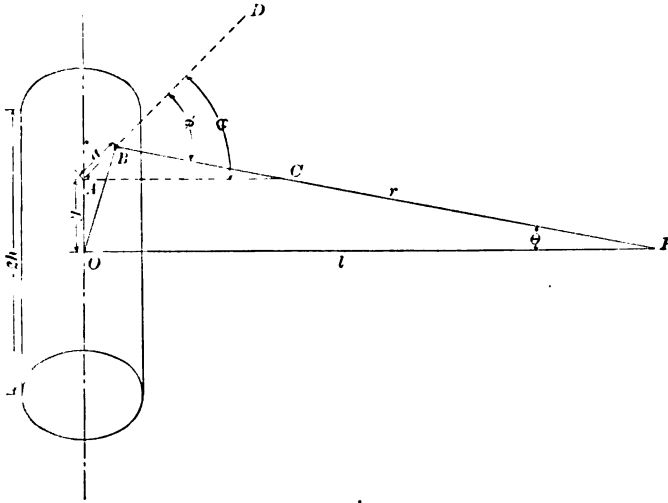


Fig. 2.

Expressing all the quantities involved in the above equation in terms of the two cylindrical coordinates a and y , and denoting the radius of the cylinder by a , and the height by $2h$, the intensity of illumination at P , at a distance, l , from the axis of the cylinder, is given by the equation,

$$J = ia \iint \frac{(l \cos a - a)(l - a \cos a)}{(a^2 + l^2 - 2al \cos a + y^2)^{3/2}} da dy \quad (2)$$

in which the limits of y are $-h, +h$, and the limits of a are

$$-\cos^{-1} \frac{a}{l}, +\cos^{-1} \frac{a}{l}$$

The integral of the above expression is,

$$J = \frac{i}{l} \left\{ a \cos^{-1} \frac{l^2 - a^2 - h^2}{l^2 - a^2 + h^2} + h \left[(q+1) \frac{1}{\sqrt{q}} \cot^{-1} \sqrt{\frac{p}{q}} - 2 \cot^{-1} \sqrt{p} \right] \right\} \quad (3)$$

in which

$$p = \frac{l+a}{l-a} \quad q = \frac{(l+a)^2 + h^2}{(l-a)^2 + h^2} \quad (4)$$

Before substituting numerical values for a and h , it is interesting to note the form which the equation assumes when h approaches infinity. Under this condition q approaches unity, and in the limit equation (3) becomes

$$J = \frac{\pi i a}{l} \quad (5)$$

Thus the intensity of illumination due to an infinitely long uniformly radiating cylinder varies inversely as the first power of the distance from the axis of the cylinder, a result which also follows from purely physical considerations of the normal flow of energy across coaxial cylindrical surfaces.

Let us now give definite numerical values to a and h in equation (3), compute the intensity of illumination at different distances, l , and compare the relative values with those obtained on the assumption of the inverse square law. Let us make $h = 10$ mm and $a = 1$ mm,

TABLE I.

Deviation from the Inverse Square Law of the Radiation of a Cylinder 20 mm long and 1 mm radius.

Distances, l	J (Equation 3)	J (Inverse Square Law)	Deviation
3000 mm	1.0000 $\times J_{3000}$	1.0000 $\times J_{3000}$	$\pm 0.00\%$
2000 "	2.2500 "	2.2500 "	+0.00 "
1000 "	9.0045 "	9.0000 "	+0.05 "
500 "	3.6040 $\times 10$ "	3.6000 $\times 10$ "	+0.11 "
200 "	2.2545 $\times 10^2$ "	2.2500 $\times 10^2$ "	+0.20 "
100 "	9.0081 $\times 10^3$ "	9.0000 $\times 10^3$ "	+0.09 "
80 "	1.4049 $\times 10^3$ "	1.4062 $\times 10^3$ "	-0.09 "
50 "	3.5593 $\times 10^3$ "	3.6000 $\times 10^3$ "	-1.13 "

since these are the approximate dimensions of an 88-watt Nernst glower, for which the case originally was solved. The results of substituting these values of a and h in equation (3) for different distances, l , are shown in Table I and Fig. 3.

In the first column of Table I are given the distances, l , for which the values of J were computed. The second column contains the

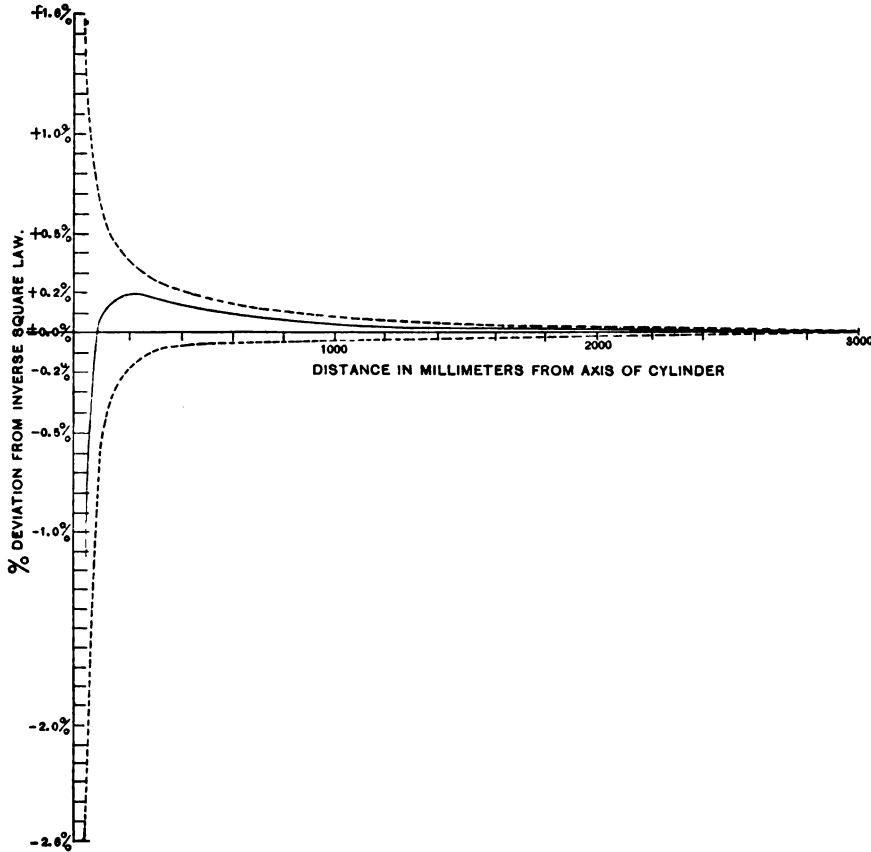


Fig. 3.—*Deviation of Radiation of Cylinder from Inverse Square Law.*

ratios of the intensities at the different distances, l , to that at $l = 3000$ mm, obtained by direct substitution in equation (3), and the third column gives the same ratios obtained by the inverse square law. The differences, expressed as percentage deviations from the inverse square law, are given in the fourth column. They are also shown in the form of a curve by the solid line in Fig. 3, in which abscissas

are distances, and ordinates are percentage deviations from the inverse square law, as deduced from the value of the intensity at $l=3000$ mm.

The value of J between $l=3000$ mm and $l=90$ mm is greater (as compared with J at $l=3000$ mm) than the value of J deduced on the assumption of the inverse square law, the maximum deviation being about $+0.2$ per cent. At distances less than 90 mm the intensity becomes very much less than the values demanded by the inverse square law, the deviation at $l=50$ mm being over 1 per cent. The reason for the peculiar form which the curve takes becomes evident when we consider the various elements that combine to determine the illumination at any distance. The greater part of the effective radiating surface is nearer the screen on which the illumination is calculated than the center of the cylinder, from which the distance, l , used in applying the inverse square law is measured. On approaching the cylinder, if this effect alone were considered, the illumination would increase more rapidly than that calculated on the assumption of the inverse square law. On the other hand, due to the increased inclination of the emitted light, both to the radiating surface elements and to the screen on which the rays impinge, the illumination falls off rapidly as the cylinder is approached. A third element, which, however, is negligible over the range of distance investigated, is the difference in the effective area of the cylinder at different distances. The combination of these various effects produces the peculiar form of the curve in Fig. 3. At great distances from the cylinder the difference in distance from the screen to the radiating surface elements and to the center of the cylinder is the most important element, and so the curve of deviation from the inverse square law rises at first. But as l becomes smaller the effect of the increased inclination of the emitted rays becomes more important, so that the curve crosses the axis at $l=90$ mm, the deviations from the inverse square law becoming negative and increasing rapidly as the surface of the cylinder is approached.

Although it is impossible to separate completely the different effects, by considering the two extreme cases of (1) a cylinder with radius a but with an infinitesimal height, and (2) a cylinder with height $2h$ but with an infinitesimal radius, we can analyze the

complex curve of Fig. 3 into two simple curves. For the first case, if we make h approach zero in equation (3) we obtain as the expression for the illumination due to the radiation from the edge of a flat circular disk,

$$J' = \frac{ih}{l^2 p} \left[(p-1)\sqrt{p} + (p^2+1)\cot^{-1}\sqrt{\frac{1}{p}} - 2p \cot^{-1}\sqrt{p} \right] \quad (6)$$

Substituting the numerical values used above, we get as the corresponding curve of deviations from the inverse square law the dotted curve in Fig. 3, lying entirely above the axis of abscissas.

For the second case, making a approach zero in equation (2), h remaining finite, we get as the expression for the illumination due to a radiating line of length $2h$,

$$J'' = 2ia \left[\frac{h}{l^2 + h^2} + \frac{1}{l} \tan^{-1} \frac{h}{l} \right] \quad (7)$$

Substituting numerical values in this equation, we get the dotted curve in Fig. 3 lying entirely below the axis of abscissas. The addition of the two dotted curves gives a curve coincident with the solid curve, which was plotted directly from equation (3).

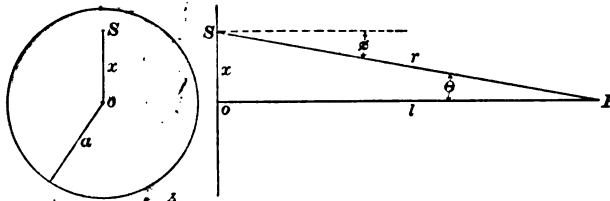


Fig. 4.

Another simple illustration of the treatment of a radiating surface, as distinguished from a point source, or aggregate of point sources, showing the errors resulting from the assumption of the applicability of the inverse square law to such a source, is had in the case of a finite plane of uniform specific light intensity, i . Suppose the plane to be a circle of radius a (Fig. 4). The illumination at P, of the screen normal to OP, where OP is perpendicular to the radiating circular disk at its center, O, is given by equation (1), integrated over the surface of the disk. Since, however, $\phi = \theta$, and all the variables can be expressed in terms of the single variable, x ,

the distance from the center of the disk to the element of surface dS , equation (1) becomes,

$$\begin{aligned} J &= \iint \frac{i \cos^3 \theta}{r^3} dS \\ &= 2 \pi i l^2 \int_0^a \frac{x dx}{(x^2 + l^2)^{3/2}} \\ &= \pi i \frac{a^2}{a^2 + l^2} \end{aligned} \quad (8)$$

From this equation the illumination at different distances, l , can be calculated, and the relative values compared with those obtained from the inverse square law, starting from the illumination at some definite distance, say $l=1000$ mm. The results of this comparison are shown in Table II and Fig. 5, analogous to Table I and Fig. 3 for the corresponding case of a radiating cylinder.

TABLE II.

Deviation from the Inverse Square Law of the Radiation of a Flat Circular Disk of 10 mm radius.

Distances, l	J (Equation 8)		J (Inverse Square Law)		Deviation
1000 mm	1.0000	$\times J_{1000}$	1.0000	$\times J_{1000}$	$\pm 0.00\%$
800 "	1.5624	"	1.5625	"	- 0.01 "
500 "	3.9988	"	4.0000	"	- 0.03 "
300 "	1.1100×10	"	1.1111×10	"	- 0.10 "
100 "	9.9020×10	"	1.0000×10^2	"	- 0.98 "
50 "	3.8465×10^2	"	4.0000×10^2	"	- 3.8 "
30 "	1.0001×10^3	"	1.1111×10^3	"	-10.0 "
10 "	5.0005×10^3	"	1.0000×10^4	"	-50.0 "
5 "	8.0008×10^3	"	4.0000×10^4	"	-80.0 "
1 "	9.9019×10^3	"	1.0000×10^6	"	-99.0 "

In each of the above special cases the expression has been deduced for the illumination at different distances along a single line or in a single plane symmetrical with respect to the radiating surface. Thus in the case of the cylinder, equation (3) gives the illumination at points in the plane normal to the axis of the cylinder at its middle

point; in the case of the disk, the illumination at points along the line normal to the disk at its center. We shall now derive the expression which will give the illumination at any point in space, though still only for a special case. Moreover, the idea of *tubes of light* will be introduced, and functions will be derived for the *light field*, analogous to *potential* and *intensity* in the electrostatic field.

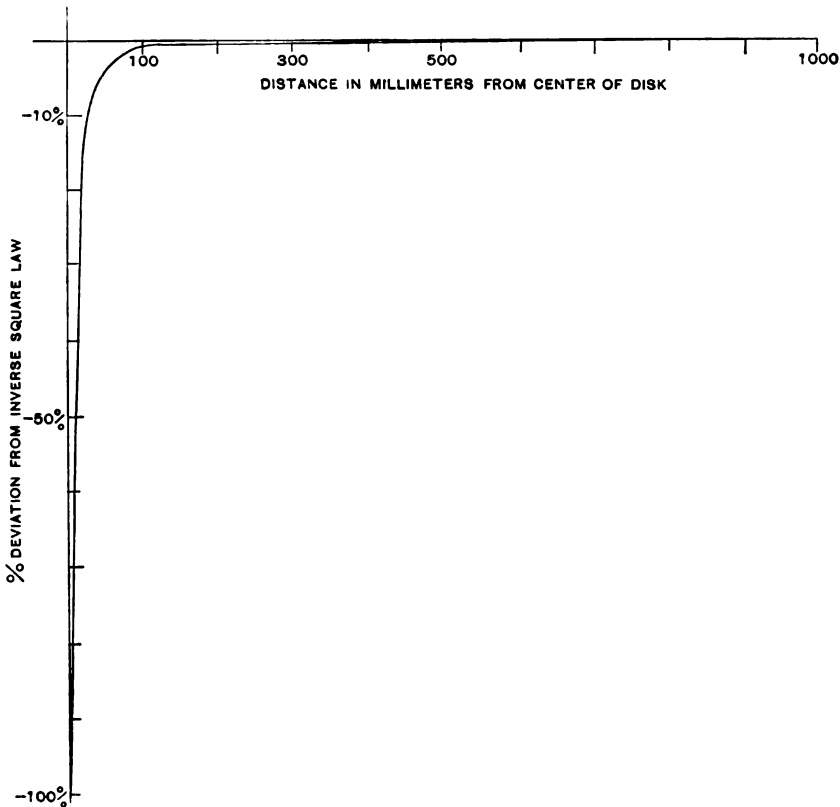


Fig. 5.—Deviation of Radiation of Circular Disk from Inverse Square Law.

Before proceeding, however, it is desirable to introduce the term *specific luminous flux* in place of the much less general term *illumination*. The *illumination* of a surface was defined by the International Congress of Electricians at Geneva in 1896 as the *luminous flux* across a given surface, divided by the area of the surface. The *luminous flux* was defined as the *intensity* of the source multiplied by the solid angle subtended at the source by the given surface.

This definition of *luminous flux* is applicable to point sources only, since the solid angle can have no meaning in the case of a source of finite dimensions. If, however, we define the *luminous flux* across a given surface as the quantity of luminous energy flowing normally across the surface in one second, we have a definition which is equally applicable to radiating surfaces and to point sources. Moreover, if the unit of *luminous flux* is defined as the quantity of luminous energy which in one second flows normally across a surface subtending a unit solid angle at a point source of unit *intensity*, the new definition of *luminous flux* is in perfect agreement with the existing definition for point sources, and is also applicable to radiating surfaces.

Intensity of illumination, which may be defined as

$$J = \frac{d\Phi}{dS} \quad (9)$$

where $d\Phi$ is the luminous flux across the surface dS , always presupposes the existence of a material screen. It is easily seen that under this condition the resultant flux per unit area across any imaginary surface may be zero, whereas the illumination on the two sides of a material screen coincident with the imaginary surface may be large, if the same on both sides of the screen. Since in the problem of the light field this unique property of the possible separation of the positive and negative flux across a surface is found, it is desirable, in considering the light field in its analogy to the electrostatic field to employ the term *luminous flux per unit area*, rather than the more special term *illumination*. At every point in space there is some definite direction in which the flux of luminous energy is a maximum. Let us define the quantity of luminous energy which in one second flows normally across a surface of unit area placed perpendicular to the direction of maximum flux, as the *specific luminous flux*, Φ_0 at the point. Φ_0 is a vector quantity, and the component in any direction equals numerically the difference in illumination on the two sides of an infinitely thin material screen placed perpendicular to the direction. If the illumination on one side is zero the *specific luminous flux* numerically equals the illumination on the other side of the screen. We shall employ the vector *specific luminous flux* in the following discussion, in which the rectilinear propagation of light is assumed.

Considering the case of an infinitely long radiating strip of uniform specific light intensity, i , let us determine the amount and direction of the specific luminous flux at every point in space. Since, however, the field will be the same in all planes perpendicular to the long dimension of the strip, it is only necessary to consider the distribution in any one of these parallel planes.

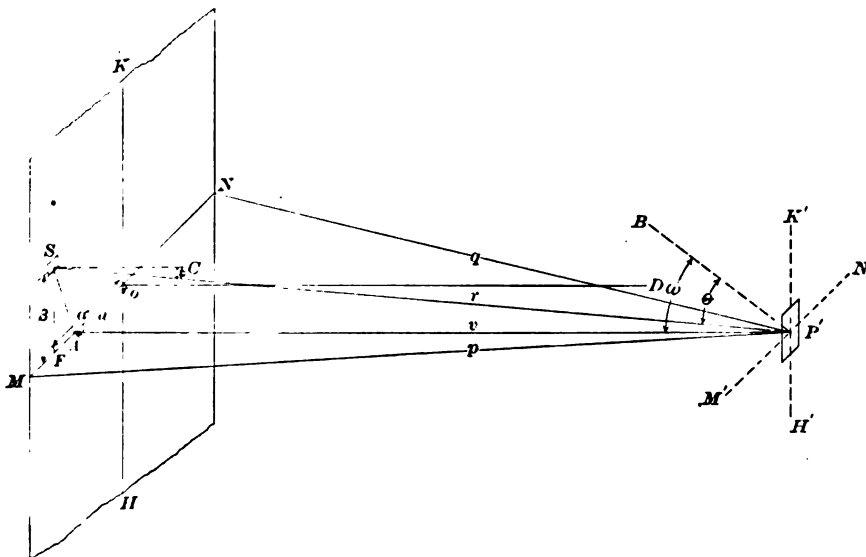


Fig. 6.

The component of the specific luminous flux at any point P (Fig. 6) across a surface, the normal to which lies in the same horizontal plane as the normal to the radiating strip, but inclined to it at an angle, ω , is (equation 1),

$$(\Phi_0)_\omega = i \int \int \frac{\cos \phi \cos \theta}{r^2} dS \quad (10)$$

taken over the entire strip. The distance from the element of surface dS at S to the point P is given by the equation,

$$r = \sqrt{\beta^2 + (a - u)^2 + v^2}$$

where a and β are the coordinates of the point S with reference to the rectangular axes KH and MN lying in the strip and having as

origin the point O. u and v are the coordinates of the point P with reference to the axes MN and OD.

Moreover,

$$dS = da d\beta$$

and

$$\cos \phi = \frac{v}{\sqrt{\beta^2 + (a-u)^2 + v^2}}.$$

$\cos \theta$ can be obtained as follows: θ is the angle between the ray PS and the normal PB to the surface at P. If the direction cosines of PB and PS with reference to the right-handed system of axes PA, PM', and PK' are, respectively, λ , μ , ν , and λ' , μ' , ν' , then

$$\cos \theta = \lambda\lambda' + \mu\mu' + \nu\nu' \quad (11)$$

where

$$\begin{aligned} \lambda &= \cos \omega & \mu &= -\sin \omega & \nu &= 0 \\ \lambda' &= \cos \phi & \mu' &= \frac{a-u}{\sqrt{\beta^2 + (a-u)^2 + v^2}} & \nu' &= \frac{v}{\sqrt{\beta^2 + (a-u)^2 + v^2}} \end{aligned}$$

Equation (11) becomes, on substitution,

$$\cos \theta = \frac{v}{\sqrt{\beta^2 + (a-u)^2 + v^2}} \cos \omega - \frac{a-u}{\sqrt{\beta^2 + (a-u)^2 + v^2}} \sin \omega$$

Substituting in equation (10) the above expressions for r , dS , $\cos \phi$ and $\cos \theta$,

$$(\Phi_0)_\omega = i \iint \frac{v^2 \cos \omega da d\beta}{[\beta^2 + (a-u)^2 + v^2]^{\frac{3}{2}}} - i \iint \frac{v(a-u) \sin \omega da d\beta}{[\beta^2 + (a-u)^2 + v^2]^{\frac{3}{2}}}$$

or

$$\begin{aligned} (\Phi_0)_\omega &= iv^2 \cos \omega \iint \frac{da d\beta}{[\beta^2 + (a-u)^2 + v^2]^{\frac{3}{2}}} \\ &\quad - iv \sin \omega \iint \frac{(a-u) da d\beta}{[\beta^2 + (a-u)^2 + v^2]^{\frac{3}{2}}} \\ &= iv^2 A \cos \omega - iv B \sin \omega \end{aligned} \quad (12)$$

where

$$A = \iint \frac{da d\beta}{[\beta^2 + (a-u)^2 + v^2]^{\frac{3}{2}}} \quad B = \iint \frac{(a-u) da d\beta}{[\beta^2 + (a-u)^2 + v^2]^{\frac{3}{2}}} \quad (13)$$

the integrals extending from $\beta = -\infty$ to $\beta = +\infty$, and from $a = -1$ to $a = +1$, the width of the strip being taken as 2 units.

Evaluating the above integrals,

$$\begin{aligned} A &= \int da \int [\beta^2 + (a-u)^2 + v^2]^{-\frac{1}{2}} d\beta = \frac{\pi}{2} \int_{-1}^{+1} \frac{da}{[(a-u)^2 + v^2]^{\frac{1}{2}}} \\ &= \frac{\pi}{2v^2} \left[\frac{a-u}{\sqrt{(a-u)^2 + v^2}} \right]_{-1}^{+1} = \frac{\pi}{2v^2} \left[\frac{1-u}{p} + \frac{1+u}{q} \right] \end{aligned} \quad (14)$$

where

$$p = \overline{PM} = \sqrt{(1-u)^2 + v^2}; \quad q = \overline{PN} = \sqrt{(1+u)^2 + v^2} \quad (15)$$

Similarly

$$\begin{aligned} B &= \int (a-u) da \int_{-\infty}^{+\infty} \frac{d\beta}{[\beta^2 + (a-u)^2 + v^2]^{\frac{1}{2}}} \\ &= \frac{\pi}{2} \int_{-1}^{+1} \frac{(a-u) da}{[(a-u)^2 + v^2]^{\frac{3}{2}}} = \frac{\pi}{2} \left[\frac{1}{q} - \frac{1}{p} \right] \end{aligned} \quad (16)$$

Substituting these expressions for A and B in equation (12)

$$(\Phi_0)_\omega = \frac{\pi i}{2} \left[\frac{q(1-u) + p(1+u)}{pq} \right] \cos \omega + \frac{\pi i v}{2} \left[\frac{q-p}{pq} \right] \sin \omega \quad (17)$$

In order to determine the direction of specific luminous flux, i. e. in order to find the value of ω , which will make $(\Phi_0)_\omega$ a maximum, put

$$\frac{\partial(\Phi_0)_\omega}{\partial \omega} = -\frac{\pi i}{2} \left[\frac{(q+p) - u(q-p)}{pq} \right] \sin \omega + \frac{\pi i v}{2} \left[\frac{q-p}{pq} \right] \cos \omega = 0$$

from which

$$\tan \omega_0 = \frac{v(q-p)}{(q+p) - u(q-p)} \quad (18)$$

It can readily be shown that this angle ω_0 is the angle APE (Fig. 7) which the normal at the point (u, v) to the ellipse passing through the point $P(u, v)$, and having as foci M and N , makes with the line PA parallel to the v -axis. (Compare Figs. 6 and 7 in which the same letters refer to corresponding points.) For if a and b are

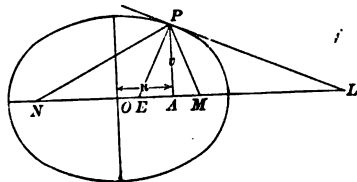


Fig. 7.

respectively the semimajor and semiminor axes of the ellipse, where $a^2 - b^2 = c^2 = \overline{OM}^2 = 1$, the slope of the tangent at P is

$$\tan \text{OLP} = \tan \text{APE} = \frac{b^2 u}{a^2 v} = uv \frac{b^2}{a^2 v^2} \quad (19)$$

But

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \quad \text{or} \quad a^2 - u^2 = \frac{a^2 v^2}{b^2}$$

Therefore

$$\tan \text{APE} = \frac{uv}{a^2 - u^2} \quad (20)$$

Similarly the expression for $\tan \omega_0$ reduces to the same form. From equation (18)

$$\tan \omega_0 = \frac{v(q-p)}{(q+p) - u(q-p)} = \frac{v(q^2 - p^2)}{(q+p)^2 - u(q^2 - p^2)} \quad (21)$$

For the ellipse with foci M, N,

$$q + p = \overline{MP} + \overline{NP} = 2a \quad (22)$$

and from equations (15)

$$q^2 - p^2 = 4u$$

Substituting these values in equation (21)

$$\tan \omega_0 = \frac{uv}{a^2 - u^2} \quad (23)$$

From equations (20) and (23)

$$\omega_0 = \text{angle APE} \quad (24)$$

or the surface through any point (u, v) across which the specific luminous flux due to an infinitely long uniformly radiating strip of width $MN = 2$ is a maximum, will be tangent to the ellipse passing through the point and having M and N as foci. Now, since the system of confocal hyperbolas having the foci M and N is orthogonal with respect to the system of confocal ellipses having the same foci, it follows that the direction of maximum specific luminous flux at

any point in the plane (being the same for all parallel planes) is in the direction of the hyperbola through the point. In other words, the direction of the vector Φ_0 at all points in space is that of the hyperbola through the point.

Suppose now we draw the system of confocal hyperbolas (Fig. 8) in such a way that the distance between the intercepts on the axis of u of successive hyperbolas is constant and equal to $1/i$.

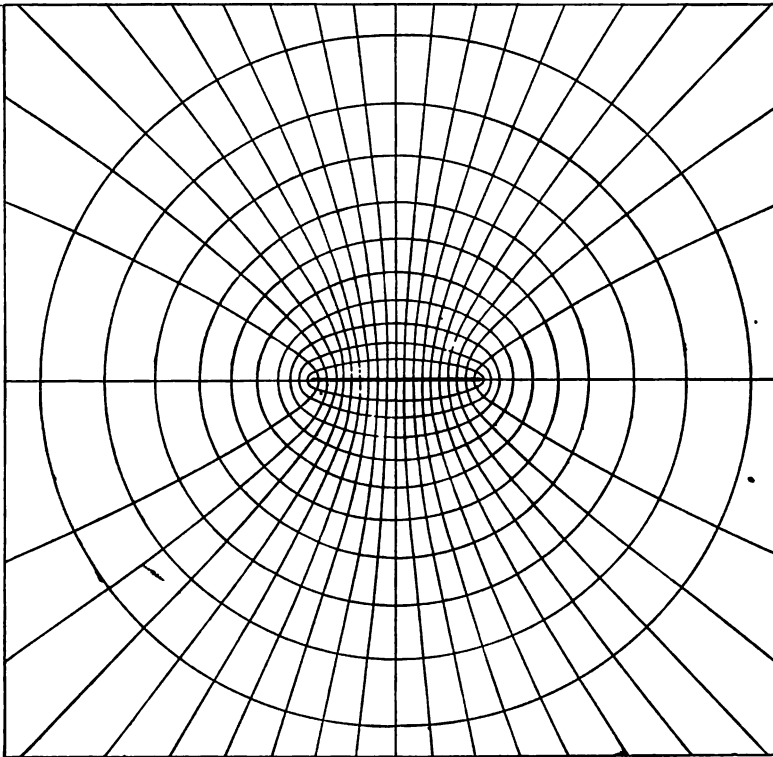


Fig. 8.—*Lines of Maximum Light Flux and Equipotential Surfaces for Uniformly Radiating Strip of Infinite Length.*

Since the specific luminous intensity, i , of a radiating surface is defined as the quantity of light per second radiated normally by a unit area of the surface, the tubes formed by the hyperbolic cylinders of unit height may be considered as unit light tubes, the flow of light being along the tube at all points, and equal to unity at the radiating surface.

It will now be shown that the specific luminous flux at any point is proportional to the number of unit light tubes per unit area of a surface perpendicular to the direction of maximum flow. In other words, the specific luminous flux Φ_0 is solenoidal, the effect at any point being the same mathematically as if the energy starting out in a light tube continued in the same tube. Of course we know physically that an eye placed at any point can see the whole radiating surface, so that the solenoidal property of the specific luminous flux is merely a mathematical property, as is probably also true in the case of Faraday tubes. But the specific luminous flux Φ_0 , and hence the intensity of illumination at any point, can be determined by the application of this property, which will now be proved, viz. that the quantity of luminous energy per second flowing across successive elliptic cylinders between two definite hyperbolic cylinders is constant.

Since the distance MN between the foci has been taken as equal to 2, the major and minor axes of the confocal ellipses and hyperbolas will be $2 \cosh x$, $2 \sinh x$, and $2 \cos y$, $2 \sin y$, respectively, where x and y are arbitrary parameters.

The u - v plane with the orthogonal system of confocal ellipses and hyperbolas corresponding to $x = \text{constant}$ and $y = \text{constant}$, respectively, may be considered as the conformal representation of the x - y plane. The relations between the variables are expressed by the equations of the ellipses and hyperbolas:

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 x} = 1 \qquad \frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1 \qquad (25)$$

from which

$$u = \cosh x \cos y \qquad v = \sinh x \sin y \qquad (26)$$

If in equation (19) we substitute for a and b , the semimajor and semiminor axes of the ellipse through any point, the expressions $\cosh x$ and $\sinh x$, and for u and v the values given in equation (26), we obtain for $\tan \omega_0$ (equation 24) the expression,

$$\begin{aligned} \tan \omega_0 &= \frac{b^2}{a^2} \frac{u}{v} = \frac{\sinh^2 x}{\cosh^2 x} \frac{\cosh x \cos y}{\sinh x \sin y} \\ &= \frac{\sinh x \cos y}{\cosh x \sin y} \end{aligned} \qquad (27)$$

from which

$$\sin \omega_0 = \frac{\sinh x \cos y}{\sqrt{\sinh^2 x + \sin^2 y}} \quad (28)$$

$$\cos \omega_0 = \frac{\cosh x \sin y}{\sqrt{\sinh^2 x + \sin^2 y}} \quad (29)$$

Substituting these values in equation (17)

$$\Phi_0 = \frac{\pi i}{2pq} \left\{ \left[(q+p) - u(q-p) \right] \frac{\cosh x \sin y}{\sqrt{\sinh^2 x + \sin^2 y}} + (q-p) \sinh x \sin y \frac{\sinh x \cos y}{\sqrt{\sinh^2 x + \sin^2 y}} \right\} \quad (30)$$

For the ellipse (equation 22)

$$q+p=2a=2 \cosh x$$

and for the hyperbola,

$$q-p=2a'=2 \cos y$$

from which

$$\left. \begin{aligned} p &= \cosh x - \cos y \\ q &= \cosh x + \cos y \\ pq &= \cosh^2 x - \cos^2 y = \sinh^2 x + \sin^2 y \end{aligned} \right\} \quad (31)$$

Substituting these values in equation (30) and putting for u its value from equation (26)

$$\Phi_0 = \frac{\pi i \sin y}{\sqrt{\sinh^2 x + \sin^2 y}} \quad (32)$$

(The denominator $\sqrt{\sinh^2 x + \sin^2 y}$ which is equal to $\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$ is the linear magnification at the point (x, y) in the transformation from the x - y to the u - v plane).

It follows from equation (32) that the ratio of the two values of maximum specific luminous flux at the points of intersection of any definite hyperbola, $y=\text{constant}$, with two ellipses x' and x'' is

$$\frac{\Phi_0'}{\Phi_0''} = \frac{\sqrt{\sinh^2 x'' + \sin^2 y}}{\sqrt{\sinh^2 x' + \sin^2 y}} \quad (33)$$

Now the ratio of the distances on the two ellipses intercepted by the two consecutive hyperbolas y and $y + \Delta y$ is readily obtained. From equations (26), for any ellipse $x = \text{constant}$

$$\Delta u = -\cosh x \sin y \Delta y; \Delta v = \sinh x \cos y \Delta y$$

and therefore

$$\Delta s = \sqrt{(\Delta u)^2 + (\Delta v)^2} = \Delta y \sqrt{\sinh^2 x + \sin^2 y} \quad (34)$$

Hence the ratio of the distances on the two ellipses x' and x'' , between the same two consecutive hyperbolas y and $y + \Delta y$, is

$$\frac{\Delta s'}{\Delta s''} = \frac{\sqrt{\sinh^2 x' + \sin^2 y}}{\sqrt{\sinh^2 x'' + \sin^2 y}} \quad (35)$$

Comparing this equation with equation (33)

$$\frac{\Phi_0'}{\Phi_0''} = \frac{\Delta s'}{\Delta s''} \quad (36)$$

or

$$\Phi_0' \Delta s' = \Phi_0'' \Delta s'' \quad (37)$$

which states that the total flux of light within a light tube remains constant. In other words Φ_0 is solenoidal, and therefore its divergence must be zero,

$$\frac{\partial}{\partial u}(\Phi_0)_u + \frac{\partial}{\partial v}(\Phi_0)_v = 0 \quad (38)$$

At $x = \infty$, where the ellipses become circles, it follows from equation (32) that

$$\Phi_0 = \frac{\pi i \sin y}{\sinh x} \quad (39)$$

and so is proportional to $\sin y$ at any circle $x = \text{constant}$. At $x = \infty$ the hyperbolas are radial, and it can easily be shown that the angle θ which the hyperbola at infinity makes with the axis of v is $\frac{\pi}{2} - y$. The slope of the tangent to the hyperbola is, in general,

$$\begin{aligned}\tan(90^\circ - \theta) &= \frac{b'^2 u}{a'^2 v} = \frac{\cosh x \sin y}{\sinh x \cos y} \\ &= \tan y \coth x\end{aligned}$$

At $x = \infty$ this becomes

$$\tan(90^\circ - \theta) = \tan y$$

and therefore

$$\theta = 90^\circ - y \quad (40)$$

Hence, since for $x = \text{constant}$, Φ_0 is proportional to $\sin y$ (equation 39) when x is so large compared with the width of the strip that the ellipses approximate circles, the number of the hyperbolic light tubes per unit arc of circle is proportional to $\cos \theta$, i. e. the distribution of light tubes follows the cosine law. In the analogous electrical problem of a charged strip of infinite length the tubes of force are uniformly distributed at $x = \infty$, but the charges are distributed over the surface of the conductor according to the cosine law, the density of charge at any point $u = u_1$ being proportional to the cosine of the angle which the tangent at infinity to the hyperbola that passes through the point $u = u_1$, $v = 0$ makes with the axis of v , i. e. $\cos \theta$.

The fundamental difference between the electrical and optical problems for an infinite strip of finite width would seem to be that in the one case the condition is imposed that the field of force shall be uniform at infinity, and the distribution of the charge over the surface of the strip is determined by this condition. In the other case the surface charge is assumed to be uniform and the distribution of the light tubes at infinity is determined.

It is easy to obtain equations for the light field analogous to those of the electrostatic field. It has already been shown that Φ_0 is solenoidal, so that

$$\frac{\partial}{\partial u}(\Phi_0)_u + \frac{\partial}{\partial v}(\Phi_0)_v = 0$$

Moreover, since the specific light flux along any ellipse ($x = \text{constant}$) is zero, the elliptic cylinders may be considered as level sur-

faces. Let Ψ be a function, holomorphic in a certain region of the x - y plane, and let it depend upon x only, so that for x constant Ψ is constant. Furthermore, let the function be of such a nature that for equal positive increments of x the changes in Ψ are equal in amount and negative in sign. Then

$$-\frac{\partial \Psi}{\partial x} = \text{constant.} \quad (41)$$

In the transformed plane Ψ becomes a function of u, v which is constant over the surface of the elliptic cylinders, and which decreases by equal amounts between successive cylinders, if the ellipses are drawn to correspond to equal increments of x . Since for x constant Ψ is constant

$$\frac{\partial \Psi}{\partial y} = 0$$

Moreover, along any hyperbola from any ellipse to the next,

$$\int \left[\frac{\partial \Psi}{\partial x} \right]_{y \text{ const.}} dx = \text{constant.} \quad (42)$$

If, now, dn be an infinitesimal distance along the normal to the ellipse at any point in the field,—i. e. along the tangent to the hyperbola through the point, since dn is proportional to dx multiplied by the coefficient of linear magnification in the transformation from the x - y to the u - v plane, it follows from equations (41) and (42) that for any point in the field

$$-\frac{\partial \Psi}{\partial n} = \frac{C}{\sqrt{\sinh^2 x + \sin^2 y}} \quad (43)$$

in which C is a constant.

Moreover, since (equation 32)

$$\Phi_0 = \frac{\pi i \sin y}{\sqrt{\sinh^2 x + \sin^2 y}}$$

it follows that

$$\Phi_0 = -C' \frac{\partial \Psi}{\partial n} \sin y \quad (44)$$

From this equation it is seen that, although in going from one hyperbola to another Φ_0 varies in proportion to $\sin y$, along any one hyperbola Φ_0 at any point is proportional to the rate of decrease of Ψ along the hyperbola. Hence, Ψ may be considered as a special form of potential function, from which the specific luminous flux is derived according to equation (44).

In general, if we denote the specific luminous flux in any direction q by $(\Phi_0)_q$, and if the unit in terms of which Φ_0 is expressed be so chosen that C' in equation (44) becomes unity,

$$(\Phi_0)_q = \Phi_0 \cos(nq) = -\frac{\partial \Psi}{\partial n} \cos(nq) \sin y = -\frac{\partial \Psi}{\partial q} \sin y \quad (45)$$

Since Ψ is a function of x only, it follows from equation (41) that

$$\Delta^2 \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad (46)$$

In the transformation to the u - v plane

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \left[\frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2} \right] \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right]$$

or

$$\Delta \Psi = h^2 \Delta' \Psi \quad (47)$$

in which h is the coefficient of linear magnification. Hence (equation 46)

$$\Delta' \Psi = \frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial v^2} = 0 \quad (48)$$

In Fig. 8 the ellipses are drawn to correspond to equal increments of Ψ .

In the earlier paragraphs of the paper two examples were given of the numerical errors incident to the application of the inverse square law to radiating surfaces. A similar numerical computation might be made for the case we have just studied. In fact, the errors incident to the application of the inverse square law might be deduced for the variation of intensity with distance in any number of directions from the strip. But since two examples of the effect of dis-

tance have already been given, and since in the case of the radiating strip the specific luminous flux in any direction at any point in the field is known, an excellent opportunity is afforded to show numerically the errors incident to assuming for a finite radiating surface as a whole the cosine law which applies only to the infinitesimal elements of the surface.

In order to determine the errors incident to the application of the cosine law to the surface as a whole let us calculate the specific luminous flux at different distances in a direction normal to the strip at its middle point, and in a direction making an angle of 45° with the normal at the middle point. For any definite distance the former, multiplied by the cosine of 45° , would equal the latter if the cosine law, which has been assumed for the elements of surface, applied to the surface as a whole. The difference between the two values gives the error for the distance used.

From equations (26)

$$u = \cosh x \cos y \qquad v = \sinh x \sin y.$$

Therefore,

$$u^2 = (1 + \sinh^2 x)(1 - \sin^2 y) \qquad \sin^2 y = \frac{v^2}{\sinh^2 x} \quad (49)$$

$$u^2 \sinh^2 x = \sinh^4 x + (1 - v^2) \sinh^2 x - v^2 \quad (50)$$

In the direction making an angle of 45° with the normal to the strip at its middle point $v = u$, and so for distances measured in this direction x and y can be expressed in terms of the single coordinate u .

Substituting u for v in equation (50)

$$\sinh^4 x + (1 - 2u^2) \sinh^2 x = u^2$$

from which

$$\sinh x = \pm \sqrt{\frac{1}{2}(2u^2 - 1 + \sqrt{1 + 4u^4})} \quad (51)$$

Also from equations (49)

$$\sin y = \frac{v}{\sinh x} = \frac{u}{\sqrt{\frac{1}{2}(2u^2 - 1 + \sqrt{1 + 4u^4})}} \quad (52)$$

For distances measured in the direction of the normal to the strip at its middle point $u=0$ and therefore $\cos y=0$. Hence we have

$$\sin y = 1 \quad (53)$$

and from equations (49)

$$\sinh x = v \quad (54)$$

By substituting for $\sinh x$ and $\sin y$ in equation (32) the values given in equations (54) and (53), the specific luminous flux at different distances $d=v$ in the direction normal to the strip at its middle point is obtained. In a similar way, by substituting the values given in equations (51) and (52), the specific luminous flux at distances $d=\sqrt{2}u^2$ is obtained for points lying on the line making an angle of 45° with the normal to the strip at its middle point. The specific luminous flux determined in this way is not, however, that in the direction of 45° , but rather the maximum specific luminous flux at the point, which is in the direction of the hyperbola through the point. The value in the direction of 45° is obtained by multiplying the maximum value by the cosine of the angle between the given direction and the tangent to the hyperbola at the point. This angle will evidently be (Fig. 6) the difference between 45° and ω_0 where (equation 28)

$$\sin \omega_0 = \frac{\sinh x \cos y}{\sqrt{\sinh^2 x + \sin^2 y}} \quad (55)$$

The differences, expressed as percentage errors, between the true value of Φ_0 in the direction of 45° and that calculated from the value of Φ_0 in the direction of the normal by multiplying the latter by $\cos 45^\circ$, are shown in the form of a curve in Fig. 9. The abscissas are distances, d , expressed in the unit in terms of which the width of the strip is 2, and the ordinates are percentage errors between the true value and that computed on the assumption of the cosine law for the strip as a whole. For every distance the true value at 45° is greater than that deduced from the value at 0° by assuming the cosine law.

It is seen from a consideration of Fig. 9 that the curve approaches the axis of distances asymptotically, so that at $d=\infty$ the error would

be zero. As d becomes smaller the error increases slowly at first, but at $d=6$ or 7 it begins to increase rapidly, reaching a maximum of 23.4 per cent at $d=1$. From this point it decreases rapidly to zero at $d=0$. That the error should be zero at $d=\infty$ and at $d=0$ might have been predicted, for at infinity the width of the strip is

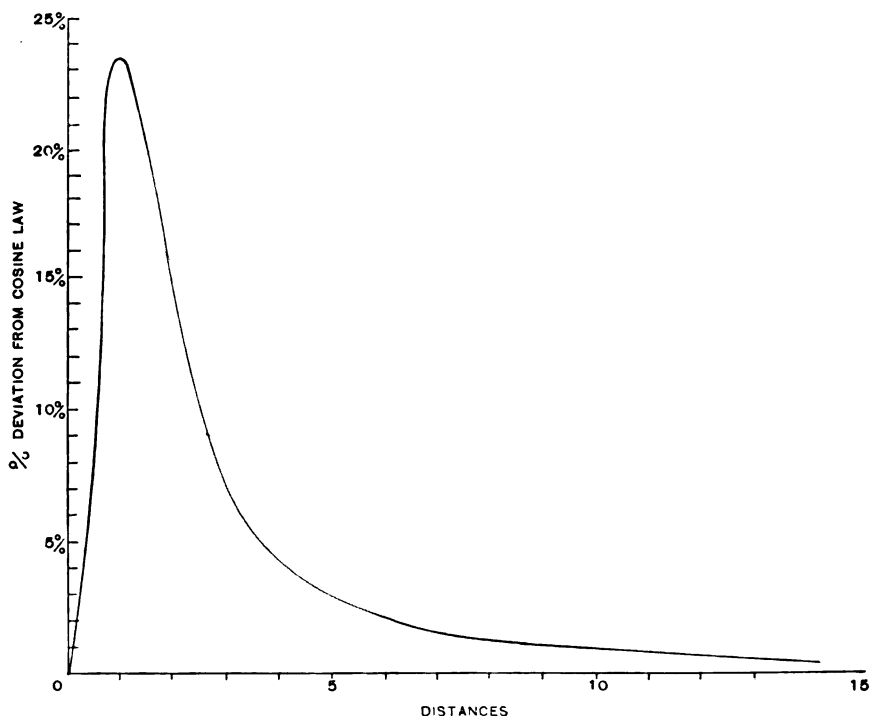


Fig. 9.—*Deviation of Radiation of Infinite Strip from Cosine Law. (45°).*

infinitesimal relative to the distance, and for a strip of infinitesimal width the cosine law by hypothesis would apply to the strip as a whole. At $d=0$ the total flux is due to the element of surface around the point $u=0$, and so the flux in any direction is equal to the maximum flux, which is normal to the surface, multiplied by the cosine of the angle of emission. In other words, at each of the two limits we are dealing with surfaces relatively infinitesimal in size, and for such surfaces we assumed the cosine law in the beginning.

THE INFLUENCE OF BASIC LEAD ACETATE ON THE OPTICAL ROTATION OF SUCROSE IN WATER SOLUTION.

By Frederick Bates and J. C. Blake.

In the polarimetric estimation of sucrose in raw sugars an indefinite amount of basic lead acetate in solution is usually added to the sugar flask before making up to volume in order to precipitate the impurities more or less completely and thus clarify the sugar solution sufficiently for reading on the polariscope. The use of this reagent is almost universal, and it is so powerful a clarifying reagent that the question whether it can be used in precision work on raw sugars is of the first importance. Except for the error due to the volume of the precipitate which is formed by the reaction it is usually regarded by commercial chemists as giving accurate results.

The evidence obtainable from the literature of the subject as to the effect of basic lead acetate on the polarization of sucrose is conflicting and very unsatisfactory, the results being usually expressed in tenths of a per cent with no indication that attention was paid to numerous minor sources of error. Von Lippmann's¹ conclusion, drawn from the results of other investigators, is that basic lead acetate exerts no influence on the rotation of sucrose in water solution. We have found this conclusion to be in error. Its great influence on the specific rotation of levulose² has long been known. Nevertheless, because of the relatively large amount of sucrose present in raw sugars as compared with the amount of levulose present, the effect of the basic lead acetate on the sucrose, though relatively less than its effect on levulose, still retains an equal importance.

The effect of basic lead acetate on the specific rotation of sucrose was overlooked by the early investigators, because of the comparatively crude polarizing apparatus and methods used, and also because

¹ *Chemie der Zuckerarten* II, p. 1185.

² *Levo-fructose*.

the temperature coefficient of pure sugar solutions was little known or entirely neglected. A few preliminary experiments soon demonstrated that it would not be safe to carry out the present investigation on an ordinary commercial saccharimeter alone. An accuracy of 0.02 Ventzke was desired. The greatest inherent difficulty in obtaining this accuracy lay in the polarizing apparatus. It was overcome by improving a Josef and Jan Frič double quartz wedge compensation saccharimeter. The instrument has glass scales and verniers, the light reaching the eye of the observer being transmitted and not reflected. As a result, the effect of temperature changes on the length of the scale is very small, and the definition of the ruling is such that the scale could readily be interpolated to hundredths of a degree Ventzke. The polarizing nicol was removed and a first-class Lippich half-shade system substituted. The light was passed through 1.5 cm of a 6 per cent potassium bichromate solution before it entered the polarizing system. An improved electric lamp of great intensity was used as the source of light, thereby permitting the use of a polarizing angle of 4.33 (circular). This angle is too small to be utilized with the ordinary mechanism for shifting the wedge. The instrument was accordingly fitted with an auxiliary screw adjustment by means of which the accurate settings were made. With the improvements as indicated above the sensitiveness of the saccharimeter was such that a difference in rotation of 0.02 Ventzke could be detected with certainty by the observer, without making a series of observations, differences of that magnitude being sufficient to produce a directly discernible change in the intensities of the two halves of the field. In Table I are given ten consecutive readings on a solution in this instrument. They were selected at random.

TABLE I.
Consecutive Readings on Sugar Solution.

(In 200 mm tube.)

Degrees Ventzke	Degrees Ventzke	Degrees Ventzke
99.98	99.99	99.98
100.00	99.99	99.98
99.99	99.98	99.98
99.99		

The surfaces of the wedges were sufficiently plane to justify this degree of refinement. The length used where the differences in rotation were measured corresponded to not more than 1:1 V of the scale. The 100 cc flasks were especially made for this work. The necks had an internal diameter of 10 mm. The volumes were known to within 0.01 cc. All temperatures were read on thermometers graduated to 0°1. The tubes were 200 mm in length and enlarged at one end. A glass tube 2.5 cm long projected from the middle of the tube and at right angles to its length. Through this side tube a thermometer was passed, its bulb being immersed in the solution. A rubber washer closed the space between the tube and thermometer. All the apparatus used was standardized for a temperature of 20°.

The sugar used in these experiments was the best commercial grade. Two samples were carefully taken from sugar purchased at different times. Each sample was thoroughly mixed. After much consideration it was concluded that practically nothing would be gained by using chemically pure sucrose. The large amount required would have been a serious difficulty. The basic lead acetate solution was prepared either by the action of the neutral acetate in solution on lead oxide at the boiling temperature or by simply dissolving a high grade product in water. The solution was made up to a density of 1.25 at 15°, and when titrated with acid gave approximately the theoretical amount of dissolved lead oxide.

Investigations of this character should be carried out in a room whose temperature can be maintained at the standard temperature of 20°. Such a room was not available at the time these experiments were made, but the results were carefully corrected for temperature changes, although these corrections were very small. The temperature coefficient of a pure sucrose solution containing 26 grams of sucrose in 100 true cc when made up and polarized at a temperature other than the standard temperature of the apparatus has been redetermined in this laboratory for both a Schmidt and Haensch polariscope and the improved Frič instrument described above. From these results, not yet published, and the fairly concordant results previously obtained by others on the Schmidt and Haensch instrument, the following values were adopted for making temperature corrections for variations from 20° not exceeding 5° (except in two instances):

1. Temperature coefficient due to cubical expansion of standard sugar solution at $23^{\circ} \pm 2^{\circ} = 0^{\circ}.029$ V.

2. Temperature coefficient of standard sugar solution made up at $23^{\circ} \pm 2^{\circ}$ read with glass tubes on a quartz wedge polariscope at $23^{\circ} \pm 2^{\circ} = 0^{\circ}.031$ V.

Although the coefficient of expansion of a standard sugar solution varies with the temperature, yet the value here given is sufficiently accurate for our purpose since the difference between the temperature of filling the flasks and the polarizing temperature was always less than 2° . Hence the error could hardly reach $0^{\circ}.01$ V.

The method of procedure finally adopted was as follows:

(1) 26.0480 ± 0.0003 grams of sugar were weighed out, all the weighings for one day being made within a short interval of time. The error due to weighing was, therefore, $\pm 0^{\circ}.001$ V.

(2) The sugar was washed into a freshly cleaned flask with ordinary distilled water and dissolved.

(3) The basic lead acetate solution was then added, usually resulting in a turbidity. This turbidity is especially troublesome for amounts of basic lead acetate solution in the neighborhood of 3 to 4 cc. It can not be filtered out with paper filters.

(4) One or two cc of a lead sulphate cream of constant composition were then added to the solution to facilitate filtration.

(5) The solution was then made up to the mark by immersing the flask in a water bath, the temperature of filling being determined by means of another flask filled with water similarly placed. The time thus required insured the complete drainage of the water in the neck of the flask. The solution was made as nearly homogeneous as possible before filling, to avoid errors due to contraction or dilution.

(6) The solution was then thoroughly shaken up and poured into a barium sulphate "Faltenfilter" (Schleicher and Schüll No. 605). The filtrate was usually clear at the start. The portion used for polariscopic reading was taken after the liquid had been on the filter almost one minute.

(7) The polariscopic tubes were filled by simple pouring and the solutions polarized on the apparatus described above. The readings were made in a room ($1.4 \text{ m} \times 2.1 \text{ m} \times 4 \text{ m}$) especially constructed for work of this nature. It was made by partitioning off one corner of the laboratory, and permitted the exclusion of all extraneous light.

It was maintained at practically the same temperature as the laboratory, by means of air circulation controlled by an electric fan. The temperature corrections were applied as follows.

The average of the temperature of the solution and the temperature read on a thermometer hanging with its bulb in contact with the polariscope was taken as the polarizing temperature. The difference between this temperature and 20° added to the difference between the polarizing temperature and the temperature of filling (which was always the smaller value) was multiplied by 0.03 and added to the polariscopic reading. The volume of the lead sulphate added was then subtracted and the corrections for the flasks and polariscopic tubes at 20° applied.

The lead sulphate added to facilitate filtration had no effect on the polariscopic reading after allowance was made for the volume of the sulphate. This is shown by the data in Table II, which contains six separate determinations, each in duplicate, three of which contained 2 cc of lead sulphate and three contained no lead sulphate. The volume of the sulphate in 1 cc of the cream was 0.045 cc. After the data given in Table III had been obtained it was found that the turbidity could be avoided by using water free from carbonates and preventing too great a dilution of portions of the lead subacetate sugar solution in filling the flask to the mark. Accordingly, much of the work was repeated without the use of any lead sulphate. The results were in complete agreement with those given in Table III.

TABLE II.

Effect of 2 cc PbSO_4 .

With PbSO_4 .			Without PbSO_4 .		
Date	Bates	Blake	Date	Bates	Blake
May 2	99°90	99°89	Apr. 28	99°90	99°92
June 12	99°895	99°90	May 3	99°895	99°89
June 28	99°895	99°895	June 9	99°920	99°895
Average without PbSO_4			99°903		
Average with PbSO_4			99°896		
Difference			0°007 V.		

The temperature coefficient of the normal sugar solution is not changed appreciably by the addition of basic lead acetate; the coefficient for a pure sugar solution is therefore used. This was determined by measuring the temperature coefficient of a normal sugar solution containing 20 cc of the basic lead acetate solution. The range of temperature was 14°. The result obtained was in complete agreement with the value given by Schönrock, who found¹ for the normal sugar solution

$$a_{20} = a_t + a_t (0.000461) (t - 20)$$

for all wave lengths. The coefficient 0.0461 ($0.000461 \times 100 = 0.0461$) was measured in preference to the coefficient 0.031, because, while both involve the temperature coefficient of the specific rotation, which alone would be appreciably affected by the presence of basic lead acetate, the 0.031 involves the temperature coefficient of the instrument and 0.0461 does not.

The data on the measurements is given in Table III. The first and second columns give the dates on which the different polarizations were made. The third column contains the number of cc of basic lead acetate added. The differences between the solution containing no basic lead acetate and one containing a given amount is given for both observers in the columns headed Sample No. 1 and Sample No. 2. The + sign indicates that the solution containing the reagent gives the higher polarization, and conversely for the — sign. The average of the differences found by both observers is given in the final column. At least five settings were made by each observer on each tube. Of the large number of observations made only three isolated ones, which were obviously in error, have been discarded. Each difference given in the table is the average of two separate determinations made simultaneously except when otherwise stated.

The results seem to be consistent within the limits of accuracy sought (0.02 V). Even when considerable variations occurred, as with 2.5 cc basic lead acetate acting on sugar No. 2, the average value fell pretty close to the smooth curve plotted in Fig. 1. The constancy of the blank determinations made on sugar No. 2 was remarkable. The value of sugar No. 1, on the contrary, continually

¹Zeit. Ver. Deut. Zuck-Ind., 58, p. 650.

decreased, the total change from March 23 to May 25 being $0^{\circ}.15$ V. The value of sugar No. 2 was, therefore, assumed constant for days on which no blank determination was made; whereas values obtained by interpolation were used under similar circumstances for sugar No. 1. The changes in No. 1 were so gradual and extended over such a long time interval that no error larger than $0^{\circ}.01$ V could have been introduced by the interpolation. The variations between the two observers are of opposite sign on the blank determinations of the two sugars, and are therefore accidental. The data are plotted graphically in Fig. 1. Owing to the length of the curve it is plotted in three sections, all on the same scale. The ordinates, in degrees Ventzke, are the differences between the polarizations of two solutions, one of which contains basic lead acetate, the other none. The zero point represents the blank solution. When the polarization given by the basic lead acetate solution is the larger, the difference is given the + sign, and inversely for the - sign. The abscissas are the volume of basic lead acetate added expressed in cc.

The curve shows beyond doubt that under the given circumstances basic lead acetate first causes a lowering of the polariscopic reading of sugar in solution amounting to more than $0^{\circ}.1$ V for normal concentration, and that further addition of the same reagent causes a continuous rise in the polarization up to the limit (63 cc) investigated. It will be observed that when about 6 cc of lead subacetate are added the polarization is not affected, the curve crossing the axis at this point.

This effect of basic lead acetate on the polarization of sugar seems to be due to the formation of soluble lead saccharates having specific rotations different from that of sugar, lead saccharates having properties which would account for the general nature of the curve having already been studied in connection with this investigation.

The elevation of the curve shows clearly the error that may be introduced in polarizations where relatively large amounts of the basic lead acetate are used. The depression in the curve, corresponding to small amounts of the reagent, is of special significance in the polarimetric estimation of sucrose in raw sugars, and is of sufficient magnitude to class it with the errors introduced by the volume of the precipitate, the temperature coefficient, and the presence of invert sugar and other impurities.

TABLE III.

Effect of Basic Lead Subacetate on Normal Solutions.

Date		Number of cc basic lead acetate added	Difference in degrees Ventske between similar solutions, one with, the other without, basic lead acetate						Final Average
Sugar No. 1	Sugar No. 2		Bates			Blake			
			Sample No. 1	Sample No. 2	Average	Sample No. 1	Sample No. 2	Average	
Apr. 18		0.5	-0.07		-0.07	-0.11		-0.11	-0.09
Apr. 12									
Mar. 25									
Apr. 12	June 1	1.0	-.09	-0.12	-.11	-.13	-0.15	-.14	-.13
Apr. 14									
Mar. 23	June 1	1.5	-.06	-.08	-.07	-.14	-.12	-.13	-.10
Apr. 14		2.0	-.09		-.09	-.16		-.16	-.13
	June 1	2.5		-.03	-.03		-.08	-.08	-.06
Apr. 11	Apr. 28								
Apr. 12	May 2	3.0	-.12	-.05	-.09	-.10	-.04	-.07	-.08
	June 6								
Mar. 24a									
Mar. 25	May 2a	4.0	-.07	-.07	-.07	-.05	-.04	-.05	-.06
Apr. 19a	May 5b								
Apr. 21									
	June 6	5.0		-.05	-.05		-.01	-.01	-.03
	June 6								
Apr. 21	June 9	6.0	-.03	-.01	-.02	.00	+.03	+.02	.00
	June 10								
	June 9	7.0		+.04	+.04		+.05	+.05	+.05
Mar. 24									
Apr. 21		8.0	+.06		+.06	+.10		+.10	+.09
	June 10	10.0		+.18	+.18		+.20	+.20	+.19
	May 3	12.0		+.31	+.31		+.28	+.28	+.29
Mar. 25	June 10	15.0	+.34	+.26	+.30	+0.25	+.31	+.28	+.29
	June 9	18.0		+.45	+.45		+.45	+.45	+.45
	May 5	20.0		+.44	+.44		+.46	+.46	+.45
	June 10	25.0		+.58	+.58		+.58	+.58	+.58
	May 5	30.0		+.61	+.61		+.63	+.63	+.62
	June 10	35.0		+.76	+.76		+.78	+.78	+.77
	May 5	40.0		+.77	+.77		+.77	+.77	+.77
	May 6	63.0					+.95	+.95	+.95

^a One determination.^b Three determinations.

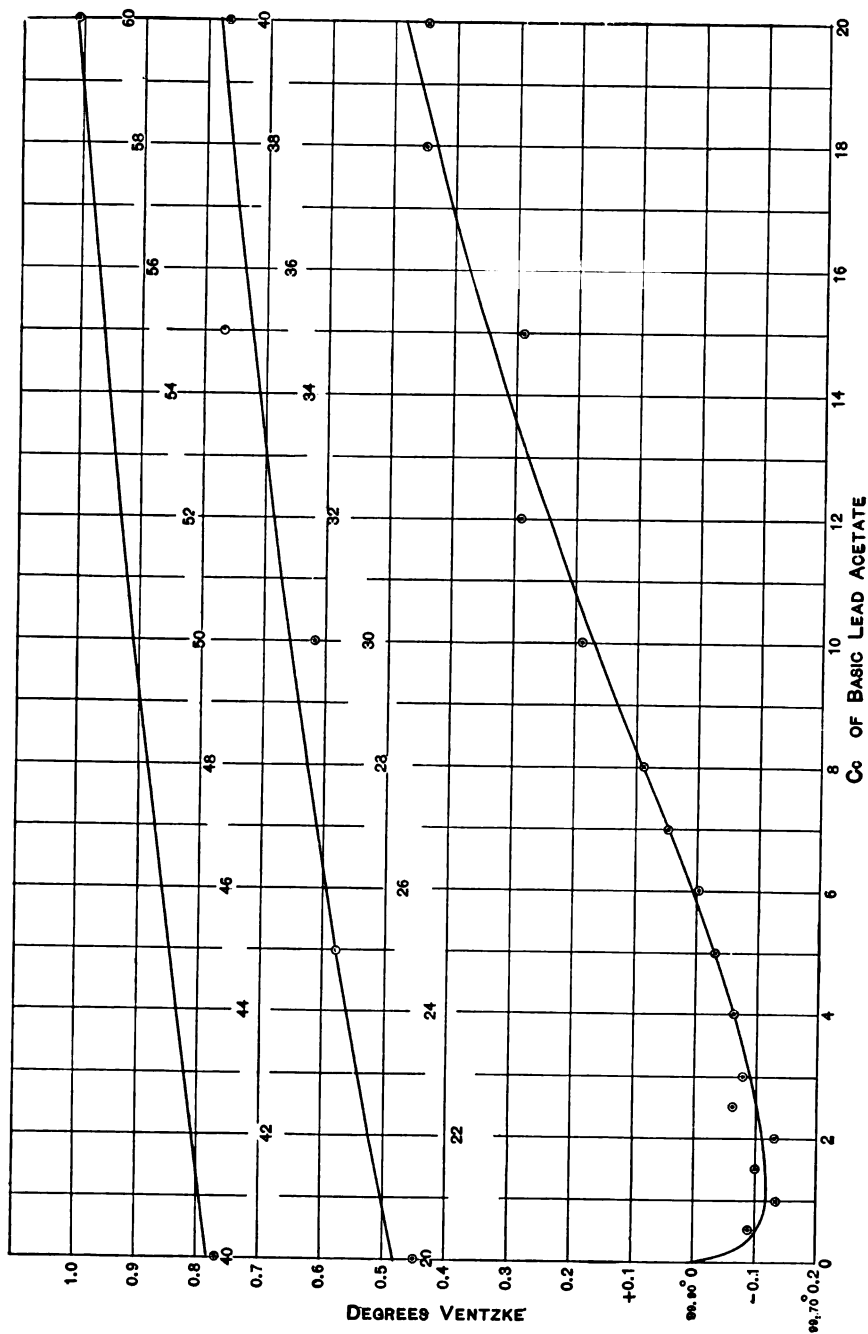


Fig. 1.

ON THE COLORIMETRIC DETERMINATION OF IRON WITH SPECIAL REFERENCE TO CHEMICAL REAGENTS.

By H. N. Stokes and J. R. Cain.

The literature of the colorimetric determination of iron is quite extensive.¹ Methods based on the color of the sulphocyanate, ferrocyanide, salicylate, tannate, sulphide, and acetonylacetate have been published. None of these has met with much consideration with the exception of the sulphocyanate methods, and the difficulties in the way of these have hitherto prevented their general adoption. It is hardly to be expected that a colorimetric method can be devised which will equal in accuracy the permanganate titration when moderately large quantities of iron are concerned. This, however, soon reaches its limit, and it is here that a satisfactory colorimetric method would find application.

The intensity of the color of ferric sulphocyanate, while very great, is extremely dependent upon the composition of the solution, and is by no means proportional to the concentration. The red color is due to the undissociated salt and to its double compounds, the ionized salt being colorless. The salt is further very prone to hydrolysis. Many substances interfere markedly with the reaction, notably fluorides, phosphates, arsenates, oxalates, citrates, tartrates, iodates, and to a less but still marked degree acetates and sulphates, the action of some of these being so strong that it is impossible to get the color even with considerable quantities of iron. In short, the intensity of the color is so influenced by the nature and concentration of the sub-

¹For a list of references up to June, 1904, see Pulsifer, *J. Am. Chem. Soc.*, **26**, p. 974. Later articles are by Mouneyrat, *Comptes Rendus*, **142**, p. 1049, 1572 (as sulfide); Leather, *J. Chem. Ind.*, **24**, p. 385 (as ferrocyanide or sulphocyanate, with Lovibond's tintometer); Marriott and Wolf, *J. Biol. Chem.*, **1**, p. 451 (as sulphocyanate in acetone solution).

stances present that, unless the test solution and the standard solution with which it is compared have identical composition and concentration, results varying many hundred or even thousand per cent from the truth may be obtained. The determination of iron in a given material would therefore presuppose the possession of a sufficient sample of the same substance either entirely iron free or of accurately known iron content; a requirement which could naturally seldom be realized. Moreover, the presence of retarding substances would often diminish greatly the sensitiveness of the reaction. The direct determination of the color of the aqueous solution, as first proposed by Thompson,² is therefore little used, and only when test and comparison solutions can be made identical. A great improvement was made by Tatlock³ in extracting the ferric sulphocyanate by ether and comparing the color with ethereal layers of the same volume and thickness containing known amounts of iron and reaching the correct result by a series of approximations. The extraction by ether is a great advance over the older method, for as fast as the sulphocyanate is removed from the aqueous solution more is formed, the result being that the greater part of the iron is extracted, although the removal is never complete, and the less so, the more disturbing substances are present. Lunge and von Kéler⁴ have improved Tatlock's method and given it a somewhat wider scope. There are certain features of their method which are not satisfactory. It can be applied to but a limited amount of material; the contents of the comparison cylinders can be given an identical concentration only by the use of iron-free material; the large amount of salt or acid used interferes somewhat with the sensitiveness, and in some cases may hinder the formation of ferric sulphocyanate almost entirely; the ether solution changes color gradually, so that comparison can be made only after several hours, while by longer waiting the color is likely to fade, and unevenly in the two cylinders; the method of approximations with standards of different strengths is cumbersome.

²J. Chem. Soc., **47**, p. 493; 1885.

³J. Chem. Ind., **6**, p. 276, 352; 1887, based on the observation of Natanson, Ann., **180**, p. 246; 1864, that the reaction is more sensitive when ether is employed as solvent.

⁴Zs. angew. Chem., 1894, p. 670; 1896, p. 3; Lunge, Chem. techn. Untersuchungs-methoden, 5^{te} Aufl., **1**, p. 385.

The abnormal behavior of the color of the ether solution, noted by Lunge, we attribute to the presence of peroxides.⁵ The color of the solution in ether which has been freed from peroxides by shaking with ferrous sulphate solution is pure pink or rose from the start; if, however, a drop of hydrogen peroxide be added, the solution becomes of a dirty yellowish pink, which becomes pure after a time, with deposition of a yellowish solid between the two layers. This is probably pseudosulphocyanogen, which is formed by the action of the peroxide on the isodisulphocyanic acid which is present. Even the best grades of ether show this behavior unless especially purified. The presence of peroxide in ether is readily detected by shaking it with freshly reduced acidified ferrous sulphate solution to which a sulphocyanate has been added, when ferric sulphocyanate is generated and is taken up by the ether. This is a good test for traces of hydrogen peroxide which might be used to determine it colorimetrically.

The method of Marriott and Wolf⁶, which consists in bringing out the color of the aqueous solution by the addition of about one and a half volumes of acetone, does not appear to afford a sufficient guarantee against bleaching, or against the effects of mass action or of inhibiting substances, and could be applied only to limited amounts of material, and to such substances as are sufficiently soluble in the aqueous acetone.

The unpleasant property of ether of giving discolored solutions led us to try other solvents. Of these, amyl alcohol proved to be the most satisfactory, giving a perfectly pure color from the start, and being a decidedly better solvent than ether for ferric sulphocyanate. A rather crude experiment with an intensely colored aqueous solution of ferric sulphocyanate, with excess of ammonium sulphocyanate and hydrochloric acid showed that ether left 3.7 times as much iron as was left by an equal volume of amyl alcohol. The relative efficiency doubtless varies with the composition of the solution, but the above figures show the decided superiority of amyl alcohol. Although we later discovered a method of inhibiting the discoloring action of peroxides on sulphocyanates we have for this reason retained the use of amyl alcohol, mixing it, however, with a cer-

⁵ See paper on Sulphocyanic Acid (this Bulletin, 3, p. 157).

⁶ J. Biol. Chem., 1, p. 451; 1905.

tain proportion of ether. Amyl alcohol is somewhat too viscous to allow rapid separation. When mercuric sulphocyanate reagent is used, ether is a relatively still poorer solvent for ferric sulphocyanate and under certain conditions has the unusual property of forming three layers, the iron being mostly concentrated in the middle layer. These objectional features are entirely obviated by using a mixture of 5 vols amyl alcohol with 2 vols ether, and it is this mixture which is meant whenever the amylic layer is spoken of below.

Having satisfied ourselves by numerous experiments that the determination of traces of iron could not be effected with sufficient accuracy in the presence of large quantities of salts, we have worked out methods whereby it can be concentrated into small bulk, practically free from interfering substances. These methods will be described below under the head of *Concentration*. We have further replaced ammonium sulphocyanate by free sulphocyanic acid, which may easily be prepared iron free in a few minutes, and which serves as a solvent for the concentrated iron. We thus obtain a solution of iron in a great excess of free sulphocyanic acid, practically free from all other substances, and so secure the most favorable conditions possible for the complete conversion of the iron into undissociated ferric sulphocyanate.

The gradual bleaching of the solution of ferric sulphocyanate has been noted by various observers and is referred to by Tatlock, by Lunge, and by Marriott and Wolf in the articles quoted. This is a very usual phenomenon even in ethereal or amylic solutions, and it is not uncommon for two identical tubes, at first matching, to show a very marked difference within less than an hour. This fading, while it may be aided by the action of light, is due to the reduction of the iron to the ferrous state by other substances than normal sulphocyanic acid. Chief among these is isodisulphocyanic acid, which is always formed when sulphocyanates are acidified, and which reduces ferric salts with great rapidity.⁷ Hydrocyanic acid and hydrogen sulphide, both of which are decomposition products of sulphocyanic acid, may possibly also take part. Hydrocyanic acid reduces traces of ferric salts, which are reoxidized by persulphate. A weak solution of ferric sulphocyanate in water or amyl alcohol is decolorized on boiling for a few moments and hydrogen sulphide can be detected in the escaping vapors.

⁷ See this Bulletin, 3, p. 159, ff.; 1907.

In order to keep the iron in the peroxidized condition and to obviate the necessity of previously oxidizing ferrous iron we invariably add a few milligrams of pure potassium persulphate to each tube. This is able to keep the iron peroxidized even in the presence of small quantities of hydrogen sulphide or sulphurous acid. The use of persulphate has one striking disadvantage which it is necessary to note. Like hydrogen peroxide, but less rapidly, it oxidizes sulphocyanic acid, forming a yellow substance which is taken up by the amyl alcohol, often rendering even an approximate comparison impossible. It appears that this yellow body does not proceed from normal sulphocyanic acid itself, but from other substances, possibly the still unknown isosulphocyanic acid, which may accompany it in small quantity. Sulphocyanic acid freshly prepared by decomposition of its silver or mercury salt by hydrogen sulphide gives but little of the yellow body with persulphate. If, on the contrary, its 5-10 per cent solution which has been allowed to stand for some time, or a freshly acidified solution of a sulphocyanate be treated with persulphate, the amylic extract is always colored. Be the cause what it may, there is always enough of the yellow substance formed by persulphate to render an accurate comparison impossible. Even without persulphate a solution of sulphocyanic acid which has stood for a few days always contains enough yellow substance to make it useless.

Fortunately we have discovered that the addition to the sulphocyanic acid of a sufficient amount of mercuric sulphocyanate to form the double compound, $\text{Hg}(\text{SCN})_2 \cdot 2\text{HSCN}$, not only totally inhibits the action of persulphate but preserves the acid indefinitely against injurious changes, while it does not appreciably diminish the sensitiveness of the reaction with ferric salts. The amylic solution has a perfectly pure color from the start and in the presence of a trace of persulphate and occasional stirring retains its intensity of color absolutely unchanged for many hours. The addition of persulphate is necessary, as mercuric sulphocyanate does not prevent the fading of the ferric sulphocyanate. The employment of mercury introduces certain complications which will be treated of in their proper places under the separation of iron from the various metals.

REAGENTS USED IN THE COLORIMETRIC IRON DETERMINATION.

Standard Iron Solution.—0.863 g ferric ammonium alum and 5 cc concentrated sulphuric acid are dissolved to 1 liter. The solu-

tion, which contains 0.1 g iron per liter, may be kept indefinitely. For use, 5 cc are diluted to 100 cc, giving a solution of which 1 cc contains 0.005 mg iron. As the dilute solution hydrolyzes and deposits iron on the glass it should be prepared fresh every day, and it is important that the measuring flask and burettes should be washed out with hydrochloric acid before using. Since persulphate is used in the cylinders the standard solution may equally well be made with the equivalent amount of ferrous ammonium sulphate, 0.702 g per liter.

Sulphocyanic Acid Reagent.—Seven per cent aqueous sulphocyanic acid, freshly prepared as directed in the following paper, is at once saturated with mercuric sulphocyanate, somewhat more of the latter than is required to form the compound $\text{Hg}(\text{SCN})_2$, 2HSCN being used, and the excess being left in the bottle. If treated with a small quantity of potassium persulphate the reagent should not impart the least yellow color to amyl alcohol, even after several hours. A slight trace of iron is occasionally observed, which comes from an impure mercuric salt. Small amounts of this are of no significance in quantitative tests, as equal quantities of the reagent are used in each cylinder. The reagent appears to keep indefinitely, but in hot weather it is well to keep it in a cool, dark place when not constantly in use.

Mercuric Sulphocyanate.—The commercial article which is prepared from the nitrate and which comes as a white or yellowish powder is not to be depended upon. It is better to prepare it by pouring a hot solution of the purest mercuric chloride (1 mol) into a solution of the purest ammonium sulphocyanate (1 mol). On cooling the sulphocyanate separates out in the form of needles, which are washed with cold water. In this case it is necessary to use twice the theoretical amount of mercuric chloride, otherwise no crystals are obtained. The excess of mercury is easily recovered by precipitation with aluminium scraps. The product is not entirely free from the double chlorine compound, but this exerts no prejudicial effect. While the yield from the nitrate is better it contains some nitrate which is undesirable because of its oxidizing action on sulphocyanic acid, and, moreover, the nitrate is not as easily obtained free from iron.

Potassium Persulphate.—This is easily obtained free from iron by

a single recrystallization, the hot concentrated solution being filtered and the crystals washed with a little cold water and carefully protected from dust.

Amyl Alcohol and Ether.—A good grade of isoamyl alcohol, such as that sold by Kahlbaum, is sufficiently pure; it need not be free from pyridine; 5 vols are mixed with 2 vols of good ether (such as Kahlbaum's 0.720). Since the discoloring action of the peroxides on sulphocyanic acid is entirely prevented by the use of mercuric sulphocyanate, no special purification is necessary.

THE COLORIMETER.

The use of the more elaborate and costly colorimeters, with lenses and prisms, is unnecessary, as the accuracy attained by the apparatus described below is quite sufficient, considering the minute amounts dealt with and the unavoidable errors involved in the methods of concentration and in working with traces of a substance so universally distributed as iron. Moreover, none of the instruments commonly in use enables one to employ an extracting liquid, as is done in our method.

Instead of graduated cylinders of equal diameters, provided with stoppers for the purpose of shaking, we use ordinary test tubes about 20 cm long and 24–25 mm diameter. These are carefully selected in pairs, with the aid of calipers. The cross sections must be as nearly circular as possible and the diameters of both tubes in a pair should not differ by more than 0.1 mm at corresponding heights, which would give a difference of 0.4 per cent in their readings. Each pair should be carefully numbered, and for ordinary purposes one or two pairs are sufficient. The mixing of the liquids is very effectively accomplished by stirrers, one of which is provided for each tube. The stirrer consists of a thin glass rod, bent as shown in Fig. 1, into the lower end of which is fused a short platinum wire, attached to a circular disk of platinum, slit radially and bent into the form of a propeller. When not in use the stirrer hangs from the edge of the test tube. It is essential to its proper functioning that the rod, when not in use, shall hang close to the side of the tube, so as not to interfere with vision; that the platinum disk shall almost, but not quite, touch the bottom, and that it shall work up and down easily. To exclude dust and prevent evaporation, it is well to provide each tube with a heavy, loosely-fitting brass cap, perforated to admit the tip

of the burette and provided with a radial slit through which the stirrer may pass (Fig. 2, a and b).

In comparing the colors in the two cylinders it is necessary to look through them horizontally, and in order to avoid the effect of the curvature of the glass and of reflection from the inside of the tubes they are surrounded by black mantles through which vertical slits are cut on exactly opposite sides. The slits have a height of about 1.5 cm and a width of 0.5 cm in the side toward the observer and 1 cm on the opposite side. Notwithstanding the curvature of the glass, slits of this width give a field of practically equal intensity, owing to the refractive action of the liquid. The mantle may be made by rolling thick black paper around the tube and pasting the edges so that the mantle may slip easily over the glass and yet retain its position through friction. Care must be taken that the centers of the slits are exactly opposite, which can be determined by marking the position with the calipers. The two cylinders are mounted in vertical position, as near together as possible, in some form of dark box such as is used in colorimetric work. Two 10-cc glass cock burettes, carrying the standard iron solution, are so mounted that their tips project into the cylinders. In reading we look against a uniformly illuminated vertical sheet of white paper, placed at an angle of about 45° with a window, the degree of the illumination being regulated by the angle. As there is generally a marked difference in the color sensitiveness of the two eyes, it is necessary, in comparing, to shield or close one eye, the other being opposite the center of the instrument. We have observed that in the great majority of persons the left eye is more sensitive to red, unless wearied. When much work involving the use of an extracting liquid is to be done it is convenient to have the mantles made of brass instead of paper and to provide them with springs which will easily hold the cylinders in place, and to mount them side by side in a permanent base. We give here a drawing and description of the instrument which we have employed in the greater part of this work.

Mantles.—The two mantles are identical in every respect. They are made of 30-mm brass tubing, of about $3/4$ mm thickness, giving an internal diameter of about 28.5 mm. The total height is 20 cm (Figs. 3 and 5, and sections Fig. 4, a and b).

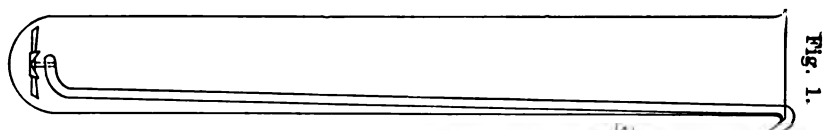


Fig. 1.

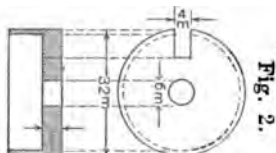


Fig. 2.

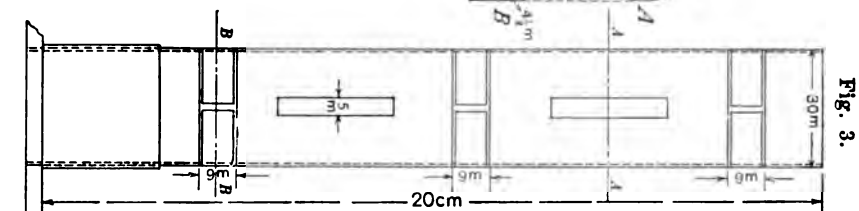


Fig. 3.

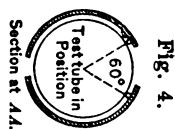
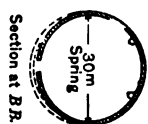


Fig. 4.



Section at B.B.

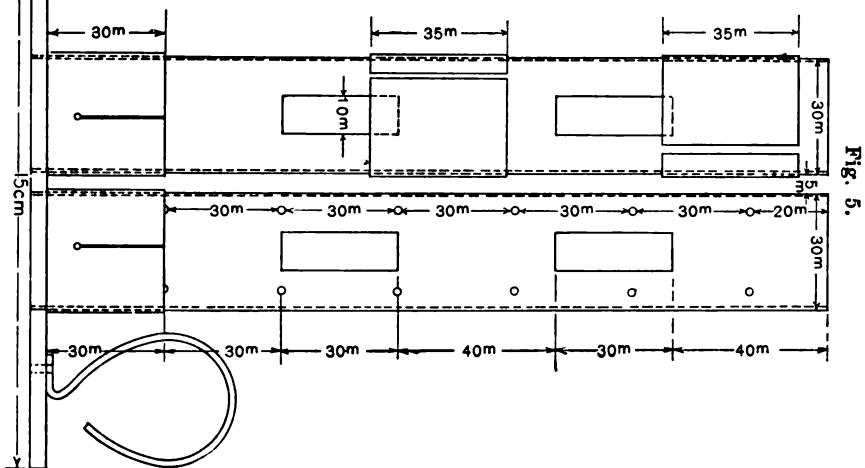


Fig. 5.

Slits.—Two pairs of slits are provided, for greater convenience in reading with varying volumes of liquid. Their length is 30 mm. The width on one side is 5 mm, on the other 10 mm; the opposite slits must exactly correspond in position, and especial care should be taken that their centers are exactly opposite with respect to the axis of the tube; the edges are cut parallel and sharp, not rounded or beveled.

Collars.—Each mantle has two thin brass sliding collars, of slit tubing, 35 mm high, the object of which is to close or vary the height of the slits. They are lined inside with black paper and must slide easily.

Guide Points.—Each mantle has 6 pairs of these, at equal distances, as indicated. Their object is to hold the cylinder exactly parallel to the axis of the mantle; it is therefore essential that they shall be exactly in line, parallel to the axis of the mantle; that they shall project into the tube exactly to the same distance, viz, about 1.5 mm, and that they shall have sharp edges or points, not rounded or flat heads. They are made by inserting brass pins or pegs through the wall of the mantle.

Springs.—These are intended to hold the cylinder in place against the guide points, and therefore parallel to the axis of the mantle. They are made in pairs, as indicated in the drawing, by cutting and bending in a portion of the tube, and should hold a 25-mm test tube filled with water firmly enough to stay in place, yet so that it can easily be shoved up or down with one hand.

Mounting.—Each mantle is supported in a socket, such as is used for microscope eyepieces, which is mounted on a brass plate 15 cm $\times$ 10 cm and not less than 4 mm thick, provided at one end with a handle for lifting. The mantles must turn easily in the sockets with one hand, and when mounted must be parallel and 5 mm apart.

Finish.—The whole instrument, except the inside of the sockets and the portion of the mantles inside them, is coated dull black, within and without.

In using the instrument, the narrow slit is turned toward the observer. In order to prevent reflection from the surface of the glass, the mantles are shielded by surrounding them with a box, made of thick, black paper open at top and bottom, 20 cm high, 10 cm deep, and 7.5 cm wide. In the front of this are cut two

openings 4 cm square, at heights corresponding to the slits; in the back at the same height are two pairs of openings 4 cm high and 1.5 cm wide, exactly coinciding with the slits in the mantles. When the box is in place, the rear openings should be invisible from the front. It is also desirable to have the two burettes mounted on a small clamp so that both can be raised or lowered at the same time. A convenient clamp is made by sawing two short pieces of brass tubing across, near the ends, and bending in the pieces so as to form springs. These tubes are soldered to a small vertical brass plate which is attached to a screw muff which supports it on a retort stand. The dimension of the clamp is such that the burette tips coincide with the centers of the cylinders. It is scarcely necessary to add that no iron is permissible, and that the apparatus should be kept free from dust.

METHOD AND ACCURACY OF COMPARISON.

The cylinders, which may be conveniently designated as the "test" and "standard," after charging in the manner described below, contain equal volumes ether-amyl alcohol mixture and equal volumes of sulphocyanic reagent diluted with equal volumes of water and a few mg potassium persulphate, and are therefore of identical composition and concentration, except that the "test" contains the iron and 2 or 3 mg manganese and oxidation products of sulphocyanic acid, an amount entirely too small to have any influence on the determination.⁸ They are placed in the colorimeter and the colors are brought to equal intensity by carefully adding to the "standard" a sufficient amount of standard iron solution from the corresponding burette.⁹

⁸To one of two carefully matched cylinders was added the manganese dioxide from 0.6 cc 1 per cent permanganate, about the quantity used in a concentration. Not the least change could be detected. After adding 40 mg sulphuric acid to one of the matched cylinders, 0.14 cc standard iron solution had to be added to the same cylinder to restore equality. It therefore appears that as little as 40 mg sulphuric acid may produce an error of - 0.0007 mg Fe. Since the sulphuric acid generated by the above amount of manganese does not exceed 1.2 mg its influence is clearly too small to be detected, amounting to only perhaps 0.004 cc standard solution.

⁹When the amount of standard added equals 1 cc, it is well to add an approximately equal volume of water to the "test" cylinder, so as to keep the volumes equal and counteract the unequal solvent action on the ether-amyl alcohol layer. If the concentration of the iron has been properly performed, the quality of the colors will be identical. A yellowish cast in the "test" is due to faulty concentration, and an accurate comparison can not then be made. As we read by looking

The difference of the two burettes is then noted, and the process is repeated three or four or more times by adding a few drops to one cylinder and bringing the other to match it. The average of the differences observed is the amount of standard iron equivalent to the iron sought: 1 cc = .005 mg Fe.

It is well known that extremely faint colors can not be matched as well as those of somewhat greater intensity. With this method it is possible to determine to within a few per cent an amount of iron so small that it scarcely gives a visible color to the amylic layer. The amylic mixture is used in multiples of 5 cc and the most favorable conditions appear to be when the volume used is roughly ten times the volume of the standard iron solution equivalent to the iron sought. Under these conditions the extreme differences between the readings should not exceed 5-6 per cent of the total iron present, a result which is much diminished by taking the mean of a series of readings.

The following experiments show the degree of accuracy attained.

1. Each cylinder was charged with 5 cc amylic mixture, 5 cc sulphocyanic reagent, and 5 cc water. The burettes were not read until the match was obtained.

Cc standard iron solution added		
Tube 1	Tube 2	Difference
0.19	0.19	0.00
.33	.33	.00
.45	.46	— .01
.64	.61	+ .03
.81	.77	+ .04
		—
Mean		+ .012
		= .00006 mg Fe.

through the amylic layer it is essential that this shall be perfectly clear and free from suspended water drops; the turbidity of the aqueous layer through suspended amyl alcohol is of no significance. If the stirring is properly performed the amylic layer becomes rapidly clear and the aqueous layer remains turbid. Whether or not this will be realized can be instantly told by observing the manner in which the separation occurs. If the churning be thorough large globules will be seen on the upper surface, which will be seen to coalesce rapidly, after the manner of bubbles, leaving a perfectly clear amylic layer, while below the mixture contains innumerable small drops which do not run together, but gradually rise, leaving a turbid aqueous layer. If, however, the churning has been imperfect the large globules are at the bottom and run together rapidly, leaving a sharply defined surface and a clear aqueous solution, while above are seen small globules, which gradually fall, leaving the amylic layer turbid. In general the former effect takes place; if it does not, even with sufficient churning, it can be brought about by adding more water to each cylinder.

2. 200 cc redistilled ammonia were evaporated to dryness in platinum in a dust-free atmosphere, and the residue taken up with sulphocyanic reagent.

Test	Standard	Difference
0.65	0.61	-0.04
.93	.95	+ .02
Mean		- .010
		= - .00005 mg Fe.

3. 200 cc distilled water treated as above.

Test	Standard	Difference
0.40	0.39	-0.01
.58	.53	- .05
.95	.96	+ .01
Mean		- .017
		= - .00008 mg Fe.

Lunge¹⁰ estimates that the permanganate method can not be depended on to give results nearer than ± 0.14 mg Fe. This would mean an error of ± 1 per cent on 14 mg or of ± 5 per cent on 2.8 mg. It appears from the above that the colorimetric method gives an error of less than ± 0.0001 mg Fe, or ± 1 per cent of .01 mg, the amount which is conveniently employed in the colorimeter. Since larger quantities of iron can be diluted to any desired extent without introducing an error of this magnitude, it follows that the colorimetric method can be used to advantage up to about .014 g in the absence of interfering substances; above this limit the permanganate method is more accurate. Where a special concentration of the iron is necessary, the error may be estimated on the basis of our results at ± 0.0005 mg in the more unfavorable cases, or ± 5 per cent of .01 mg. In such cases the colorimetric method would be applicable, with suitable dilution, up to about .0028 g Fe.

CONCENTRATION METHODS.

The problem of separating a few ten-thousandths of a milligram of iron from several grams of material in a form suitable for determination in the colorimeter is one which must necessarily vary with the nature of the material under examination. We have worked out the conditions which will be found applicable in most cases. In

¹⁰ Zs. angew. Chem., 1896, p. 3.

a few we have met with no success whatever, and must await the result of the future experiments. The details will be given below.

In all work with traces of iron it is necessary to exclude dust most carefully, especially where operations which consume considerable time are carried on. All utensils should be carefully rinsed with strong hydrochloric acid just before using; it is well to keep them under hydrochloric acid when not in use as far as practicable; all funnels and dishes should be kept covered with watch glasses, which, when removed, should never be placed on the table, but set concave side up, on small glass supports having three projecting glass points; reagent bottles should be kept covered with caps and the contents removed by pipetting rather than by pouring.

CONCENTRATION BY EVAPORATION.

When the material is volatile at a sufficiently low temperature and does not attack the vessels the iron may be concentrated by evaporation. This method is applicable to hydrochloric, nitric, sulphuric and acetic acids, ammonia, and other substances of a similar order of volatility. Substances like ammonium sulphate or oxalate and oxalic acid can not be so treated, as they attack the vessels appreciably. Concentration by evaporation alone is seldom sufficient for bringing the iron into a suitable form for colorimetric determination. Even the purest acids and ammonia are likely to contain traces of colored substances which pass over into the amylic layer, and so prevent accurate comparison. In general, after driving off the greatest part of the volatile material, or bringing to dryness and redissolving the residue in a few drops of hydrochloric acid, the iron must be precipitated by one of the methods given below.

The vessels used for evaporation may be of platinum, porcelain, or Jena glass. Except for ammonia and hydrofluoric acid, for which platinum should be used, this metal is unsatisfactory. All platinum contains a small amount of iron, either present as an original impurity or derived from the tools used for working the metal or from materials which have been previously contained in the vessels. This is quite sufficient to cause an appreciable error when acids or ammonium sulphide are evaporated. Berlin porcelain is better than platinum, and Jena glass superior to porcelain. The following data will show the relative values of the different materials:

100 cc of concentrated c. p. hydrochloric acid (*a* and *b*), nitric acid (*a* and *b*), and sulphuric acid were used in each case, the figures being the mean of the number of determinations indicated.

	HCl <i>a</i> mg Fe	HCl <i>b</i> mg Fe	HNO ₃ <i>a</i> mg Fe	HNO ₃ <i>b</i> mg Fe	H ₂ SO ₄ mg Fe
Evaporation in platinum.....	.0102 (4 det.)	.0084 (3 det.)	.0095 (2 det.)	.0034 (2 det.)	.0226 (3 det.)
Evaporation in Berlin porcelain....		.0084 (3 det.)	.0083 (10 det.)	.0034 (3 det.)	.0162 (4 det.)
Evaporation in Jena glass.....		.0074 (4 det.)		.0022 (5 det.)	.0130 (4 det.)
Direct neutralization and precip. of Fe	.0070 (8 det.)	.0068 (4 det.)	.0088 (6 det.)	.0026 (3 det.)	.0130 (4 det.)

Assuming, as we must, that higher results indicate contamination by iron contained in the material of the vessels, it will be seen that platinum is markedly superior to Jena glass in the case of sulphuric acid and, in a less degree in the case of nitric acid and hydrochloric acid, Berlin porcelain is more resistant than platinum and inferior to Jena glass. The results obtained by evaporation in Jena glass are close to those obtained by direct neutralization with redistilled ammonia and subsequent concentration of the iron by precipitation.

The individual data, which we can not quote here, show that the results obtained in Jena glass are much more uniform than those given by porcelain and platinum. We have used several platinum dishes, partly new ones which were reserved exclusively for this work, with the same result. Considering that platinum utensils are likely to be used for all sorts of purposes, and are therefore extremely subject to contamination, we must unqualifiedly recommend the use of Jena glass for evaporation in testing acids for iron. We have tried nonsol glass with unsatisfactory results, as the dishes are liable to crack with the high temperature and sudden changes involved in rapid evaporation. Doubtless vessels of fused quartz would give still better results, but unless extreme accuracy is desired their costliness renders their employment unnecessary. Probably crucibles or dishes of fused quartz might be used with advantage in driving off the more volatile salts. Evaporations should be made as rapidly as possible, in order to diminish the time of action upon the vessel, and entirely out of contact with the external air, so as to avoid dust. It is inadmissible to use any form of apparatus in which the condensed liquid can run or drop back into the vessel.

We have found the following form of apparatus entirely satisfactory: A circular disk of asbestos is placed upon a Chaddock's porcelain burner, and upon it is placed an inverted set of porcelain water-bath rings. Upon these rests a nine-inch funnel, the stem of which is drawn out and bent down. Through the stem is forced a current of air which has been filtered by passing through a long tube or series of tubes filled with cotton. A bent calcium chloride tube filled with cotton is directly connected with the stem of the funnel. The speed of the air current should be such as to prevent condensation in the stem of the funnel. The porcelain rings are slightly inclined so that the condensed liquid running down the sides of the funnel drops off at one point into a beaker. The whole rests on a glass plate. By this means it is possible to evaporate 100–200 cc concentrated sulphuric acid quietly and rapidly without danger of contamination from dust; 200 cc distilled water and 200 cc redistilled ammonia evaporated in this apparatus gave no trace of iron. (See page 127.)

The ordinary hemispherical Jena glass evaporating dishes with flat bottoms do not well bear the strain of this treatment. A suitable dish 9.5 cm wide by 4.5 cm high is conveniently made by cutting off the lower part of an 800 cc Jena-Griffin beaker, and making a lip in it. The dish rests on the porcelain rings, leaving an air space between it and the asbestos. In some cases it is desirable to finish the evaporation on a steam bath. For this purpose a Berlin porcelain steam bath is used, with the same funnel. If the steam, as is likely, carries over water containing iron in suspension, it should be passed through a separator, which is conveniently made of a large calcium chloride tube, half filled with beads and provided with an overflow for the water. Combustible liquids like acetic acid can not be evaporated over the free flame and are evaporated either on the steam bath or on a small electric hot plate fitted up as above.

CONCENTRATION BY PRECIPITATION.

In by far the greater number of cases it is necessary to concentrate the iron by precipitation. An almost indefinitely small quantity of iron may thus be determined in an indefinitely large amount of material, the only limit being the solubility of the iron precipitate in the solution. It is obviously impossible to collect on a filter

traces—say a thousandth of a milligram of ferric hydroxide or sulphide distributed through a considerable volume of an otherwise clear liquid. We therefore employ the method which has been occasionally used successfully in other cases, of mechanically carrying down the precipitate by a relatively large amount of another precipitate, which, when practicable, is generated simultaneously with the iron precipitate. We may designate this secondary precipitate as the “collector.” Various substances suggest themselves as collectors; their number is limited by the following considerations. A collector must be sufficiently insoluble, so that but a small amount of possibly impure foreign substance need be introduced; it must be of such physical consistency as to enable it to carry down all suspended precipitates, and must therefore be amorphous and flocculent, not granular or crystalline; it should not be gelatinous or otherwise difficult to wash out in the filter, neither should it be of such consistency as to run through the filter on washing; it must be easily soluble in 7 per cent sulphocyanic acid and must neither interfere with the ferric sulphocyanate reaction nor in the presence of mercuric sulphocyanate impart a color to amyl alcohol, or, if it does not meet these requirements, it must be capable of easy separation from the iron. Aluminium hydroxide would be the ideal collector were it not for the fact that it dissolves slowly and imperfectly in sulphocyanic acid, and thus frequently prevents complete solution of the accompanying ferric hydroxide. Repeated experiments have satisfied us that it is not to be depended on, and we have therefore employed it only in special cases where it was removed before final treatment of the precipitate with sulphocyanic acid. We precipitate the iron either as sulphide or as ferric hydroxide. The hydroxide precipitation is employed in the absence of materials which have a solvent action, such as citrates, tartrates, sugar, and many other organic substances, pyrophosphates, arsenites, arsenates, antimonates, etc. The usual collector for ferric hydroxide is hydrated manganese peroxide. The sulphide precipitation is used when from the presence of any of the just mentioned substances, hydroxide would remain in solution. It is also used when other sulphides insoluble in ammonium or sodium sulphide are practically absent. The best collector for iron sulphide is cadmium sulphide. In this case the cadmium sulphide is redissolved and the iron reprecipitated as hydroxide with manganese dioxide as collector.

In many cases the choice between the methods is optional. When there is reason to fear the presence of traces of organic matter, as in the case of materials which have been treated in wooden vessels in the process of manufacture, or when arsenic or other prejudicial substances may be present, as in the cruder reagents, the sulphide method is more accurate. For example, pure sodium chloride gave identical results by either method, while a sample of the best commercial chloride gave decidedly too low results with the hydroxide method.

Special care is necessary in sampling the substance, and wherever practicable duplicate determinations should be made on portions of the same solution, as it frequently happens that different samples, especially of crystallized substances, taken from the same bottle show widely varying results, owing to the irregular distribution of the iron.

APPARATUS AND REAGENTS USED IN CONCENTRATING.

The apparatus used for evaporations has been described above. Only the best ashless filters, Schleicher and Schüll's No. 590, should be used, and as even these contain very appreciable quantities of iron they must be moistened in the funnel with 1-1 hydrochloric acid, allowed to stand at least fifteen minutes, and then washed with water to which a few drops of ammonia are finally added. Only Jena or nonsol beakers or dishes, or platinum dishes, should be used, Jena glass dishes for evaporating strong acids, and these, or Jena beakers, for cadmium sulphide concentration, and for manganese dioxide precipitations when the time of heating is brief. For longer heating, or when sodium hydroxide is used, platinum is employed. Small pipettes are used for transferring the reagents from the bottles, and pouring should never be resorted to, as the lips of bottles are almost invariably dirty.

Reagents which are used in large amounts must be specially freed from iron; we have therefore limited these to the smallest possible number and to those most easily purified. Reagents which are used in very small amounts need not be specially purified, provided the amount of iron present is insufficient to affect the results.

Ammonia.—As even the best c. p. ammonia contains notable amounts of iron, it must always be redistilled. The washed ammonia gas is conducted into a cooled ceresine-lined bottle containing

water, in which it is kept. Only the best white ceresine should be used and care taken to coat the bottle uniformly up to but not into the neck.

Hydrochloric Acid.—The best c. p. hydrochloric acid invariably contains iron. We therefore always use carefully washed hydrochloric acid gas, prepared by dropping pure concentrated sulphuric acid upon pure ammonium chloride or concentrated hydrochloric acid. Rubber tubing should be avoided as much as possible, and that which is necessary should be washed out with acid. When practicable the gas is conducted directly into the solution; when aqueous acid is required, it should be freshly prepared.

Hydrogen Sulphide.—The use of hydrogen sulphide directly made from iron sulphide is inadmissible and may lead to gross errors. The gas is prepared by dropping acid into a sodium sulphhydrate solution. A stock solution of this is made by saturating 33 per cent sodium hydroxide with hydrogen sulphide and diluting four or five times before using.

Ammonium Sulphide.—In general the sulphhydrate is used and is always freshly prepared by saturating redistilled ammonia with hydrogen sulphide prepared as above.

Bromine Water is used to oxidize arsenious and antimonious oxides and to dissolve metals. As iron-free bromine and bromine water, kept in glass vessels, rapidly become contaminated with iron, the bromine water should be prepared as needed by drawing out a clean test tube so as to form a small retort, sucking in two or three cc bromine by alternate warming and cooling, and distilling it over into water.

Nitric Acid.—When more than one or two cc is required it should be redistilled from a small test-tube retort into a test tube placed in a beaker of water.

Potassium Permanganate.—A 1 per cent solution of the best c. p. grade is used. It furnishes the manganese dioxide used as collector for ferric hydroxide, and at the same time serves to oxidize traces of organic matter which might hold it in solution. If not sufficiently free from iron it may be purified by a manganese concentration.

Cadmium Sulphate and Chloride are used to supply the cadmium sulphide which serves as collector for iron sulphide, the chloride being used when barium, strontium, or calcium salts are present.

A one-fourth molecular stock solution is made, and freed from iron by making a manganese precipitation as described below. The presence of a slight excess of permanganate in the solution has no prejudicial influence.

Sulphurous Acid, Ammonium Sulphite, Sodium or Ammonium Formate in 1 per cent solution and Alcohol are used in small quantities to reduce permanganate to manganese dioxide. They need not be specially purified.

Sodium Potassium Tartrate is used to hold up alumina or chromic oxide in concentrating iron from their salts. A 20 per cent stock solution is made and freed from iron by a cadmium sulphide precipitation, as described below under tartrates. The alkaline solution should be neutralized to prevent its action on the glass.

Sodium Hydroxide is used in special cases, and its 5 per cent solution must be freed from iron by a manganese concentration. It should be freshly purified unless kept in platinum bottles.

CONCENTRATION BY MANGANESE DIOXIDE.

This is applicable in nearly all cases where substances which have a solvent action on ferric hydroxide, or more than traces of alumina and chromic oxide, are absent. There are certain special modifications of the method which will be given below, and we here give only the procedure in the simpler cases. The amount and concentration of the substance operated on seem to be immaterial; we often operate with as much as 50 g and in solutions as strong as 20 per cent. If the solution is not precipitated by ammonia and contains no substances capable of reducing permanganate to manganese dioxide, it is made weakly alkaline with ammonia, about 10 drops of permanganate are added, and then one to three drops of a reducer, such as 1 per cent formate, sulphurous acid, or occasionally alcohol, and the solution is then heated a few minutes until the manganese dioxide has separated in flocculent form. It is well to have a slight excess of permanganate. If the substance is one which is precipitated by ammonia, such as zinc, lead, or cadmium, just enough of this is added to form a slight permanent precipitate, and the manganese precipitation is made as above. The precipitate is collected on a 5.5 or 7 cm washed filter, and washed a few times with water. 2.5 cc of the sulphocyanic reagent are placed in the beaker to dissolve the pre-

precipitate adhering to the sides, and then dropped carefully around the top of the filter so as to dissolve the manganese dioxide and accompanying ferric hydroxide, the filtrate being run directly into the test cylinder. From 5 to 20 cc ether-amyl alcohol are added¹¹ according to the amount of iron present. The beaker is washed out with exactly 10 cc water, which is poured carefully through the filter. The standard cylinder is charged with 2.5 cc sulphocyanic reagent, 10 cc water, and as much ether-amyl alcohol as was used for the test. Finally a few mg potassium persulphate are added to each cylinder, and they are transferred to the colorimeter.

Solubility of Ferric Hydroxide.—The accuracy of the above method depends upon the degree of insolubility of ferric hydroxide in the weekly ammoniacal solutions of the various salts. In general this could be determined only by operating with solutions either iron free or containing an accurately known quantity of iron. In most cases this was impracticable owing to the lack of any method other than the one in question of accurately determining iron and to the difficulty of obtaining absolutely iron-free material. In the case of ammonium salts of volatile acids, however, the amount of iron can be accurately checked by evaporating the acid and using redistilled ammonia. The following are some of the results:

	Found	
	Fe mg	Fe ₂ O ₃ mg
100 cc c.p. HCl, sp gr 1.19, evaporated in Jena glass (4 dets.)	.0074	= .0106
The same neutralized with NH ₄ OH and concentrated by Mn (4 dets.)	.0068	= .0097
100 cc c.p. HNO ₃ a, sp gr 1.42, evaporated in porcelain (10 dets.)	.0083	= .0119
The same neutralized with NH ₄ OH and concentrated by Mn (2 dets.)	.0071	= .0102
100 cc c.p. HNO ₃ b, sp gr 1.42, evaporated in Jena glass (5 dets.)	.0022	= .0031
The same neutralized with NH ₄ OH and concentrated by Mn (3 dets.)	.0026	= .0037
100 cc c.p. H ₂ SO ₄ , sp gr 1.84, evaporated in Jena glass (4 dets.)	.0130	= .0187
The same neutralized with NH ₄ OH and concentrated by Mn (4 dets.)	.0130	= .0187
50 cc redistilled glacial acetic acid + .0100 mg Fe evaporated in Jena glass	.0107	= .0153
The same neutralized with NH ₄ OH and concentrated by Mn	.0108	= .0154

¹¹ The ether-amyl alcohol should be added before adding the water, as otherwise there is likely to be a separation of mercuric sulphocyanate. The amount to be added can be judged by the color; it is better to add too little than too much, as more can be added later, if desired. If the amount of iron is very considerable, so as to require more than 5 cc standard iron solution, the filtrate can be diluted in a measuring flask and an aliquot portion taken, a fresh portion of sulphocyanic reagent being used.

Whence, in strong hot weakly ammoniacal solution,

		Fe	Fe ₂ O ₃
100 g	NH ₄ Cl dissolved	.0009 mg	.0013 mg
100 "	NH ₄ NO ₃	.0009	.0013
100 "	(NH ₄) ₂ SO ₄	none	none
100 "	NH ₄ C ₂ H ₃ O ₂	none	none

These figures, which are possibly high, if anything, owing to the solvent action of the evaporating acid on the vessel, can lay no claim to accuracy; but they at least show that the loss through solubility will not affect the percentage result of a determination nearer than the seventh decimal place, and may therefore be set off against the slight sources of contamination through dust, solvent action of the reagents on the vessels, etc.

CONCENTRATION BY CADMIUM SULPHIDE.

Cadmium sulphide is used as a collector for iron in the form of sulphide and is applicable in nearly all cases in which the substance under the examination either gives little or no precipitate with ammonium sulphide, or one soluble in an excess. Its chief use is to remove the iron as sulphide from solutions which exert a solvent action on ferric hydroxide, and from aluminium and chromium salts. The following is the method of procedure in the simpler cases.

To the cold solution contained in a Jena beaker, and which should not contain much free acid, are added 2 cc cadmium solution and then a slight excess of fresh ammonium sulphhydrate, or in case of sulphides soluble in an excess, enough to dissolve these. The liquid is allowed to stand in the cold for about half an hour, with frequent stirring, and the cadmium sulphide, which carries the iron, is then collected on a washed filter and washed a few times with water containing a little ammonium sulphhydrate. The precipitate can not be directly treated with sulphocyanic acid reagent, as much iron would be retained and much mercuric sulphide formed; neither can it, as we have found, be dissolved in bromine water with satisfactory results. It is therefore dissolved by carefully dropping hot 1-1 hydrochloric acid around the top of the filter, the solution and wash water being run back into the original beaker. The solution, which contains some free hydrogen sulphide, is treated as in a man-

ganese concentration, somewhat more than enough permanganate being added first, to oxidize the hydrogen sulphide, and then ammonia, the manganous salt in alkaline solution acting as the reducer of the permanganate. In general, the manganese dioxide comes down at once in good form or on gentle heating. The precipitate is collected on the same filter, or if this contains a residue, on a fresh one, and is further treated as described under manganese concentration. In the case of tartrates, oxalates, and other organic substances interfering with the manganese concentration, traces of which remain in the cadmium sulphide precipitate, hot 1-1 nitric acid is used in place of hydrochloric acid, and the organic matter is destroyed in the filtrate by adding permanganate to the hot acid solution until the color is permanent, when the solution is made ammoniacal as before. Special precautions to be observed in particular cases will be given below.

Solubility of Iron Sulphide.—In the absence of the disturbing factors above referred to, determinations made by both methods give practically identical results, from which it may be concluded that the solubility of iron sulphide, like that of the hydroxide, is negligible. There are certain exceptions to this, especially in the case of stannic salts, which will be noted below.

In the following we give the details of the determinations for a considerable number of reagents. It was obviously impossible to cover the whole field and we have chosen those cases which seemed of most importance as being typical or as most likely to cause difficulty. A few of the cases have presented difficulties which we have not yet been able to solve, and a few of the important types are omitted as it seemed undesirable to longer postpone publication. Some of the determinations were made before the details of the method were fully worked out and the results are therefore less accurate and less complete than would be the case now. In general the data for each substance were obtained from the same solution or liquid unless otherwise stated.

Hydrochloric Acid, c. p.—100 cc or 200 cc are evaporated down to a few cc in a Jena glass dish (page 130). In the residue the iron is concentrated by manganese. Finishing the evaporation on the steam bath and taking up the residue with sulphocyanic acid gives a somewhat discolored amylic solution which can not be sharply compared.

Instead of evaporating, 50 cc may be diluted with an equal volume of water, neutralized with redistilled ammonia, and treated with manganese. Results are identical in either case and identical with those obtained by the cadmium method. Hence it would appear that there is no appreciable loss of ferric chloride on evaporation.

Hydrochloric acid, c. p., sp gr 1.19. Two lots, *a* and *b*, the analysis labels of which indicated 0.0002 per cent Fe, gave:

Cc acid		Fe	per cent Fe	Method
50	<i>a</i>	0.0024 mg	0.000040	evaporation in porcelain
"	"	0.0021	0.000035	
"	"	0.0024	0.000040	
		Mean	0.000038	
50	"	0.0022	0.000037	Mn conc. after neutralizing
"	"	0.0024	0.000040	
		Mean	0.000038	
50	"	0.0023	0.000039	Cd conc. after neutralizing
"	"	0.0021	0.000035	
"	"	0.0020	0.000034	
		Mean	0.000036	
100	<i>b</i>	0.0074	0.000062	evaporation in Jena glass
"	"	0.0075	0.000063	
"	"	0.0071	0.000060	
"	"	0.0077	0.000065	
		Mean	0.000062	
50	"	0.0031	0.000052	Mn conc. after neutralizing
"	"	0.0044	0.000074	
"	"	0.0028	0.000048	
"	"	0.0039	0.000066	
		Mean	0.000060	

Nitric Acid, c. p.—The treatment of nitric acid is identical with that of hydrochloric acid. The results with evaporation, manganese, and cadmium concentrations coincide.

Nitric Acid, c. p., sp gr 1.42, the analysis label of which indicated 0.0002 per cent Fe, gave:

Cc acid	Fe	per cent Fe	Method
100	0.0022 mg	0.0000016	} evaporation in Jena glass
"	0.0022	0.0000016	
"	0.0024	0.0000017	
"	0.0022	0.0000016	
"	0.0022	0.0000016	
	Mean	0.0000016	
50	0.0013	0.0000018	} Mn conc. after neutralizing
"	0.0012	0.0000018	
"	0.0015	0.0000021	
	Mean	0.0000019	

Sulphuric Acid, c. p.—100 cc may be rapidly evaporated in Jena glass (page 130) and a manganese concentration made on the residue, or 25–50 cc may be diluted with 2–3 vols water, neutralized with redistilled ammonia, and the iron concentrated by manganese or cadmium.

Sulphuric acid, c. p., sp gr 1.84, gave :

Cc acid	Fe	per cent Fe	Method
100	0.0134 mg.	0.0000073	} evaporation in Jena glass
"	0.0122	0.0000066	
"	0.0137	0.0000074	
"	0.0129	0.0000070	
	Mean	0.0000071	
25	0.0029	0.0000063	} Mn conc. after neutralizing
"	0.0035	0.0000077	
"	0.0035	0.0000077	
"	0.0031	0.0000077	
	Mean	0.0000071	

Acetic Acid.—The iron may be determined in acetic acid either by evaporating to small volume in Jena glass on the steam bath or electric stove and precipitating in the residue with manganese, or by directly neutralizing with redistilled ammonia and concentrating with manganese or cadmium. It is essential to wash out the acetate thoroughly from the manganese precipitate as even small quantities interfere seriously with the sulphocyanate reaction.

100 cc glacial acetic acid "Kahlbaum" gave on evaporation 0.0075 mg Fe = 0.0000071 per cent.

50 cc redistilled glacial acetic acid gave on evaporation 0.0015 mg Fe.

50 cc of the same to which 0.0200 mg Fe had been added gave by manganese concentration 0.0216 mg Fe.

Ammonia.—This is best evaporated in platinum, but may also be directly neutralized with gaseous hydrochloric acid. The evaporation residue usually contains something which discolors amylic solutions; it is therefore taken up with a few drops of hydrochloric acid and precipitated by manganese.

200 cc redistilled ammonia which had been kept in a ceresine lined bottle gave no iron on evaporation.

C. p. pyridine-free ammonia, sp gr 0.90, lots *a*, *b*, *c*, taken directly from the shipping bottle, gave on evaporation:

Cc ammonia	Fe	per cent Fe	Mean per cent Fe
200 <i>a</i>	0.0035 mg		0.000020
200 <i>b</i>	0.0169	0.000094	0.000092
" "	0.0160	0.000089	
" "	0.0168	0.000094	
200 <i>c</i>	0.0171	0.000095	0.000093
" "	0.0155	0.000086	
" "	0.0178	0.000099	

All samples were free from sediment. It appears that the best c. p. ammonia after keeping in glass is likely to contain as much iron as the best grades of acids.

Sodium and Potassium Hydroxides.—These hydroxides frequently contain a little iron which gradually separates from the solution on standing, or as sulphide, on saturating with hydrogen sulphide. To determine this it may be concentrated directly by adding 2 drops sulphurous acid and 10 drops permanganate and heating in a platinum dish until the manganese dioxide has separated. This method is, however, likely to give somewhat too low results, and the precipitates occasionally give discolored amylic solutions; it must, however, be resorted to in certain cases where an iron-free caustic alkali is required. Better is it to neutralize with gaseous hydrochloric acid or carefully distilled acetic acid and to make a cadmium or manganese concentration. As shown by the following data, a 10 per cent sodium hydroxide solution gradually becomes weaker in iron apart from that which may be deposited as sediment, probably because a portion is taken up by, or at least firmly attached to, the glass of the bottle.

Two 10 per cent solutions of ordinary "pure" stick sodium hydroxide, lots *a* and *b*, were prepared and filtered; *a* was kept in a Jena glass bottle and tested at intervals, the bottle being shaken before each determination to disseminate any sediment.

100 cc *a* gave:

	Fe
fresh; direct Mn concentration.....	{0.0217 mg 0.0216
after 7 days; direct Mn concentration.....	0.0103
after 3 days; Cd conc. in acetate.....	0.0111
after 4 days; Cd conc. in acetate.....	0.0103
after 7 days; Cd conc. in acetate.....	0.0078

100 cc *b* fresh gave:

direct manganese concentration.....	{0.0231 0.0236
manganese concentration in acetate.....	{0.0264 0.0260
cadmium concentration in acetate.....	{0.0278 0.0257

Sodium and Potassium Carbonates.—The 10 per cent solution may either be treated directly with permanganate or neutralized with hydrochloric acid gas and treated by the manganese or cadmium methods, the results being practically the same.

Kahlbaum's potassium carbonate gave:

K ₂ CO ₃	Fe	per cent Fe	Method
2.5 g	0.0085 mg	0.000340	direct Mn concentration
5.0	0.0179	0.000358	Mn concentration in chloride
2.5	0.0085	0.000340	Mn concentration in chloride
5.0	0.0173	0.000346	Cd concentration in acetate
Mean		0.000346	

Insoluble Carbonates.—The substance is covered with water and decomposed by leading in hydrochloric acid gas, or if this is excluded, by iron free acetic or nitric acid; the iron is concentrated from the solution by the usual methods with due reference to the nature of the base.

Calcium carbonate "Kahlbaum" gave:

CaCO ₃	Fe	per cent Fe	Method
5.0 g	0.0201 mg	0.000402	Mn concentration in chloride
5.0	0.0206	0.000412	Cd concentration in chloride
Mean		0.000407	

Sulphides.—Soluble sulphides are decomposed by leading in hydrochloric acid gas and making a cadmium or manganese concentration in the chloride solution. Or we may add the cadmium solution directly to the sulphide solution as it carries down the iron sulphide completely on sufficient stirring. Insoluble sulphides are either decomposed by iron-free acid or when practicable directly dissolved in ammonium sulphide.

Sodium sulphide from Kahlbaum, in 10 per cent solution gave:

Cryst. sulphide	Fe	per cent Fe	Method
10.0 g	0.0044 mg	0.000044	direct concentration by Cd
"	0.0035	0.000035	
"	0.0039	0.000039	
Mean		0.000039	
10.0	0.0034	0.000034	Cd concentration in chloride
"	0.0038	0.000038	
Mean		0.000036	

*Tartates, citrates, and other organic substances which hold up ferric hydroxide*¹² must be treated by the cadmium method, which gives entirely satisfactory results as far as tested. Substances of this class containing metals whose sulphides are insoluble in ammonium sulphide or whose salts precipitate in ammoniacal solution in the cold have not been examined, and special methods would have to be devised for each. In some cases the organic matter might be destroyed with iron-free permanganate or other oxidizers, in others the metal might be removed by hydrogen sulphide. In considering the latter, it is important to bear in mind that sulphides frequently carry down iron from acid solution; we have occasionally found that over half the iron is precipitated in this way. Some substances, as calcium tartrate, are sufficiently soluble in the cold in ammonium salts to admit of the concentration being made. In any event, when such organic substances are present the cadmium sulphide precipitate must be washed as thoroughly as practicable, dissolved in nitric acid, and the trace of organic substance destroyed by heating with permanganate as described under the cadmium concentration (p. 136).

¹² For a list of such substances see Roszkowski, Z. anorg. Chem., 14, p. 1; 1897. The statements in his article apply to larger quantities of iron and must not be unqualifiedly accepted with regard to traces.

The cadmium precipitate should stand at least a half hour, with frequent stirring, as otherwise very gross errors may result.

Sodium potassium tartrate in 10 per cent solution gave:

KNaC ₄ H ₄ O ₆ .4H ₂ O	Fe	per cent Fe	Method
5.0 g	0.0105 mg	0.000210	Cadmium concentration
5.0	0.0104	0.000208	
5.0	0.0116	0.000232	
5.0	0.0106	0.000212	
	Mean	0.000215	

Oxalic Acid.—Oxalic acid does not rank as a substance capable of holding up ferric hydroxide in alkaline solution; as a matter of fact, however, it does do this to a slight extent and it has been found impossible to recover a trace of iron by the manganese method from solutions of ammonium oxalate to which small quantities of iron had been added. The cadmium method gives satisfactory results, but it is necessary to bear in mind that even traces of oxalate interfere seriously with the sulphocyanate reaction. The last traces remaining in the cadmium sulphide precipitate must therefore be destroyed by dissolving in nitric acid and oxidizing with permanganate. Insoluble oxalates may be decomposed by careful heating and subsequent treatment as under carbonates, care being taken to oxidize any remaining traces of oxalate with permanganate.

C. p. ammonium oxalate gave:

(NH ₄) ₂ C ₂ O ₄	Fe	per cent Fe	Method
3.33 g	0.0056 mg	0.000168	Cadmium concentration
3.33	0.0050	0.000150	
5.00	0.0082	0.000164	
5.00	0.0075	0.000150	
	Mean	0.000156	
10.00	0.0001	0.000001	effect of not destroying last traces of oxalate

Salts of the Alkali Metals.—A very large class of salts of the alkali metals may be equally well treated by either the cadmium or manganese methods without special precautions, provided interfering substances are absent. Such are the chlorides, bromides, nitrates, nitrites, sulphates, sulphites, phosphates, sulphocyanates, acetates, and many others. Especially excluded are chromates and permanganates, which must be treated by the manganese method, and pyrophosphates, arsenites and arsenates, oxalates, salts of organic

acids containing alcoholic hydroxyl or mixtures containing these or carbohydrates, which require a cadmium precipitation. We give below a few cases which may be considered typical of this group.

Sodium Chloride.—C. p. sodium chloride from Kahlbaum, lots *a* and *b*, in 25 per cent solution gave:

NaCl	Fe	Per cent Fe	Method
lot <i>a</i>			
25 g	0.0019 mg	0.000076	Manganese concentration
"	0.0020	0.000080	
"	0.0023	0.000092	
	Mean	0.000083	
"	0.0023	0.000092	Cadmium concentration
"	0.0024	0.000096	
"	0.0022	0.000088	
"	0.0023	0.000092	
	Mean	0.000092	
lot <i>b</i>			
25	0.0030	0.000120	Manganese concentration
"	0.0031	0.000124	
"	0.0030	0.000120	
	Mean	0.000122	
"	0.0031	0.000124	Cadmium concentration
"	0.0030	0.000120	
"	0.0028	0.000112	
"	0.0030	0.000120	
	Mean	0.000119	

A 25 per cent solution of the best commercial sodium chloride gave the following, in which the different results by the two methods indicate a possible contamination interfering with the precipitation of iron as hydroxide:

NaCl	Fe	Per cent Fe	Method
25 g	0.0118 mg	0.000472	Manganese concentration
"	0.0100	0.000400	
"	0.0085	0.000340	
"	0.0106	0.000424	
"	0.0099	0.000396	
	Mean	0.000406	
"	0.0174	0.000696	Cadmium concentration
"	0.0163	0.000652	
	Mean	0.000674	

Orthophosphates.—Alkali orthophosphates give equally good results by either the manganese or cadmium methods, provided pyrophosphate is absent. All trace of phosphate should be thoroughly washed out.

Sodium phosphate from Kahlbaum gave:

$\text{Na}_2\text{HPO}_4, 12 \text{ H}_2\text{O}$	Fe	Per cent Fe	Method
10 g	0.0094 mg	0.000094	Manganese concentration
"	0.0096	0.000096	
	Mean	0.000095	Cadmium concentration
"	0.0093	0.000093	
"	0.0096	0.000096	
	Mean	0.000094	

Insoluble phosphates which are soluble in ammonia may be concentrated by manganese; aluminium phosphate may be dissolved in acid, a tartrate or citrate (iron free) and ammonia added, and a cadmium concentration made. Calcium phosphate is best treated according to Glaser,¹³ being dissolved in hydrochloric acid, the calcium precipitated by sulphuric acid and alcohol, and the iron removed from the filtrate by cadmium. One g tricalcium phosphate gave 0.0167 mg Fe (0.00167 per cent) and 0.0148 mg Fe (0.00148 per cent), and the precipitated calcium sulphate was free from iron.

Pyrophosphates.—Ferric pyrophosphate dissolves readily in ammonia in presence of an excess of alkali pyrophosphate; the manganese method is therefore inapplicable, and the cadmium precipitation must be employed.

Ten g Kahlbaum's sodium pyrophosphate gave 0.0114 mg Fe or 0.000114 per cent.

Sulphocyanates.—Sulphocyanic acid being the reagent *par excellence* for iron, the detection and determination of iron in its salts and their complete purification from it become points of considerable importance. All specimens of sulphocyanates which we have examined contain traces of iron. It is present even when the salt appears to be absolutely colorless, and such colorless salts not infrequently contain more iron than those which are slightly pinkish. Sulphocyanic acid is a reducer of ferric salts, as may be seen by

¹³ Zs. angew. Chem., 1889, 636.

boiling a pink solution, when the color rapidly disappears. The same bleaching may be observed in the cold, and is due to the formation of colorless ferrous sulphocyanate. Moreover, the presence of a trace of free alkali or ammonia suffices to prevent the iron from betraying its presence. Whether a sample of sulphocyanate carrying iron changes from pink to colorless or *vice versa* depends on whether the reducing action of the salt or the oxidizing action of the air gets the upper hand. A convenient test for iron in sulphocyanate is to pour on a gram or two of the solid salt, contained in a test tube, an amount of redistilled ethyl alcohol insufficient to cover it; the alcohol at once becomes more or less pink even if the salt be absolutely colorless.

We have attempted to obtain iron-free ammonium and potassium sulphocyanate by careful recrystallization from water or alcohol, but without success; the point is soon reached where the contamination from dust, filter papers, and apparatus counteracts the effect of recrystallizing. Fortunately, purification by this means is unnecessary, and the chemist would do well to remove the last traces of iron himself rather than demand it of the manufacturer. For all but the roughest sort of qualitative work the solution should be purified after making up by adding a few milligrams of alum, making faintly ammoniacal and filtering off the alumina, which acts as collector for the iron. In the alkaline solution any ferrous hydroxide is at once oxidized by the dissolved air and carried down. That the iron is completely precipitated as hydroxide is shown by the following, in which an alumina concentration was made and a cadmium concentration in the filtrate.

50 cc of a 10 per cent ammonium sulphocyanate solution, lot *a*, the analysis label of which gave 0.004 per cent Fe, were used:

NH ₄ SCN	Fe	Per cent Fe	• Fe in filtrate	Method
5g	0.0034 mg	0.000068		Mn concentration
"	0.0035	0.000070		Al concentration
"	0.0036	0.000072	0.0002 mg	
"	0.0034	0.000068	0.0002	
"	0.0032	0.000064	0.0002	
	Mean	0.000068		

From these data it appears that the iron is entirely removed by the alumina method, the slight amount found in the filtrate being

within the limit of error of the method. It is also noteworthy that the amount of iron present was not more than one-sixtieth of that stated on the analysis label. If it is essential that no foreign substance be introduced a few drops of a solution of aluminium hydroxide in aqueous sulphocyanic acid may be used, and after purifying the solution may be neutralized with sulphocyanic acid. The removal of the iron by the manganese method is also practicable, though in this case oxidation products are introduced. Purification by extracting the iron as sulphocyanate with amyl alcohol is impracticable, as this solvent does not remove ferric sulphocyanate from very strong solutions of alkali sulphocyanate.

In the quantitative determination of iron in sulphocyanates the alumina method is not to be recommended, the precipitate being far too slowly soluble in sulphocyanic acid. We use either the manganese or the cadmium methods. The manganese concentration is carried out in the usual manner, except that as sulphocyanate in alkaline solution at once reduces permanganate to manganese dioxide no reducer is necessary, the solution must be made ammoniacal before adding permanganate and then gently heated to promote precipitation.

Ammonium sulphocyanate, lot *b*, an ordinary c.p. grade, gave the following:

NH ₄ SCN	Fe	Per cent Fe	Method
5 g	0.0039 mg	0.000078	Manganese concentration
"	0.0044	0.000088	
	Mean	0.000083	
"	0.0046	0.000092	Cadmium concentration
"	0.0042	0.000084	
"	0.0036	0.000072	
	Mean	0.000083	

Iodides.—Soluble iodides give somewhat too low results by the manganese method, but this may be used when the cadmium method is inapplicable. No reducer is necessary, as permanganate at once gives a precipitate in the ammoniacal solution.

Potassium iodide from Kahlbaum gave:

KI	Fe	Per cent Fe	Method
10 g	0.0013 mg	0.000013	Manganese concentration
5	0.0001	0.000002	
5	0.0006	0.000012	
10	0.0030	0.000030	Cadmium concentration
5	0.0012	0.000024	
	Average	0.000027	

Calcium, Strontium, Barium.—The soluble salts are treated like sodium chloride, the carbonates as described under insoluble carbonates, while the sulphates are heated to dull redness for some time in a porcelain crucible in a current of hydrogen, the resulting sulphides being dissolved in hydrochloric acid and treated as described under sulphides.

Magnesium.—The manganese and cadmium methods are applicable in most cases. In the manganese concentration enough ammonia is first added to give a faint precipitate, or at least to make the solution alkaline; for the cadmium precipitation enough ammonium salt should be present to prevent the separation of magnesia.

Magnesium sulphate from Kahlbaum gave:

MgSO ₄ ·7H ₂ O	Fe	Per cent Fe	Method
10 g	0.0208 mg	0.000208	Manganese concentration
"	0.0214	0.000214	
"	0.0210	0.000210	Cadmium concentration
	Mean	0.000211	

Zinc.—The cadmium method is naturally inapplicable, and we employ the manganese method, adding enough ammonia to produce a slight permanent precipitate, or, at least, an alkaline reaction; the solution is then treated in the usual manner with permanganate and a reducer. Enough ammonia to redissolve the zinc hydroxide may be added, but in this way we get somewhat too low results.

Zinc sulphate from Kahlbaum gave:

ZnSO ₄ ·7H ₂ O	Fe	Per cent Fe	Method
10 g	0.0363 mg	0.000363	Supersaturation with ammonia
10	0.0397	0.000397	Neutralization with ammonia
5	0.0191	0.000382	
	Mean	0.000389	

Manganese.—A little hydrochloric acid is added and then ammonia to alkaline reaction and the solution heated with addition

of either permanganate or a few drops of bromine water. In this case the manganous hydroxide itself serves as a reducer.

Manganese sulphate from Kahlbaum gave:

MnSO ₄ ·7H ₂ O	Fe	Per cent Fe
5 g	0.0121 mg	0.000242
"	0.0133	0.000266
"	0.0131	0.000262
Mean		0.000257

The salt gave no iron reaction when treated directly with sulphocyanate.

Permanganates.—Iron may be separated from permanganates by causing a manganese dioxide precipitate to form in the neutral or alkaline solution.

Nickel.—This metal presents a peculiar difficulty. Nickel sulphocyanate is insoluble in amyl alcohol; nickel mercuric sulphocyanate on the contrary is readily extracted from its aqueous solution by amyl alcohol, giving a green solution, the color of which, even in small amounts, partially neutralizes the pink of ferric sulphocyanate, and moreover imparts to it an impure tint. As the sulphocyanic reagent contains mercury it is absolutely essential to remove every trace of nickel from the manganese precipitate. One or two drops of ammonia are added to the nickel solution, enough to produce a slight permanent precipitate, or at least to make the solution alkaline; permanganate and a reducer are then added and the solution boiled. The precipitate is well washed, finally with strong ammonia, dissolved in hydrochloric acid and reprecipitated by permanganate, the second precipitation being also well washed with ammonia. The effect of the double precipitation is shown in the following figures:

Cobalt-free nickel sulphate from Kahlbaum gave:

NiSO ₄ ·7H ₂ O	Fe	Per cent Fe	Method
1 g	0.0303 mg	0.00303	Single precipitation; color impure
"	0.0325	0.00325	
"	0.0334	0.00334	Double precipitation; color pure

Cobalt.—The behavior of cobalt with sulphocyanic acid is the exact reverse of that of nickel. Cobalt salts give with strong sulphocyanic acid or sulphocyanates a blue salt which is dissolved by

amyl alcohol, giving a deep blue solution (Vogel's reaction). Mercuric sulphocyanate reagent gives a beautiful blue crystalline double salt, practically insoluble in water and insoluble in amyl alcohol. When this reagent is used not a trace of cobalt passes over into the amylic layer. To the cobalt solution one or two drops of ammonia are added to produce a slight permanent precipitate, and then permanganate, after which the liquid is heated. A reducer is not needed, as the permanganate generates cobaltic oxide, the dark precipitate consisting of a mixture of this with manganese dioxide. This must be washed with ammonia, redissolved in hydrochloric acid, and reprecipitated. The object of the second precipitation is to get rid of most of the cobalt, which gives rise to the blue mercuric salt above referred to, which otherwise remains in the filter and holds back not a little of the iron, as the following figures show.

Cobalt sulphate from Kahlbaum gave:

CoSO ₄ ·7H ₂ O	Fe	Per cent Fe	Method
0.25 g	0.0093 mg	0.00372}	Single precipitation
"	0.0089	0.00356}	
"	0.0208	0.00832}	Double precipitation
"	0.0211	0.00844}	
	Mean	0.00838	

Copper.—Cupric mercuric sulphocyanate dissolves in amyl alcohol with a yellow color, and even traces of copper in the manganese precipitate suffice to make the color of the amylic solution so impure that no accurate comparison can be made; such traces can be removed neither by washing with ammonia nor by double precipitation. The following method is effective. Enough ammonia is added to give a slight permanent precipitate, and the solution is heated after adding permanganate and a reducer. The washed precipitate is dissolved in hydrochloric acid, run back into the same beaker, the trace of copper precipitated by hydrogen sulphide, and the liquid filtered through the same filter. The iron is thrown out of the filtrate in the usual manner by permanganate, the hydrogen sulphide serving as reducer. The colors obtained in this way are perfectly pure. If necessary, the copper salt may be dissolved in ammonia.

Copper sulphate (Kahlbaum's I) gave:

CuSO ₄ .5H ₂ O	Fe	Per cent Fe
1.0 g	0.0075 mg	0.000750
2.5	0.0176	0.000704
	Mean	0.000727

Lead.—The manganese precipitation is made in a solution of lead salt in the usual manner, after adding enough ammonia to produce a slight permanent precipitate, or to make the solution alkaline. On treating the precipitate, which contains lead dioxide and manganese dioxide, on the filter with sulphocyanic reagent, the small amount of lead sulphocyanate remains behind.

Cadmium.—One or two drops of ammonia are added to produce a slight permanent precipitate, and then permanganate and a reducer, and the solution is heated. The addition of ammonia in excess, to redissolve the cadmium hydroxide, presents no advantages.

1.6 g 3CdSO₄.8H₂O gave 0.0058 mg Fe or 0.00036 per cent.

Bismuth.—We have been unable to devise a method of satisfactorily concentrating iron from bismuth salts. On account of the tendency to give basic salts a neutral or slightly alkaline solution in which a manganese precipitation may be made can not be obtained. Attempts to separate the bismuth as basic salt by dilution failed, because a large part of the iron, often as much as one-half, is carried down with the precipitate. On precipitating as sulphide from acid solution as much as two-thirds of the iron was carried down with the sulphide and could only be extracted by repeated precipitations.

Mercury.—Preliminary experiments indicate that iron may be removed from mercuric chloride solution by adding enough ammonia to give a slight precipitate, and then concentrating with manganese. Difficultly soluble or insoluble salts are dissolved in an iron-free sodium sulphide solution and the iron concentrated by cadmium. The results obtained were approximate only and much mercury was carried down with the cadmium sulphide.

Aluminium.—Iron is separated from aluminium salts by adding an equal weight of sodium potassium tartrate, which has been freed from iron, and making a cadmium concentration. The well-washed precipitate must be dissolved in nitric acid and the trace of tartrate destroyed by permanganate as described under cadmium sulphide concentration. Other methods, which appear to give less satisfactory results, are to dissolve in an excess of purified sodium hydroxide and concentrate with manganese, or to add an excess of purified sodium

pyrophosphate, make ammoniacal, and concentrate with cadmium. Ammonia alum from Kahlbaum gave:

$\text{NH}_4\text{Al}(\text{SO}_4)_2 \cdot 12\text{H}_2\text{O}$	Fe	Per cent Fe	Method
5 g	0.0217 mg	0.000434	Cd conc. in tartrate solution
"	0.0223	0.000446	
Mean		0.000440	
"	0.0172	0.000344	Mn conc. in sodium hydrate solution
"	0.0153	0.000306	
"	0.0200	0.000400	
"	0.0163	0.000326	Cd conc. in pyrophosphate solution

Chromium.—Ammonia precipitates chromic hydroxide from chrome alum even in the presence of a large excess of tartrate; the precipitate does not redissolve on boiling. The same occurs if the chrome-alum solution be first boiled and then cooled before adding tartrate. If, however, the chromic salt and tartrate be boiled together for a moment, ammonia gives no precipitate. In order to separate iron from chromium we add an equal weight of purified sodium potassium tartrate, boil for a moment, cool and concentrate the iron with cadmium, taking care to destroy every trace of tartrate in the washed precipitate by dissolving in nitric acid and oxidizing with permanganate as directed.

Chrome alum from Kahlbaum gave:

$\text{KCr}(\text{SO}_4)_2 \cdot 12\text{H}_2\text{O}$	Fe	Per cent Fe
0.25 g	0.0223 mg	0.00892
"	0.0224	0.00896
Mean		0.00894

Chromates.—Iron may be separated from chromates soluble in ammonia by the manganese method.

Arsenic.—As ferric hydroxide is readily soluble in ammonia in the presence of either arsenites or arsenates in excess, the manganese precipitation is inapplicable. Cadmium sulphide dissolves to a considerable extent in a solution of an arsenite in ammonium sulphhydrate, as does also iron sulphide. In order to remove the iron completely as sulphide the arsenic must be in the form of an arsenic compound. This is best effected by oxidizing the solution of arsenious acid or acidified arsenite with a slight excess of bromine water and making the cadmium precipitation in the usual manner. The pre-

caution of adding bromine water should be observed even in the case of arsenates. Arsenious sulphide may be dissolved in yellow ammonium sulphide containing enough polysulphide to convert it into the sulpharsenate. The presence of arsenic in the solution in which the second or manganese precipitation is made must be carefully avoided. The cadmium sulphide precipitate is dissolved on the filter in 1-1 hydrochloric acid as usual and any arsenic sulphide running through must be carefully filtered off.

Solubility of iron sulphide in sulpharsenites.—To a solution of 1 g carefully resublimed arsenious oxide was added 0.0100 mg Fe, and a cadmium precipitation made in the usual way. Only 0.0010 mg Fe was recovered.

Resublimed arsenious oxide, to which a definite amount of iron was added, gave the following results on oxidizing with bromine water and concentrating with cadmium:

As ₂ O ₃	Fe added	Fe recovered
2 g	0.0150 mg	0.0167 mg
1	0.0100	0.0093
1	0.0200	0.0210
1	0.0200	0.0210

Sodium arsenate from Kahlbaum, without addition of bromine water, gave:

Sodium arsenate	Fe	Per cent Fe
5 g	0.0082 mg	0.000164
"	0.0070	0.000140
"	0.0066	0.000132
	Mean	0.000145

Antimony.—An ammoniacal solution of antimony trioxide dissolves small but appreciable amounts of ferric hydroxide. An ammoniacal solution of common potassium antimonate to which iron has been added dissolves traces only, but if it has been previously oxidized with bromine water to remove traces of the lower oxide no iron can be detected in the solution. This fact can not be made use of, however, for a manganese concentration, as the oxide must be dissolved in tartaric acid. Ammoniacal sulphantimonite solution dissolves notable amounts of iron sulphide,¹⁴ and the solution has not the green color characteristic of colloidal iron sulphide; it also dissolves cadmium sulphide. Ammoniacal sulphantimonate, on the

¹⁴ See Storch, Ber., 16, p. 2015; 1883.

contrary, is colored green by a trace of iron sulphide, and this is easily extracted by agitating with cadmium sulphide. On diluting an acid solution of antimonious acid or precipitating by hydrogen sulphide, the greater part of the iron is carried down with the precipitate.

These facts indicate the method to be followed in extracting traces of iron. The antimony compound is dissolved in iron free sodium potassium tartrate, oxidized with bromine and cadmium solution added, followed by ammonium sulphhydrate. Antimony trisulphide is dissolved in yellow, antimony pentasulphide in colorless ammonium sulphide, and cadmium solution added. The mixture is allowed to stand half an hour with frequent stirring. The cadmium precipitate is washed with water containing ammonium sulphide, dissolved in nitric acid, and precipitated by manganese, care being taken to destroy in the acid solution any remaining traces of tartaric acid.

We have been unable to obtain absolutely iron-free antimonious chloride, as some iron passes over on distillation, and we have therefore used the ordinary article, dissolved in sodium potassium tartrate.

SbCl ₃	Fe	Per cent Fe
2.5 g	0.0079 mg	0.000316
"	0.0074	0.000296
"	0.0070	0.000280
	Mean	0.000297

Tin.—We have been unable thus far to discover a satisfactory method of separating traces of iron from tin. The manganese concentration is impossible, as ammonia precipitates the hydroxide before neutrality is reached. A solution of stannic hydroxide in potassium hydroxide readily dissolves ferric hydroxide which is not precipitated on boiling. If stannic chloride be added to an excess of ammonia, a clear colloidal solution is obtained, but it appears to be impossible to produce a manganese precipitate in this. The sulphide concentration also offers difficulties, as a solution of stannic sulphide in ammonium sulphide dissolves both cadmium and iron sulphides readily under certain conditions.

Platinum.—The concentration of iron from platinum solution as sulphide is impracticable, owing to the difficult solubility of platinum sulphide in ammonium sulphide. We are therefore limited to concentration as hydroxide. The manganese method is unsatisfactory,

because manganese dioxide carries down a considerable amount of platinum from which it is very difficult to free it completely, either by reprecipitation, by hydrogen sulphide, or by reduction with formic acid. Traces of platinum exercise an extremely prejudicial effect on the colorimetric determination as they invariably pass over into the amylic layer and impart a yellowish color to the ferric sulphocyanate. We have found the following method to be rapid and to give satisfactory results. To the hot platinic chloride solution, which should be sufficiently dilute not to deposit ammonium chloroplatinate, are added a few milligrams alum and ammonia in excess. The washed alumina precipitate is dissolved in hydrochloric acid, enough sodium hydroxide is added to redissolve the alumina, and the iron is carried down by manganese dioxide. The color obtained from this precipitate is entirely pure.

0.2 g platinic chloride (10 per cent solution from Kahlbaum) gave 0.0205 mg Fe or 0.01025 per cent Fe.

20 cc 10 per cent platinic chloride solution was freed from iron by alumina, a definite amount of iron was added to the filtrate and concentrated as before.

Fe added	Fe found
0.0150 mg	0.0161 mg
0.0150	0.0150

SUMMARY.

1. The ferric sulphocyanate is extracted from its aqueous solution with a mixture of amyl alcohol and ether, using two cylinders, one of which contains the iron to be determined; standard iron solution is then added to both cylinders from two burettes, the difference of the readings indicating the amount of iron sought. The final result is reached by averaging a series of readings.

2. A form of colorimeter is described, adapted to the use of extracting liquids.

3. The iron is concentrated and freed from all interfering substances by evaporation or by precipitation as hydroxide or sulphide, using a suitable substance to collect and carry down the trace of precipitate; it is then dissolved in free sulphocyanic acid.

4. The fading out of the color of the ferric sulphocyanate is prevented by the use of persulphate, and the formation of discolored

amylic solutions is prevented by saturating the sulphocyanic acid with mercuric sulphocyanate.

5. Special directions are given for concentrating traces of iron from various substances, with analytical data.

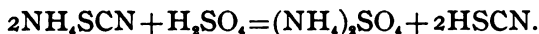
ON SULPHOCYANIC ACID.

By H. N. Stokes and J. R. Cain.

An aqueous solution of sulphocyanic acid may be prepared by distilling an acidified solution of a sulphocyanate, by decomposing the barium or lead salts with sulphuric acid or by decomposing the lead¹ or mercuric² salts under water with hydrogen sulphide and removing the excess of the latter by a stream of air or carbon dioxide. All these methods have their drawbacks, involving either distillation or the use of salts which are not to be found in every laboratory, and which are expensive and not to be depended on for purity. The decomposition of the lead salt by hydrogen sulphide is slow and the removal of the excess of the latter tedious.

In connection with our work on the colorimetric determination of iron³ we have employed a process by which any desired amount of aqueous sulphocyanic acid, free from iron and all other nonvolatile impurities, may be prepared in a few minutes and which, in view of the increasing importance of this body as an analytical reagent, we communicate under a separate heading, together with a few observations on its properties and those of some of its derivatives.

One hundred parts coarsely powdered ammonium sulphocyanate, which need not be iron free, are dissolved in a stoppered graduated cylinder in a cool mixture of 65 parts by weight of concentrated sulphuric acid with 100 parts water. These correspond approximately to the equation



There is a marked fall of temperature. After noting the volume,

¹ Zimmermann, Ann., **204**, p. 226; 1880.

² Rosenheim and Cohn, Zs. anorg. Chem., **27**, p. 288, note; 1901.

³ This Bulletin, **3**, p. 115; 1907.

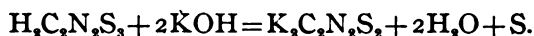
the solution is transferred without delay to a separating funnel and shaken out once with three-fourths its volume of amyl alcohol. We here notice the peculiar phenomenon that iron, if present, remains entirely in the acid layer, which is usually colored pink, while the amylic layer is colorless. In order to obtain this result, however, it is necessary to adhere to the proportions given. If a more concentrated acid is used, there is considerable decomposition of the sulphocyanic acid, while if as much as an equal volume of amyl alcohol be employed, some of the iron passes over into the latter. The amylic solution, which is found by titration with silver nitrate to contain 20–21 per cent sulphocyanic acid, is unstable and soon becomes yellow, with ultimate deposition of isopersulphocyanic acid. It is therefore at once shaken out twice with an equal volume of water. As we have determined, sulphocyanic acid distributes itself about equally between water and amyl alcohol; the first aqueous extract therefore contains about 10 per cent and the second 5 per cent, or the united extracts about 7.5 per cent, while the amyl alcohol retains about 5 per cent, in which concentration it is relatively stable and may be kept or used for making a weaker aqueous acid. It gradually turns yellow from formation of isopersulphocyanic acid, but this is entirely retained on extraction with water.

The aqueous acid thus prepared contains about one-fortieth of amyl alcohol, which, however, is without significance for most purposes. The 7.5 per cent acid generally becomes yellowish, but the color may be removed by shaking with a small quantity of amyl alcohol. In this strength it is used in the colorimetric iron determination, as described in the preceding paper, after saturating with mercuric sulphocyanate.

Aqueous sulphocyanic acid prepared by the action of mineral acids on an alkali sulphocyanate is a substance of very varying properties, depending on its age and concentration. It contains, as does also a freshly acidified solution of alkali sulphocyanate, a small quantity of a substance which increases on keeping, which rapidly gives with hydrogen peroxide, bromine, or persulphate a yellow amorphous oxidation product, scarcely soluble in water, but easily soluble in amyl alcohol, thus interfering seriously with colorimetric iron determinations. This substance, which appears to be colorless, can be removed for the greater part by shaking out with a small

quantity of amyl alcohol. It appears to be absent at first in acid prepared by decomposing the silver salt with hydrogen sulphide, but in this, too, it gradually makes its appearance. It differs from isodisulphocyanic acid, which is also present, as the latter gives an oxidation product insoluble in amyl alcohol. Normal sulphocyanic acid is likewise rapidly oxidized by hydrogen peroxide, with marked rise of temperature, and also by bromine water, but the oxidation products are soluble and colorless. Judging from the analogous behavior of the unknown body and isodisulphocyanic acid it may be as yet unisolated isosulphocyanic acid $\text{HN}:\text{CS}$. The formation of yellow oxidation products by persulphate is entirely prevented by saturating the acid with mercuric sulphocyanate.

Isodisulphocyanic acid, $(\text{HN}:\text{CS})_2$, is one of the products of the action of mineral acids on sulphocyanates or sulphocyanic acid⁴ and of the spontaneous change of sulphocyanic acid itself. Hence it is present in both aqueous sulphocyanic acid and in the acidified sulphocyanate solutions used in colorimetric iron determinations. Our experiments show that to this substance in part at least is to be attributed the gradual bleaching of the color of ferric sulphocyanate so often referred to by writers on this subject, and also the formation of a precipitate in the ether solution of the same. We prepared this acid by decomposing isopersulphocyanic acid crystallized from amyl alcohol, with potassium hydroxide:⁵



On acidifying, the amorphous yellow acid is obtained, which is slightly soluble in water and easily soluble in amyl alcohol, giving a yellow solution. It is distinguished from isopersulphocyanic acid by giving with stannous chloride an amorphous yellow precipitate which does not turn red and crystalline, is soluble in amyl alcohol, dissolves in boiling water, and does not separate on cooling. Its behavior toward ferric salts is also characteristic. Its aqueous solution is colored red-brown by a small amount of ferric salt, but the intensity of the color is by no means comparable with that of ferric sulphocyanate. The color is partly taken up by amyl alcohol, giving a red-brown solution, but its solubility in this does not seem to exceed that in water. Peculiarly noteworthy is it that the color of

⁴ Klason. J. pr. Chem. (2), **88**, p. 383; 1888.

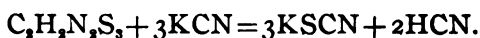
⁵ Fleischer. Ann., **179**, p. 204; 1875.

either the aqueous or the amylic solution of the ferric salt fades out in a few minutes, and if a little ferric sulphocyanate be present this is bleached at the same time. Addition of a trace of persulphate instantly restores the color temporarily. Isodisulphocyanic acid is very rapidly oxidized by persulphate, hydrogen peroxide or bromine water, giving a pulverulent yellow body, which is insoluble in and separates from either water, amyl alcohol, or ether. The properties of this substance agree with those of pseudosulphocyanogen.

Isopersulphocyanic acid, $\text{H}_2\text{C}_2\text{N}_2\text{S}_3$, which is formed by the decomposition of sulphocyanic acid, especially when concentrated, after the equation



is frequently met with. It imparts a yellow color to the amylic or aqueous solutions and is deposited as yellow needles. It is not always distinguished from pseudosulphocyanogen or the other yellow oxidation product referred to above, nor from isodisulphocyanic acid, all of which are yellow and which may be encountered in the course of analytical operations with sulphocyanic acid. Isopersulphocyanic acid is not in itself an oxidation product, although it may accompany these. It may be recognized by its giving sulphocyanic acid on warming with potassium cyanide:⁶



A further characteristic reaction which we have observed is its behavior toward stannous chloride.⁷ An amylic solution, shaken with a few drops 10 per cent stannous chloride solution, becomes of a stronger yellow, and soon, or at once, according to the concentration, deposits a yellow precipitate. This rapidly becomes darker and denser, and soon turns to a dark red, heavy, crystalline sediment. This redissolves in amyl alcohol on heating, forming a yellow solution which on cooling may either give the red or yellow form, the latter, however, soon turning red. This change is promoted by gentle warming. The salt is decomposed by much acid, but may be regenerated by further addition of stannous chloride. Impure

⁶ Steiner. Ber., 15, p. 1603; 1882.

⁷ Völckel (Ann., 48, p. 76) states that it gives a yellow precipitate with stannous chloride.

material shows the same behavior, but the change to red is slower. Isopersulphocyanic acid may be detected in aqueous sulphocyanic acid by extracting with a small amount of amyl alcohol and treating the latter as above.

SUMMARY.

1. A rapid method of preparing pure sulphocyanic acid.
2. Some properties of the solution.
3. Isodisulphocyanic acid, its occurrence in sulphocyanic acid, its behavior toward oxidizing agents and its agency in the bleaching of ferric sulphocyanate.
4. Isopersulphocyanic acid, its occurrence in sulphocyanic acid and a characteristic reaction.

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DEPARTMENT OF COMMERCE AND LABOR

Vol. 3

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No. 2

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

MAY, 1907

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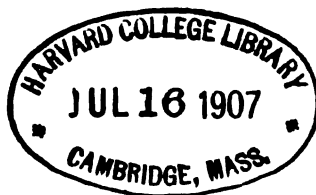
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From the
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THE RADIATION FROM AND THE MELTING POINTS OF PALLADIUM AND PLATINUM.

By C. W. Waidner and G. K. Burgess.

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I. INTRODUCTION.

The scale to which all measurements of temperature, from the lowest attainable to 1200°C , are now referred is that of the gas thermometer. A consideration of the best experimental data now available leads to the conclusion that the scale of the nitrogen gas thermometer, which is the standard above 200° , is probably in agreement with the absolute thermodynamic scale to within the limits of accuracy at present attainable in high temperature gas thermometry (above 500°). The present upper limit of the gas scale is about 1200° , i. e., where the limit of reasonably certain accuracy is 5° . With facilities that are now attainable this limit could perhaps be raised some 300° or 400° . For the range of temperatures above 1500° the scale must be based, for the present at least, on extrapolation by means of the radiation laws, which have some theoretical support, and can be tested within the range of the gas scale.

In the practical establishment and use of the scale it is convenient to have certain easily reproducible fixed points. Two standard temperatures of reference that have been made the basis of numerous investigations are the melting points of palladium and platinum. The best determinations of these melting points have differed by some 70° within the past two years, and the true values may still be regarded as uncertain by 40° .

In the present paper are described the results of experiments on the melting points of these metals, as determined by several different methods, and in particular the results obtained by the application of the Wein¹ equation giving the relation between the absolute temperature of a black-body and the intensity of any monochromatic radiation.

The first part of the paper is devoted to a description of the calibration of the optical pyrometer and accessories used in the investigation, and includes a discussion of the practical realization of blackness in certain furnaces used as light sources. It was also found necessary to study in detail the effect, on the temperature measurements, of the lack of monochromatism in the glasses used with the pyrometer. The observations on the monochromatic radiation from palladium and platinum strips, which is next undertaken, using red, green, and blue light, give results in very close agreement when applied to the determination of melting points. These optical determinations also agree with the more exact optical measurements made on the melting points of the metals placed within an iridium tube furnace which approximates a black-body. The values of the melting points as determined by the usual thermoelectric method are some 45° lower than those found by the two optical methods.

II. STANDARDIZATION OF OPTICAL PYROMETER.

The optical pyrometer used in the present investigation is of the Holborn-Kurlbaum type² in the form in which it was originally

¹ The Wien equation $J=c_1 \lambda^{-5} e^{-\frac{c_2}{\lambda T}}$ is a sufficiently close approximation to the more general equation of Planck $J=c_1 \lambda^{-5} \left(e^{\frac{c_2}{\lambda T}-1} \right)^{-1}$ within the visible spectrum and for any attainable temperature.

² Holborn and Kurlbaum, *Berichte Berlin Akad. d. Wiss.*, p. 712; 1901. *Ann. d. Physik*, 10, p. 225; 1903.

constructed by Siemens and Halske, embodying the disappearing filament principle.³ The suitability of this instrument for temperature measurements, the order of accuracy attainable, and the constancy of the lamps have been discussed in a previous paper.⁴

Experimental Black-Body.—The only perfectly definite radiation independent of the physical properties of the radiator, and dependent only on the temperature, is the radiation of a Kirchhoff black-body. In the absence of definite knowledge concerning the law of radiation for each particular substance, it is not possible at present to measure the true temperature on the thermodynamic scale by the intensity of the radiation of any such substance. It is only feasible at present to express very high temperatures on the black-body scale,⁵ and it is therefore necessary for the realization of this scale, to construct a black-body that can be used experimentally for the calibration of optical pyrometers.

The black-body used in this work is a modified form of the electrically heated black-body devised by Lummer and Kurlbaum.⁶ The modification consists in the addition of supplementary heating coils, on independent electrical circuits, wound on a porcelain tube inclosing the black-body. These coils, which project 8 cms beyond each end, are wound so as to give a very uniform temperature distribution throughout the black-body, a condition which can be realized with a few trials. The winding of these secondary coils is very close about the ends of the black-body and very open about the center.

The accompanying Table I will serve as an illustration of the uniformity of temperature obtainable throughout the interior of this compensated black-body. Temperatures were measured by an exploring thermocouple. The distance from the central diaphragm used as a radiating source to the opening is 19 cms.

³ Morse, U. S. Letters Patent, 696878, 696916; 1902.

⁴ Waidner and Burgess, Optical Pyrometry, this bulletin, 1, p. 189; 1905.

⁵ The black-body temperature, s° , is the temperature at which the ideal black-body radiator of Kirchhoff would emit radiation of the same intensity as a substance at a temperature t° for the particular wave length under consideration. Black-body temperatures might conveniently be indicated by $t^\circ K$ where $^\circ K$ means the Kirchhoff absolute scale. When it is necessary to specify the wave length, this could be done as follows ($t^\circ K_\lambda$): For example, the expression "1000° Abs. black-body temperature for wave length 0.650 μ ," becomes 1000° $K_{0.650}$ or 727° $K_{0.650}$ C.

⁶ Lummer and Kurlbaum, Verh. Deut. Phys. Ges., 17, p. 106; 1898. Ann. d. Physik. 5, p. 829; 1901.

TABLE I.
Temperature Distribution Within Black-Body.

Distance in cm from central diaphragms		0	2	4	8	12
Temperature	Series I	621°	621°	620°	612	
	Series II	1041	1042	1042	1032	
	Series III	1308	1310	1311	1300	
	Series IV	1244.9	1244.9	1244.7	1244.7	1244.8

In order to determine the temperature of the radiating wall accurately, it is important that its front and back surfaces be at approximately the same temperature. Two thermocouples were therefore used with the junctions in contact with these surfaces.

The ordinary form of black-body uniformly wound with platinum ribbon does not realize with the highest attainable accuracy the two necessary conditions for ideal black-body radiation, namely, uniform temperature distribution throughout the radiating inclosure and a determinable temperature of the radiating wall. This is illustrated in Fig. 1, where A is such a black-body,⁷ provided with diaphragms both in front and back of the radiating wall. The plot immediately below A gives the temperature distribution as measured by thermocouples in the manner indicated. It will be seen that the temperature distribution is not uniform, and that the back and front of the radiating wall differ considerably in temperature.

If the diaphragms back of the radiating wall are removed, as in B, the temperature distribution in front of the radiating wall may be improved, but the actual temperature of the radiating wall itself is impossible of exact determination. The temperature distribution in the ordinary black-body (A of Fig. 1) can be improved without the use of supplementary heating coils, by winding it with a platinum ribbon cut so that its width diminishes as the ends are approached.

After the calibration of the pyrometer lamps in terms of the radiation of the nearly ideal black-body described above, they were applied to an investigation of the degree of approximation to blackness of various forms of electrically heated furnaces. Measurements

⁷ The black-body radiator alone is shown in the illustration. In use this was surrounded by concentric porcelain tubes.

up to 1300° made on ordinary Heraeus resistance furnaces, in which the porcelain heating tube (2 cm diam.; length of winding equals 25 diameters) is uniformly wound with platinum ribbon, gives an approximation to black-body radiation to within 5° , for a radiating surface at the center of the furnace. Experiments with the interior of the furnaces, both coated with black oxides and uncoated, gave no certain difference. When thermocouples are used in furnaces coated with metallic oxides, which conduct at lower temperatures than porcelain, care must be taken to avoid leakage from the heating

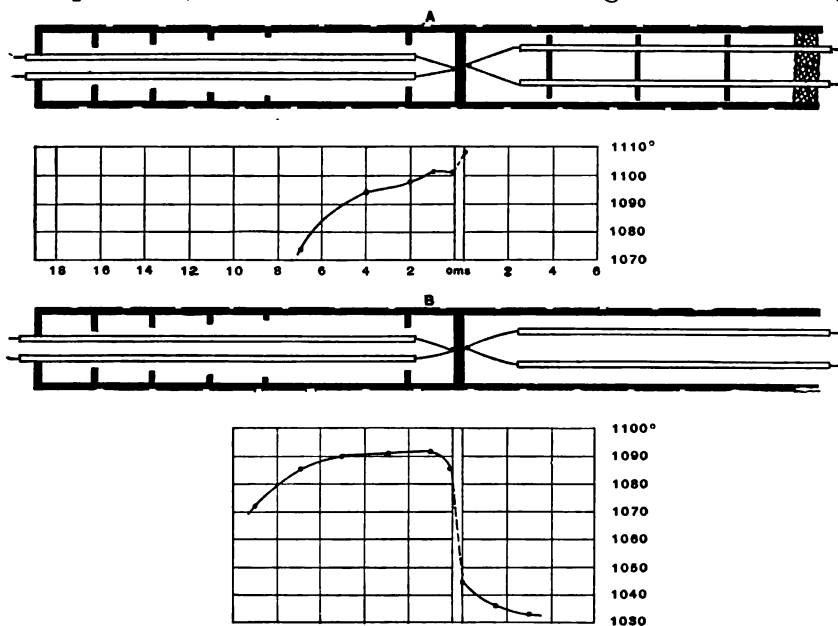


Fig. 1.—*Temperature Distribution within Black-Body.*

circuit if direct current is used, and to avoid shunt circuits on the couple.

With a view to calling attention to the errors that may arise in the experimental realization of black-body radiation, several cases may be cited. The effect of the temperature gradient in the radiating wall, of the method of measuring the temperature where a thermocouple is used, and of the ratio of size of opening to length of radiator are illustrated in Fig. 2. Here the numbers given below each illustration are the values of the temperature indicated by a

thermocouple minus the temperature indicated by the optical pyrometer. The arrows indicate the direction in which the optical pyrometer was sighted.

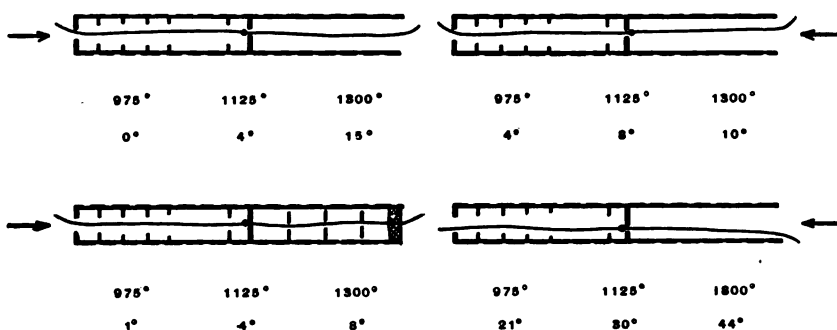


Fig. 2.—Approximations to a Black-Body.

Lamp Calibration.—The pyrometer lamps used in the Holborn-Kurlbaum pyrometer were four-volt carbon filament lamps, which were aged at a temperature of about 1800° for a period of twenty-five hours previous to their calibration. These lamps were calibrated by measuring the current when the filament had the same brightness (for red light) as the compensated black-body described above, whose temperature was measured by two thermocouples. The milliammeter with its shunt was calibrated frequently during the progress of the work, and its temperature coefficient determined. The calibration of lamp No. 140 is cited as an illustration (Table II).

TABLE II.

Calibration of Lamp 140.

Temperature (Observed)	Current (Observed)	Temperature (Computed)	Temp. obs.—Temp. calc.
920°	0.4486	921°	—1°
1087.5	.5305	1087.5	0
650	.3357	649	+1
1221	.6023	1221	0
692	.3525	692.5	—0.5
1285	.6393	1285	0
1089	.5309	1088.5	+0.5

The observations are represented by the equation,

$$I = 0.1681 + 0.0001482t + 0.0000001700t^2$$

where I = current in amperes through the lamp filament, and t = the black-body temperature for red light.

In this table are given the observed currents through the lamp and the corresponding temperatures of the compensated black-body; also the temperature, corresponding to the observed currents, computed from the above equation. In the last column are given the differences between the observed and computed temperatures.

The calibration equations of the five pyrometer lamps are given in Table III.

TABLE III.
Calibration Equations of Pyrometer Lamps.

Lamp No	Equation	No. of Cal. Points	Average dev. of an obs.
132	$I = 0.1736 + 0.0_3130t + 0.0_61865t^2$	8	1.0
140	$I = .1681 + .0_31482t + .0_61700t^2$	7	0.4
145	$I = .1600 + .0_31863t + .0_61422t^2$	9	1.2
168	$I = .1829 + .0_66010t + .0_62348t^2$	9	1.7
171	$I = .1785 + .0_31179t + .0_62112t^2$	10	2.2

The deviation of a single observation from the mean exceeds 3° at only three points, and the average is 1.3 .

The above calibration equations of the lamps, although determined with red glass before the eyepiece of the pyrometer, hold also in practice for any color if a black-body is observed which, it may be remarked, is evidence that a carbon filament is a "gray" body within the limits of the visible spectrum. This is shown in Table IV.

During the progress of this investigation, the calibration of the lamps was frequently checked by use of the black-body, and no evidence of change in the lamps was detected. For example, near the

⁸ A body may be said to be gray throughout any spectral region when the ratio of the intensity of its radiation to that of a black-body is the same for all wave lengths within that region.

conclusion of this investigation a recomparison of the pyrometer lamps with the compensated black-body gave—

Mean Temperature of Black-Body.

By 5 Pyrometer Lamps.	By Two Thermocouples.
1018°0	1017°6
1154.5	1154.9

TABLE IV.

Calibration of Pyrometer Lamps for Red and Green Light.

Lamp No	Obs. temp. of the black-body	Observed current, using:		Computed temps., using:	
		Red light	Green light	Red light	Green light
145	1261°	0.6208	0.6188	1260°5	1256°5
168	1277	.6221	.6229	1276	1277.5
171	1267.5	.6675	.6686	1267.5	1269
Mean	1268°5			1268°2	1267°8
140	1088°5	0.5306	0.5303	1088°5	1087°5
145	1093.0	.5325	.5323	1091.0	1090.5
168	1084.2	.5242	.5248	1084.7	1086.2
Mean	1088°6			1088°1	1088°1

Absorption Factors.—In order to determine very high temperatures by extrapolation based on Wien's equation it is usually necessary to diminish the intensity of the radiation in a known ratio by means of absorbing glasses, mirrors, or sector disks placed in front of the optical pyrometer. The absorption factor K is given by the equation⁹

$$\log K = c_3 \frac{\log e}{\lambda} \left(\frac{1}{T_1} - \frac{1}{T_2} \right)$$

where λ is the wave length in μ of the radiation employed, c_3 is taken as 14,500 for a black-body, and T_1 and T_2 are, respectively,

⁹ If J_1 and J_2 are the intensities of the light of wave length λ , corresponding to the black-body temperatures T_1 and T_2 (abs.), then the absorption coefficient is—

$$K_\lambda = \frac{J_2}{J_1} = \frac{c_2 \lambda^{-5} e^{-\frac{c_2}{\lambda T_2}}}{c_2 \lambda^{-5} e^{-\frac{c_2}{\lambda T_1}}} = e^{\frac{c_2}{\lambda} \left(\frac{1}{T_1} - \frac{1}{T_2} \right)}$$

the apparent black-body temperatures (absolute scale) when the radiating source at constant temperature is viewed with and without the absorbing screen.

In the melting point determinations, use was made of the absorbing mirrors of the Holborn-Kurlbaum pyrometer, shown diagrammatically in Fig. 3.

The intensity of the incident light is diminished by two reflections from black glass. The absorbing glasses furnished by Pellin with the Le Chatelier optical pyrometer were also used.

The absorption factors were determined by sighting the pyrometer on an electrically heated black-body whose temperature was maintained constant to a few tenths of a degree, as indicated by a thermocouple. A single measurement consisted in a series of six to ten observations of T_2 between two series of observations of T_1 . With the mirror form of absorber, the pyrometer has to be resighted for each set of observations, and as the neighboring parts of the radiating surface may differ somewhat in temperature, there is a small uncertainty introduced into any given value of K from this cause.

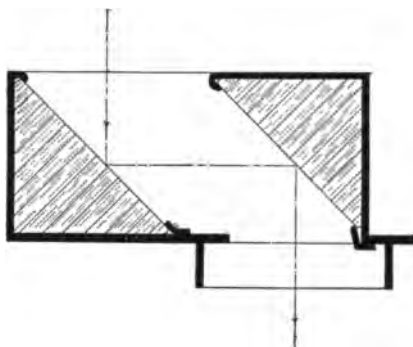


Fig. 3.—Absorption Mirrors.

The results of the measurements of the absorption factor K of the mirrors, made with red and green light, are given in Table V.

Very small errors in the temperature measurements will cause considerable errors in the resulting value of K ; thus an error of 1° in T_2 produces 0.8 per cent error in K , and 1° in T_1 an error of 2 per cent in K . Again, two observers may not agree in their estimations of equal brightness at different temperatures; for measurements in the red the two observers differ consistently by 3 per cent in K and in the green by over 5 per cent. The probable errors of K correspond to an uncertainty in the platinum melting point of 1° for the red and 2° for the green.

TABLE V.
Absorption Factor of Mirrors.

$\lambda=0.662\mu$ (Red)				$\lambda=0.547\mu$ (Green)			
Lamp No.	T_1	T_2	K	Lamp No.	T_1	T_2	K
132	1108.5	1519°	208.4	171	1204.5	1591.7	211.7
"	1109	1513	194.9	"	1199.5	1592	232.1
171	1150.8	1592.6	196.1	168	1208	1598	212.2
"	1152	1592	191.3	"	1205	1599	225.9
168	1153	1595	193.1	140	1202.5	1598	233.6
"	1151	1595.8	201.0	"	1200	1596.5	240.8
140	1151	1597.5	203.9	145	1203	1593	220.4
"	1149.5	1599	212.3	"	1199.5	1593	234.6
145	1152	1597	199.9				
"	1149	1594.5	205.7				
			Mean: 200.6 $\pm$ 1.5				Mean: 226.4 $\pm$ 3.0

Since the mirrors show slight selective absorption and the maximum of visual intensity of the transmission band of the red glasses varies with the temperature (see p. 175), the value of K must be corrected to correspond to the wave length used. The $\lambda_{\max}$ for the two red glasses usually used at the eye-piece of the pyrometer, has been found to be 0.666μ and 0.668μ at black-body temperatures of 1550° and 1750° , respectively, for which the corresponding values of K may be taken as 200 and 199. For the monochromatic combination 2 Cobalt+1 Jena No. 2745, whose $\lambda_{\max}$ is 0.720μ at 1550° , K becomes 186.5.

If the absorption is strongly selective, as is the case for most transmission glasses, this method of determining K becomes complicated by the fact that the effective wave length is different with and without the absorption screen, owing to the lack of monochromatism of the colored glasses used before the eyepiece of the pyrometer, and Wien's equation is no longer directly applicable for the computation of K .

As a check on this method of measuring K , the absorption factor of a sector disk of known opening was determined. The results are given in Table VI.

TABLE VI.
Absorption Factor of Sector Disk.

Red $\lambda_{\max}=0.660\mu$			
Lamp No	T_1	T_2	K
145	1225	1524.6	33.7
168	1223	1526.5	35.6
168	1225.5	1528	34.8
145	1220	1525.5	36.8
			Mean: 35.2

The value of K for this disk¹⁰, from measurements on a circular dividing engine, is 35.45. The agreement is better than we have any reason to expect, as it corresponds to an accuracy of a few tenths of a degree in the temperature measurements T_1 and T_2 . The sector disk is free from selective absorption, and the value of $\lambda_{\max}=0.660\mu$ taken, is that corresponding to $T_2=1525^\circ$ abs. (1250°C) for the two red glasses used.

A few observations were also made with two absorbing glasses furnished by Pellin with a Le Chatelier optical pyrometer. The absorption factor of the two together was 101 for $\lambda=0.650\mu$, and about 30 for $\lambda=0.547\mu$, showing very decided selective absorption.

Effect of Polarization on Absorption Factor.—Owing to the polarization by reflection from polished surfaces, the value of the absorption factor of the mirrors may vary through wide limits if the incident light is polarized, as is the case for light emitted by incandescent platinum¹¹ and palladium, the amount of polarization in the emitted radiation being zero for normal incidence and increasing with the angle of incidence.

¹⁰ This Bulletin, 2, p. 23; 1906.

¹¹ The effect of polarized light emitted by platinum on the indications of the Wanner pyrometer is discussed by the authors in this Bulletin, 1, p. 251; 1905.

In Table VII are given the results of measurements of the absorption factor of the mirrors when the platinum radiating surface is viewed at an angle of 45° , and when the plane of incidence of the mirrors is, (1) parallel to the plane of emission, and (2) perpendicular to this plane.

TABLE VII.

Polarization and Absorption Factor.

Red ($\lambda=0.660\mu$)		Green ($\lambda=0.547\mu$)	
$\parallel$	$\perp$	$\parallel$	$\perp$
235	165	264	188

The absorption factors for normal incidence when the polarization is zero, as previously given, is 200 and 226 for red and green, respectively.

Monochromatic Glasses.—If the light used in optical determinations of temperature is not monochromatic, there may be error introduced into the temperature measurements from this cause. Instruments of the spectrophotometer type such as the Wanner pyrometer, using a definite narrow spectral band, are free from this error; but in all instruments employing colored glasses this source of error may become quite serious, varying in general with the temperature, the kind of glass used, and the method of observing. All so-called monochromatic glasses transmit one or more spectral bands of considerable width, variable with the intensity of the incident light. But more important is the fact that the position of maximum brightness, as estimated by the eye, shifts with the intensity and so with the temperature of the source. When intensities are measured with a bolometer or other instrument whose indications are proportional to the incident energy, the magnitude of the apparent change in temperature for a given change in wave length can be calculated from Wien's equation; but where intensities are estimated by the eye, account must be taken of the variation in sensibility of the eye, which decreases rapidly toward each end of the spectrum. When the wave length used varies with the temperature, Wien's equation can not be applied directly.

In view of the above considerations it was deemed important to investigate the optical homogeneity of the glasses used in this work. For this purpose an electrically heated platinum strip, whose temperature was measured by the optical pyrometer, was mounted before the slit of a Fuess spectrometer, by means of which the position of maximum intensity— $\lambda_{\max}$ —and the limits of the transmission bands were studied for different temperatures, and for different thicknesses of glass. The results are given in Table VIII.

TABLE VIII.
Monochromatism of Glasses.¹²

Glass	Thickness of Glass mm	Tempera- ture of Source	λ $\max$	Limits of Transmission Band
1 Jena red #2745.....	3.04	1000°	0.645 μ	0.698 μ — 0.610 μ
		1250	.650	.731 — .602
		1450	.656	.772 — .598
2 Jena red #2745.....	6.05	1450	.661	.753 — .608
2 Pyrometer red glasses.....	5.38	1050	.655	.711 — .613
		1450	.664	.755 — .613
2 Jena + 1 Pyrometer red.....	8.72	1450	.667	.749 — .624
2 Jena + 2 Pyrometer red.....	11.43	1450	.675	.745 — .634
1 Cobalt blue (red bands).....	2.31	1250	.708	.742 — .683
			.620	.643 — .598
2 Cobalt.....	4.62	1250	.716	.741 — .694
		1450	.716	.754 — .689
3 Cobalt.....	7.10	1450	.721	.755 — .701
		1250	.716	.738 — .697
2 Cobalt + 1 Jena red.....	7.66	1450	.719	.750 — .693
2 Jena green #431 ¹¹¹	6.18	1150	.547	.602 — .532
		1450	.546	.631 — .468
		1320	.462	.500 — .421
1 Jena blue #3086.....	4.32	1470	.462	.511 — .408

The limits of the transmission bands here given are defined by the disappearance of the cross-hair of the eyepiece at either edge, although in some cases, notably on the infra red side of the red bands there is a considerable region of vanishing brightness. Independent settings by two observers on $\lambda_{\max}$ agree on the average to better than 0.002 μ .

¹² The glasses designated "pyrometer red" were furnished with the Holborn-Kurlbaum pyrometer and are probably of the same composition as the Jena red glass No. 2745. The Jena glasses are described by R. Zsigmondy Zs. Instrkunde 21, p. 97; 1901; see also Holborn and Kurlbaum, Ann. der Phys. 10, p. 227; 1903.

An examination of the data shows that as the temperature of the source, i. e., the intensity of the incident light, increases, the width of the transmission band increases for all the glasses. An increase in thickness of a glass cuts down the width of all bands, and in the case of cobalt glass eventually cuts out the second red band. The most important effect, however, for the work in hand, is the shift of the position of maximum brightness in the red glasses toward *longer* wave lengths with increase in temperature, accompanied by a shift of the maximum in the same direction with increased thickness. For the green and blue glasses the maximum does not shift with temperature within the range studied, but determinations with these glasses, especially with the blue, are relatively less satisfactory.

By the combination of two cobalt blue glasses which give but one red band ($\lambda_{\max}=0.716\mu$) with one Jena red glass No. 2745, which serves to cut out the green and blue bands transmitted by the cobalt glasses, a glass is obtained which gives a very much narrower band in the red, whose maximum ($\lambda_{\max}=0.716\mu-0.719\mu$) shifts less with temperature than any other red monochromatic glass that has come to our attention.

Whether the use of $\lambda_{\max}$, as determined by the eye, for the effective wave length in Wien's equation, is a sufficiently close approximation depends on the variation of intensity and on the sensibility of the eye within the spectral band. This procedure can not lead to errors that are significant for the work in hand. An error of 0.005μ in wave length involves an error of only about 5° at the temperature of the melting point of platinum. In the extrapolation to find $\lambda_{\max}$ at 1545° and 1750° , corresponding to the melting points of palladium and platinum, the wave lengths 0.666μ and 0.668μ were used, instead of the values given by a linear extrapolation (Table VIII), as the shift of $\lambda_{\max}$ must become less rapid at high temperatures.

III. THE RADIATION FROM PLATINUM AND PALLADIUM.

The relation of the radiation (red and green) of platinum and palladium to black-body radiation was determined in the temperature range $700^\circ-1300^\circ$. The black-body temperature of the incandescent metal was measured with the optical pyrometer. Three methods were used to estimate the actual temperature of the radiating surface.

In the *furnace method*, Fig. 4, a platinum ribbon 4 mm wide was closely wound on a porcelain or quartz tube of about 4 mm external diameter, the space between successive turns being of the order of 0.1 mm. This small electric furnace was maintained at the desired temperature by a suitable electric current. A thermocouple, threaded through the tube, with its junction at the center of the furnace, was assumed to give the temperature of the platinum ribbon immediately surrounding it. In all, some ten furnaces were used

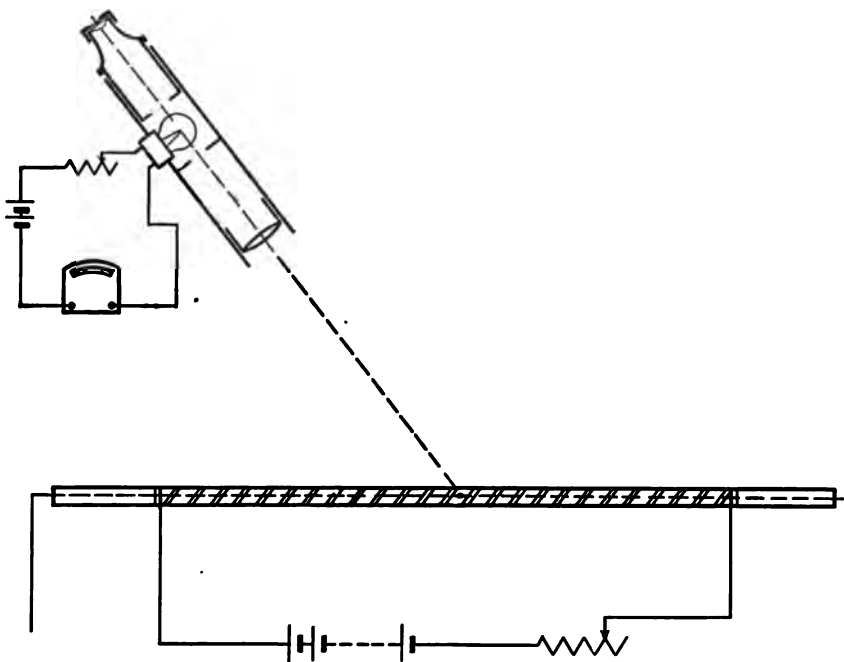


Fig. 4.—Radiation by Furnace Method.

in these experiments. The furnaces were worked practically in the open air to eliminate any possible effect due to light reflected from surrounding walls, blackened asbestos shields affording sufficient protection against sudden temperature fluctuations. The results of the separate determinations are given in Fig. 5 where abscissae are black-body temperatures, s , and the ordinates are values of $t-s$, where t is the true temperature. The points on the plot designated by circles were obtained with palladium wound furnaces.

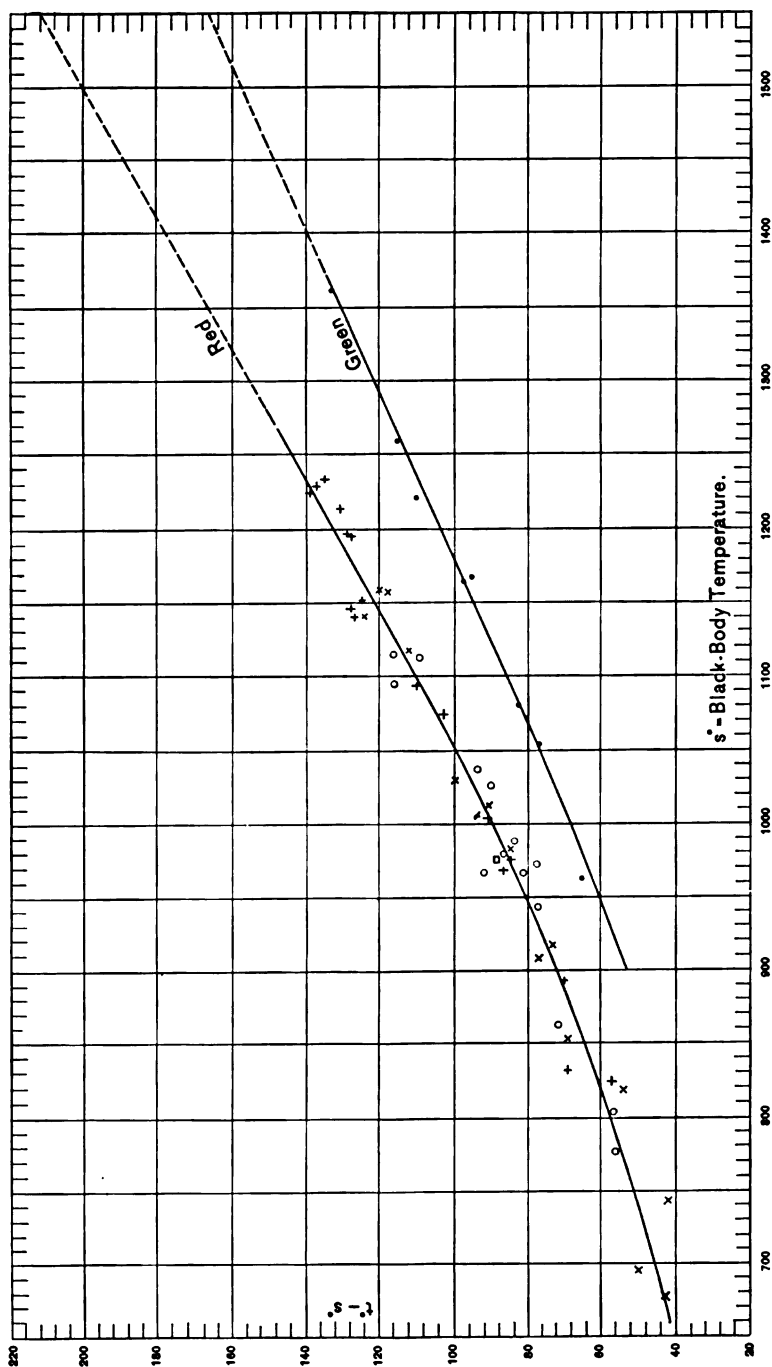


Fig. 5.—Radiation from Platinum and Palladium.

To obtain some idea how nearly the assumption made in this method, viz, that the temperature of the radiating surface is given by the almost completely inclosed thermojunction, the indications of the thermocouple were noted at the instant a minute piece of gold (a few thousandths of a milligram) was observed to melt on the surface of the platinum or palladium ribbon when viewed with a microscope. The following observations on a radiator wound with palladium ribbon will serve as an illustration:

$$1064^{\circ} \quad 1053^{\circ} \text{ }^{18} \quad 1063^{\circ} \quad 1066^{\circ}$$

It will be seen that for this radiator the mean reading of the inclosed thermocouple agrees very well with the melting point of gold (1064°). Unfortunately, the first two radiators examined seemed to substantiate the above assumption so thoroughly that this test was not applied to the remaining radiators. Toward the close of the work, when most of the radiators were no longer available, evidence developed that part of the discrepancies observed may have been due to failure to comply with the above assumption.

In the *melting-point method* the black-body temperature of an electrically heated platinum or palladium strip was measured with an optical pyrometer at the instant of melting of a minute fragment of gold, as described above. The results of some observations made on the palladium strip by this method are given in Table IX.

This method involves the assumption that the temperature of the platinum (or palladium) ribbon is equal to the melting point of gold at the instant the minute fragment of gold on the ribbon is observed to melt. The agreement of observations made with fragments of very different size indicates that the assumption is quite exactly fulfilled.

The black-body temperature (for $\lambda = 0.655\mu$) of platinum (or palladium), as determined from the mean of all the observations by this method, is 975° at the melting point of gold (1064°), corresponding to

$$t^{\circ} - s^{\circ} = 1064^{\circ} - 975^{\circ} = 89^{\circ}.$$

This point is indicated by a square ($\square$) on the curve in Fig. 5.

¹⁸ Thermojunction was probably displaced from under the portion of the palladium ribbon on which the gold melted.

By the same method the black-body temperature (for $\lambda = 0.666\mu$) of platinum at the melting point of palladium was found to be 1383° . On account of the almost exact equality in the radiation from these two metals this determination was very difficult.

TABLE IX.

Radiation from Palladium at the Melting Point of Gold.

Lamp No.	Black-body temperature of palladium 8° for $\lambda = 0.655\mu$	$T^\circ - 8^\circ$
140	980°	84°
"	975	89
132	976	88
"	974	90
145	976	88
"	976	88
168	979	85
"	978	86
"	976	88
	977°	87°

As a further check on the suitability of the preceding methods for determining the temperature of the radiating surface, a third method, which may be designated as the *equal radiation method*, was employed. In this method, which is illustrated in Fig. 6, the junction T_1 of a platinum, platinum-rhodium thermocouple is heated in a colorless Bunsen flame. To protect the flame from convection currents, it is inclosed in a metal cylinder C . The current through the radiator, Pt is then slowly increased until the brightness of Pt is equal to that of the platinum wire of the thermocouple T_1 , which is indicated by the disappearance of the junction when viewed against the radiator Pt as background. At this instant, temperature measurements of T_1 and T_2 are taken alternately. To insure equal temperature distribution at T_1 , the platinum and the alloy wires were fused together without appreciable enlargement at the junction. In Table X are given some observations by this method.

TABLE X.
 Temperature of Radiating Platinum Surface.

Temperature given by—	
Thermocouple T_1 in flame	Thermocouple T_2 in radiator
1344°	1345°
1339	1341
1329	1327
1332	1332

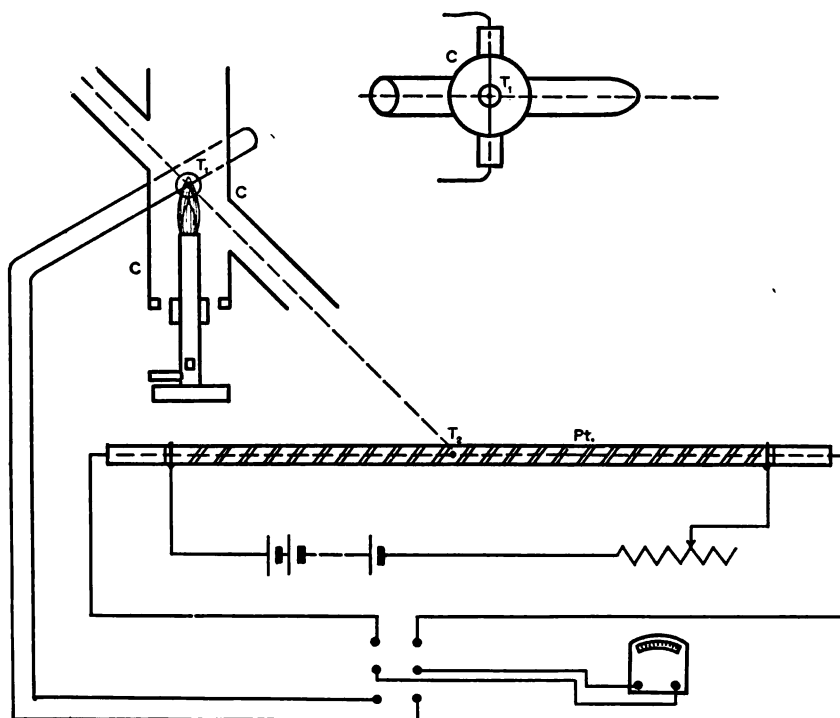


Fig. 6.—Equal Radiation Method.

For this radiator, therefore, the surface temperature of the platinum ribbon was sensibly identical with the temperature indicated by the inclosed thermocouple.

Some experiments were also made on the *radiation from platinum in hydrogen*, using the furnace method. For this purpose the platinum radiator was inclosed within a metal inclosure, blackened on the inside, and the radiation observed through a mica window. Alternate series of measurements made in hydrogen and in air at about 900° , 1100° , and 1300° , gave values of $t^{\circ}-s^{\circ}$ about 7° larger than in air. This difference may be due in part to the absence of oxidation and in part to the greater loss of heat from the surface in hydrogen.

The full curves in Fig. 5 show the relation between s° and $t^{\circ}-s^{\circ}$, for the radiation of platinum and palladium in air, for red light, and also for green ($\lambda=0.547\mu$). The curve for red light does not apply for strictly monochromatic radiation, as the effective wave length transmitted by the red glasses used before the eyepiece of the pyrometer increases from about 0.65μ at 800° to 0.66μ at 1300° (See p. 175.)

Some previous observations¹⁴ based on the assumed melting points of salts and metals, including platinum and palladium, when corrected to the temperature scale to which the present investigation has led, viz, 1546° for the melting point of palladium and 1753° for platinum, are in satisfactory agreement with the results of the later work. The value of $t^{\circ}-s^{\circ}$ for the earlier observations after correction are 9° higher at 800° , in exact agreement at 1100° , and 9° lower at 1500° than the values of $t^{\circ}-s^{\circ}$ given in Fig. 5.

On account of the great difficulty in determining the temperature of a radiating surface, even under the most favorable conditions, as when platinum is used, and the unavoidable changes in character of the surface, a high order of accuracy in the determination of $t^{\circ}-s^{\circ}$ is very difficult of attainment. The effect of constant errors likely to be present in the first two methods employed would be to give low values of $t^{\circ}-s^{\circ}$.

IV. MELTING POINTS OF PALLADIUM AND PLATINUM.

In the work incident to the establishment of the high temperature scale during the past three years evidence was obtained, before we had proceeded far, that the scale based on the usual thermoelec-

¹⁴ This Bulletin, 1, p. 189; 1905.

tric extrapolation was not in agreement with that defined by the Wien equation, temperatures on the thermoelectric being lower than on the optical scale. Accordingly the determination of the melting points of palladium and platinum was undertaken with the object of finding the magnitude of this difference and, at the same time, establishing suitable fixed points to which the high temperature scale may at any time be referred. In the present communication the results obtained by three different methods are given: (1) Black-body method using monochromatic radiation; (2) Surface radiation method; (3) Thermoelectric method.

The palladium used in these experiments was obtained from Heræus and from Kahlbaum under specifications calling for the highest attainable purity. Wires of 0.3 mm and 0.6 mm were used as well as ribbon 4 mm wide by 0.02 mm thick. The platinum was the purest obtainable from Heræus. Samples of the same metal purchased at different times showed no measurable differences in the melting point. The same kind of platinum has been used recently by Nernst and Wartenberg, by Harker, and by Holborn and Valentiner. The temperature coefficient of electrical resistance of this platinum has been found to be higher than that of any other platinum used, which is the most sensitive test of its purity.¹⁵

BLACK-BODY METHOD.

The metals were melted in an Heræus tube furnace of pure iridium. The temperature measurements were made with an Holborn-Kurlbaum optical pyrometer, with which absorbing mirrors and a sector disk were used to reduce the intensity of the light by known amounts. The temperatures were computed from the Wien equation (p. 170).

Iridium Furnace.—The iridium tube I (Fig. 7) was 25 cm long, 2 cm in diameter, and 0.25 mm wall thickness. To avoid destructive evaporation of the iridium and contamination of the platinum and palladium, the iridium tube was coated inside and out with the

¹⁵ The purity of these metals is discussed by Mylius and Dietz, *Berichte Deutsch. Chem. Ges.* **31**, p. 3187, 1899.

refractory earth mixture¹⁶ due to Nernst. The interior of the furnace was recoated several times during the progress of the work.

The iridium tube was platinum-soldered at each end to a platinum disk which was bolted to silver alloy terminal springs *A*. This construction allows the iridium tube to expand freely and so prevents buckling of the iridium tube.

The furnace was heated by means of an alternating current supplied by two 3-kw transformers with primaries P_1 and P_2 in parallel and secondaries S_1 and S_2 in series. The transformers were operated from an alternator connected directly with a motor which was run from a

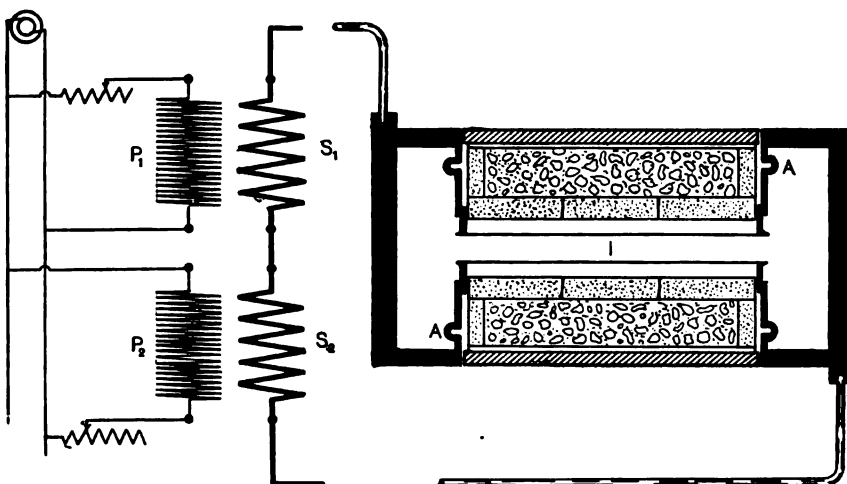


Fig. 7.—Iridium Tube Furnace.

storage battery, so that very exact temperature regulation was possible by the use of suitable rheostats in the primary circuits. The furnace was used in a horizontal position. To prevent convective circulation and to give a more uniform temperature distribution, one end was closed with an asbestos plug.

Approximation to Black-Body Radiation.—As the pyrometer lamps were calibrated by the use of a nearly ideal black-body (p. 165), it was possible to test how closely the iridium furnace as

¹⁶10 grams zircon-oxychloride and 2 grams yttrium-nitrate are dissolved in 60 cc of water, and pure finely ground thorium oxide is added to form a thin paste which is applied with a brush. The furnace should be heated to 1600° and recoated several times.

used satisfied this condition. This was done for several temperatures. The black-body temperature of melting gold was found to be 1063° from six determinations all lying between 1062° and 1064° , which agrees very closely with the true temperature, 1064° . Again, the black-body temperature of a coated iridium disk near the center of the furnace, as measured with these lamps, was 1306° , and the true temperature, as given by the thermocouples, was 1307° . Other corroborative measurements were made at intermediate temperatures.

From the final differences found between the extrapolated thermoelectric scale and the Wien scale (p. 205), it follows that the differences between these two scales at 1300° C is about 2° or 3° . These experiments therefore indicate that the departure of the radiation of the iridium furnace from the practically ideal black-body radiation in terms of which the pyrometer lamps are calibrated, is 1° at 1064° and about 3.5° at 1300° . The corrections for "lack of perfect blackness" at the melting points of palladium and platinum are therefore about 6° and 8° , respectively. As the surrounding walls of the furnace are at a slightly higher temperature than the radiating body under observation, the lack of blackness is in part offset by reflection from that body.

Methods of observation.—To determine the possible effect of contamination of the metals by iridium, some melts were taken with the metal (platinum) completely inclosed in tubes of fused magnesia;¹⁷ others were taken with the pyrometer sighted on the bare platinum hammered into a disk; and a few melts were also taken by sighting upon the coated iridium disk, immediately behind which the platinum was placed. The instant of melting for the first and last methods was determined by the interruption of an electric circuit, while for the second the melting could be seen. The above methods gave no measurable difference. The rate of rise of temperature at the melting point was varied within the limits 2° to 15° per minute, the usual rise being about 5° .

Reduction of an observation.—The following example will serve to illustrate the method of determining the temperature at the instant of melting. When the temperature had approached to within

¹⁷ The authors are greatly indebted to Mr. J. A. Heany of York, Pa., for tubes of fused pure magnesia.

15° or 20° of the melting point, observations were begun and continued up to or, where possible, a few degrees beyond the melting point. The time-temperature plot was then drawn (Fig. 8) and the temperature corresponding to the instant of melting was found by interpolation. This procedure gives a considerably higher accuracy than a single observation.

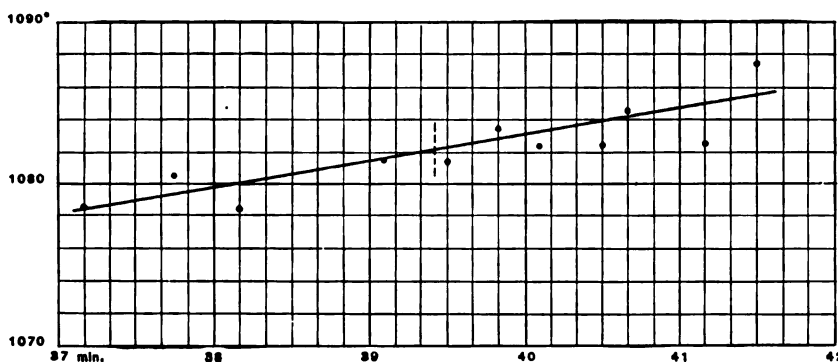


Fig. 8.—Melting Point by Interpolation.

The plot gives 1082° ($T_1 = 1355^\circ$ abs.) for the black-body temperature of the melting metal (platinum) as observed through the absorbing mirrors whose coefficient of absorption, K , equals 200. Substituting the above values of T_1 and K in the Wien equation,

$$\log K = c_2 \frac{\log e}{\lambda} \left(\frac{1}{T_1} - \frac{1}{T_2} \right),$$

where $c_2 = 14500$ and $\lambda = 0.668\mu$, we have $T_2 = 2023^\circ$ abs. $= 1750^\circ$ C.

Palladium.—Four series of observations were taken on the melting point of palladium, two with the absorbing mirrors, one with the ten degree sector disk, and one without any absorption screen, Table XI. The second series with the absorbing mirrors was taken several months after the first. The series without any absorbing screens gives a determination of the melting point by a method which is not based on Wien's equation but on direct extrapolation by the calibration equations (p. 169) of the pyrometer lamps. The series with the sector disk was taken after the work on platinum when the furnace had developed local irregularities in its temperature distribution.

TABLE XI.

Melting Point of Palladium in Iridium Furnace.

BY WIEN'S EQUATION.

With Absorbing Mirrors $K=200$, $c_2=14500$, $\lambda=0.666\mu$.

Lamp No	Current in Lamp	Observed M. P. Black-body Temperature	Melting Point
132	0.4827	984°	1538°
"	.4838	986	1542
145	.4808	983.5	1537
"	.4823	987	1544
171	.4991	984	1538
"	.4989	983.5	1537
168	.4704	986	1542
"	.4711	987	1544
			1540°
132	.4822	983	1536
"	.4827	984	1538
"	.4842	987	1544
168	.4709	986.5	1543
"	.4692	983.5	1537
171	.4982	983	1536
"	.4997	985	1540
			1539°

With Sector Disk $K=35.4$, $c_2=14500$, $\lambda=0.666\mu$.

171, 132, 145, 168, 11 Observations, 1124°5. 1540°

BY LAMP CALIBRATION.

Lamp No	Current in Lamps	Melting Point
132	0.8220	1547°
"	.8219	1547
145	.7895	1548
"	.7900	1549
171	.8654	1546
"	.8665	1547
		1547°

TABLE XII.

Melting Point of Platinum in Iridium Furnace.

BY WIEN'S EQUATION.

With Absorbing Mirrors $K=199$, $c_1=14500$, $\lambda=0.668\mu$ (Red).

Lamp No	Current in Lamp Amperes	Observed M. P. Black- body Temperature	Melting Point
145	0.5280	1081.5	1749° C
168	.5227	1082	1750
"	.5227	1082	1750
171	.5526	1081	1748
"	.5516	1079	1744
"	.5531	1081	1750
"	.5526	1081	1748
"	.5526	1081	1748
"	.5516	1079	1744
145	.5271	1080	1746
"	.5281	1082	1750
132	.5321	1079.5	1745
"	.5306	1076.5	1738
"	.5311	1077.5	1741
"	.5321	1080	1746
145	.5245	1074.5	1735
			1746°
145	.5247	1075°	1736° C
"	.5271	1080	1746
"	.5257	1077	1740
132	.5338	1082.5	1751
"	.5323	1080	1746
"	.5334	1082	1750
"	.5311	1077.5	1741
			1744°

With Absorbing Mirrors $K=228$, $c_1=14500$, $\lambda=0.547\mu$ (Green).

145	0.5653	1156°	1747° C
132	.5748	1158	1750
"	.5746	1157.5	1749
168	.5646	1153.5	1742
171	.5948	1152.5	1740
"	.5945	1152	1739
145	.5641	1153.5	1742
			1744°

TABLE XIII.

Melting Point of Platinum in Iridium Furnace.

BY WIEN'S EQUATION.

With Sector Disk $K=35.4$, $c_2=14500$, $\lambda=0.668\mu$ (Red).

Lamp No	Current in Lamp Amperes	Observed M. P. Black- body Temperature	Melting Point
168	0.6189	1241°	1743° C
"	.6189	1241	1743
132	.6224	1240	1741
"	.6241	1242.5	1745
"	.6246	1243.5	1747
145	.6105	1242	1744
"	.6101	1241	1743
171	.6508	1242	1744
"	.6525	1245	1750
"	.6518	1243.5	1747
			1745°

With Sector Disk $K=35.4$, $c_2=14500$, $\lambda=0.547\mu$ (Green).

171	0.6983	1314°5	1745° C
"	.6948	1309	1739
145	.6463	1306.5	1736
"	.6478	1309.5	1740
132	.6643	1309	1739
"	.6658	1311.5	1741
			1740°

With Sector Disk $K=35.4$, $c_2=14500$, $\lambda=0.462\mu$ (Blue).

132	0.6968	1361°	1736° C
"	.7018	1369	1746
168	.7020	1364	1740
"	.7028	1366	1742
			1741°

Platinum.—Three series of observations of the platinum melting point were taken with the absorbing mirrors, Table XII, two with red and one with green light. Three series were taken with the sector disk, Table XIII, using respectively red, green, and blue light. In the first series with red light, using the mirrors, Table XII, the platinum was completely inclosed in small tubes of fused magnesia; in the second, the platinum was hammered out into a small disk and observed directly; in the third series, with green light, the platinum was protected with magnesia. The results in the table show that there was no appreciable contamination of the exposed platinum. Most of the observations with the sector disk, Table XIII, were taken with uncovered platinum. The results of the measurements of both platinum and palladium are discussed subsequently (pp. 202 et seq.).

SURFACE RADIATION METHOD.

In this method the black-body temperature of exposed strips of platinum or palladium at its melting point was observed by means of the optical pyrometer. The relation between the true temperature and the corresponding black-body temperature is assumed to be that given by extrapolation of the relation investigated experimentally to 1250° C. (p. 178). This method is not susceptible of the accuracy of the preceding one, and we regard the results only as corroborative evidence that the usual extrapolated thermoelectric relation gives too low temperatures.

Although the black-body temperature of a radiating surface may be measured with very considerable accuracy, it is difficult to pass from this to the true temperature, as it is necessary to know for the material under observation the relation of its radiation to black-body radiation. This is difficult to determine within the range of known temperatures (about 1200°), and is therefore attended with greater uncertainty when extrapolation must be resorted to. The surface changes due to very high temperatures, such as pitting, oxidation, and crystallization, might be expected to increase the "blackness" of the radiation (i. e., $t-s$ would increase less rapidly, due to this cause), and consequently this method would give a maximum value for the true temperature of the melting point.

It is evidently possible to determine experimentally the relation between t° and s° above 1250° by the use of the extrapolated thermoelectric scale, but this would then not be a method independent of the thermoelectric scale. In this way Holborn and Henning¹⁸ were led to 1729° and 1549° for the melting points of platinum and palladium, respectively.

In the present experiments the pyrometer was sighted on the hottest portion of a metallic strip, 4 mm wide, 0.02 mm thick, and about 8 cm long. The strip was slowly raised to the melting point by means of an electric current, and the reading of the pyrometer taken at the instant of melting.

Palladium.—In Table XIV are given the results of the experiments on the melting point of palladium in terms of the black-body temperatures for red, green, and blue light by extrapolation of the lamp calibration equations; and also for red light by extrapolation of the Wein equation, using the absorbing mirrors.

By linear extrapolation of the curve giving the relation between the black-body temperature and the true temperature (p. 178) the value of $t^\circ - s^\circ = 174$ for $s = 1387^\circ$, so that this method gives for the corresponding temperature of the melting point of palladium 1561° for the observations with red light. Similarly the observations with green light give $t = 1425 + 144 = 1569^\circ$. Experimental evidence indicates that the surface of palladium deteriorates more than that of platinum at high temperatures, so that the linear extrapolation gives too high values for $t - s$ for palladium.

Platinum.—In the experiments on platinum, Table XV, the same procedure was followed as for palladium. In the experiments based on Wien's equation the absorbing mirrors were used in two positions, with the line of sight normal to the radiating surface and at an angle of about 45° . As has already been shown (p. 174), the coefficient of absorption of the mirrors is different in these two cases owing to the polarization of the light. Two absorbing glasses from Pellin (p. 173) were also used by this method. The use of the lamp calibration equations gives a slightly higher result than the Wien equation, as was also found by the furnace method (p. 187).

¹⁸Holborn and Henning, *Berichte d. Berlin Akad.*, **12**, p. 311; 1905.

TABLE XIV.

Melting Point of Palladium with Palladium Ribbon.

BY WEIN'S EQUATION.

With Absorbing Mirrors $K=203$, $c_1=14500$, $\lambda=0.652\mu$ (Red).

Lamp No.	Number of Observations	Mean Current in Lamps	Observed Black-body Temperature	Melting Point, Black-body Temperature
140	6	0.4473	918° C.	1391° C.
168	6	.4353	916.5	1389
145	6	.4501	916	1388
				1389°

BY LAMP CALIBRATION.

Lamp No.	Number of Observations	Mean Current in Lamps	Wave Length λ	Melting Point, Black-body Temperature.
171	8	0.7456	0.662 μ	1383°
"	6	.7495	"	1388
168	6	.7167	"	1385
"	8	.7150	"	<i>i</i> 1383
132	12	.7132	"	1386
145	6	.6958	"	1393
"	6	.6925	"	1388
140	6	.7027	"	1390
5 lamps	20		"	<i>k</i> 1385
	20		"	<i>h</i> 1384
				1386°
145	6	.7128	.547	1422°
"	4	.7128	"	1422
132	4		"	1429
140	2		"	1428
168	4		"	1429
				1425°
132	14		.462	1450°
168				
140				

i Palladium melted on Platinum strip. *k*=Kahlbaum Pd; *h*=Heraeus Pd.

TABLE XV.

Melting Point of Platinum with Platinum Ribbon.

BY WEIN'S EQUATION.

Lamp. No.	Wave Length λ	Absorbing Screen	Abs. coef. K	Number of Observations.	Observed Black-body Temperature.	Melting Point, Black-body Temperature.
132	0.666 μ	Mirrors $\perp$	200	9	983°5 C	1537° C
140	"	"	"	6	979	1527
145	"	"	"	6	982	1534
"	"	Mirrors 45°	164	10	996	1533
140	"	"	"	8	996	1533
"	.650	2 Pellin glasses	101	8	1044	1537
145	"	"	"	6	1044.5	1538
						1534°
145	.547	Mirrors $\perp$	226	8	1070	1579
140	"	"	"	8	1071	1581
145	"	Mirrors 45°	188	4	1077	1568
						1578°

BY LAMP CALIBRATION.

Lamp No.	Wave Length λ	Number of Observations	Melting Point, Black-body Temperature
171	0.666 μ	8	1542° C
168	"	6	1548
132	"	6	1544
			1544°
132	.547	6	1592
168	"	6	1595
			1593°
168	.462	6	1632
171	"	6	1623
			1627°

By the same method used for palladium, we find for the temperature of the melting point of platinum, from the experiments with red and green, respectively:

$$t = 1539 + 209 = 1748^{\circ} \text{ (red).}$$

$$t = 1585 + 172 = 1757^{\circ} \text{ (green).}$$

The experiments with red are entitled to a much greater weight than those with green light.

THE THERMOELECTRIC METHOD.

Palladium.—The wire method, frequently used in the calibration of thermocouples at the melting points of silver and gold, was used in these determinations. A short length (3 to 6 mm) of palladium wire was fused between the platinum and the platinum-rhodium or platinum-iridium wires of the thermocouple. The couple was mounted with this junction near the center of an electric furnace. The heating current through the furnace was then increased, slowly raising the temperature, while readings of the E. M. F. of the couple were taken rapidly on the potentiometer until the palladium connecting link melted. Suitable commutator switches, designed to reduce stray thermal E. M. F.s to a minimum, permitted the reversal of the potentiometer current and the connections of the couple during the heating.

The furnaces used in this work were modified Heræus resistance furnaces, with porcelain tubes, about 20 mm internal diameter, wound over a length of about 25 cms with platinum ribbon about 20 mm wide and 0.007 mm thick.

Various methods of mounting the couple were tried. In some of the earlier experiments the couple was stretched through the furnace in a horizontal position, without, however, being in contact with its walls; in others, the wires of the couples were run through Berlin porcelain tubes (2 mm internal, 4 mm external diameter). On account of the unavoidable slight differences in temperature between the two ends of the short palladium wire, a vertical arrangement of the furnace,¹⁹ shown in Fig. 9, was used in the later experiments. This mounting also eliminated the possible effect of tension on the wire.

¹⁹ Although shown in a horizontal position in the illustration, the furnace was used in a vertical position.

At these high temperatures porcelain begins to soften and also becomes conducting, so that care must be taken to eliminate the effect of any leakage of the heating current to the thermocouple circuit. Some preliminary experiments, made in a small porcelain tube furnace 5 mm internal diameter with the wires of the couple resting on the porcelain, gave quite discordant results due to this cause. When the couple was mounted, as indicated in Fig. 9, a reversal of the furnace heating current showed entire absence of leakage. In some of the earlier experiments with other forms of mounting there was a slight leakage, which, however, was always determined by taking observations of the E. M. F. of the couple, a few degrees below the melting point, with the heating current alternately reversed. The correction for leakage never exceeded 3° ,

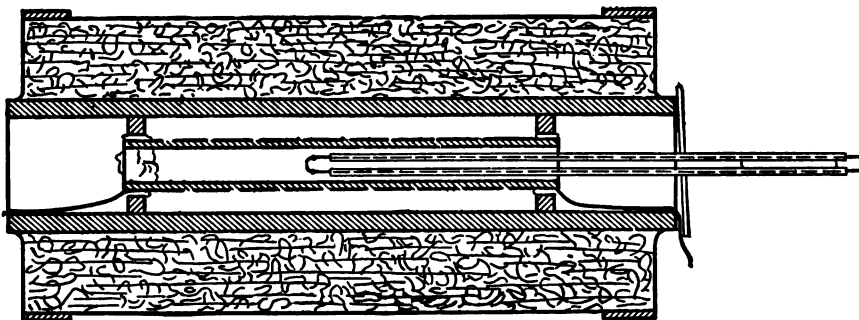


Fig. 9.—Furnace for Palladium Melting Point.

and the results obtained, when this correction is applied, agree with the later determinations.

The palladium used was in the form of wire and twisted strips, furnished as chemically pure by Heræus and by Kahlbaum. Wires of two sizes (0.6 mm and 0.3 mm) were used, which, together with the strips, allowed any possible effect of strain in interrupting the circuit near the melting point to be determined. With the mounting finally adopted no such effect was observed.

Calibration of Thermocouples.—Five thermocouples were used in this determination of the melting point of palladium, designated as P_1 , P_2 , S_1 , W_1 , and W_2 , respectively, the letters indicating the classification and use of the couples in the laboratory as primary, secondary, and working standards. The nature and source of the couples

are given in Table XVII. The five thermocouples form a part of the set of sixteen standard platinum couples of the Bureau, all of which have been frequently intercompared and which define the temperature scale of the Bureau in the range 500° to 1200° C.

The couples were calibrated at the freezing points of pure zinc (419°), antimony (630.5°), and copper (1084°) in a reducing atmosphere.²⁰ The metals were contained in graphite crucibles, 16.5 cms high and about 200 ccms capacity, thus insuring ample depth of immersion. The surface of the metal was covered with powdered graphite to prevent oxidation. The calibrations were carried out in both gas and electric furnaces, and at the beginning and after the conclusion of this work. For this purpose chemically pure zinc and copper from Kahlbaum, Eimer and Amend, and Baker and Adamson were used. The freezing points of these two metals from the different sources agreed to within 0.2. The antimony used was "Kahlbaum" antimony.

The couples were further checked during their use by determinations of the melting points of gold and silver by the wire method. Thirty-two such determinations on gold, the average value of which was 1064.0 and five on silver which gave 953.4 , showed that the couples remained constant to within two or three tenths of a degree, notwithstanding that small lengths of the wire at the junction were frequently cut off. This indicates that the wires were homogenous and that the exposures to over 1500° in air produced no measurable changes.

Before calibration the wires of the couples were explored for homogeneity by passing them step by step through a short electric furnace made of small bore porcelain tubing, the wires being in circuit with a sensitive D'Arsonval galvanometer. The wires were thoroughly annealed electrically at white heat previous to calibration.²¹ In all measurements with thermocouples the cold junctions were kept at 0° C.

Two of the couples used in this investigation were recently calibrated at the Physikalisch-Technische Reichsanstalt and another about four years ago. These couples were also compared in an

²⁰ Holborn and Day, *Am. Jl. Sci.*, (4), 10, p. 171; 1900.

²¹ The results obtained at the Bureau during the past three years on the treatment and homogeneity of wires for thermoelectric use will be published later.

electric furnace with a couple calibrated at the National Physical Laboratory of England. These calibrations agree with our own to within 2° at 1500° .

Potentiometer.—The measurements of the electromotive force of the thermocouples were made with a Leeds and Northrup type K potentiometer, modified by the addition of two shunts, so that the range was extended from 1.6 to 0.000001 volt. As one division of the bridge wire, which consists of 10 turns of manganin wire wound on a grooved marble cylinder about 15 cm diameter, corresponds to a length of about 2.75 mms on the divided circular scale, estimates to within one-tenth of a microvolt can easily be made. A twenty-step dial enabled the working current (0.02000 amp) to be adjusted to any value of the Weston (unsaturated) cadmium cell (1.0180 to 1.0199 volts). The potentiometer was calibrated by the electrical division of the Bureau immediately after the potentiometer was received from the makers and again during the progress of this investigation.

All measurements were referred to two Weston cadmium cells, one used as a working standard, the other as an occasional reference standard, these cells being frequently compared with the standard Clark cells of this Bureau.

Reduction of observations.—As an illustration of the method of reduction of the observations we may take the determinations in the vertical furnace with couple W_3 . The calibration of W_3 is given by the following data:

Freezing point of	Zn	Sb	Cu
Temperature	419°0	630°5	1084°0
E. M. F. (microvolts)	5966	9502	17702

This leads to the equation:

$$(a) \quad E = -498.3 + 14.569 t + 0.0020490 t^2$$

where E is expressed in microvolts and t in degrees centigrade, if the cold junction is at 0° C.

The observations at the melting point of palladium were as follows:

<i>E. M. F.</i> (microvolts)	<i>Deviation.</i>
26602	—15
611	—28
575	+08
580	+03
540	+43
590	—07
Mean=26583	17
Potentiometer correction= +8	
26591	

By substitution in the above equation, the melting point of palladium = $1530^{\circ}1$, as given by W_3 .

Results.—The results of the thermoelectric determination of the melting point of palladium are given in Table XVI.

TABLE XVI.

Melting Point of Palladium—Wire Method.

Thermocouple	Number of Observations	Melting Point of Palladium	Remarks
W_1	5	1531.0	Horizontal furnace; bare wires.
S_1	6	1530.5	Horizontal furnace; porcelain tubes.
W_3	4	1530.0	" "
P_1	5	1529.5	" "
P_1	2	1530.0	Vertical furnace; see Fig. 9.
S_1	2	1530.5	" "
P_1	2	1530.0	" "
W_1	3	1530.5	" "
W_3	6	1530.1	" "
Mean = 1530.2			

The average deviation of a single observation of the melting point from the mean is $1^{\circ}2$ in the thirty-five determinations here summarized, and is considerably less for the later determinations, i. e., when the mounting of Fig. 9 was used. Some months later, additional

determinations were carried out in the iridium furnace (p. 184), which was heated by an alternating current and in which the couples were insulated with fused magnesia tubes. These later measurements, which are entitled to less weight, gave 1528°. Observations with the three samples of palladium used can not be distinguished from one another.

Platinum.—Each of the optical determinations of the melting point of platinum carried out in the iridium furnace was accompanied by a thermoelectric determination. For the couples in which one wire was pure platinum, the E. M. F. was observed at the instant the platinum wire melted. For couples W_4 and W_9 , which were each made of a 10 per cent and of a 20 per cent alloy of rhodium with platinum, the wire method already described (p. 194) was used. For the experiments with the iridium, iridium-ruthenium couples an indicator circuit, consisting of two platinum-rhodium wires joined by a short connecting link of pure platinum placed very near the junction of the thermocouple, served to define the instant of melting of the platinum by the interruption of an electric circuit. The results are summarized in Table XVII. From two to five melts were made with each couple.

The equations in the table are those given by a calibration made after the melting point determinations. Some of the couples were also calibrated before the determinations and the equation (a) computed (see p. 201). The melting points given in the table by equation (a) are obtained by assigning greater weight (3:1) to the final calibration.

The two series of calibrations gave the following differences in E. M. F. at 1000°:

Couple	Platinum-Rhodium			Platinum-Iridium	
	P_2	P_3	S_1	W_6	W_3
$E_{\text{new}} - E_{\text{old}}$ Microvolts	-7	-5	-46	+3	+53

The effect of exposure to very high temperature is in general to produce a decrease in E. M. F. for a given temperature, due to distillation of the rhodium and iridium. The opposite effect in the iridium couples (W_4 and W_9), as well as the large effect in S_1 , is

probably due to the inhomogeneity of the wires. After each melt a short piece of wire was cut off, so that a different junction was calibrated.

TABLE XVII.

Melting Points of Palladium and Platinum by Thermoelectric Method.

Couple	Calibration Points	Equation	Melting Point of Palladium	Melting Point of Platinum
P ₁ Heraeus.....	Zn, Sb, Cu.....	$E = -266 + 7.991t + 0.001747t^2$	1521°	1701°
Pt, 90 Pt-10 Rh.....	Zn, —, Cu.....	$E = 2.658t^{1.184}$	1548	1745
	—, Sb, Cu.....	$E = 2.464t^{1.186}$	1542	1738
P ₂ Heraeus.....	Zn, Sb, Cu.....	$E = -299 + 8.088t + 0.001699t^2$	1537	1715
Pt, 90 Pt-10 Rh.....	Zn, —, Cu.....	$E = 2.618t^{1.187}$	1560	1753
	—, Sb, Cu.....	$E = 2.460t^{1.186}$	1556	1748
S ₁ Heraeus.....	Zn, Sb, Cu.....	$E = -272 + 8.085t + 0.001620t^2$	1532	1708
Pt, 90 Pt-10 Rh.....	Zn, —, Cu.....	$E = 2.807t^{1.176}$	1561	1754
	—, Sb, Cu.....	$E = 2.634t^{1.185}$	1558	1747
P ₄ Johnson and Matthey.....	Zn, Sb, Cu.....	$E = -490 + 9.868t + 0.001617t^2$	1524	1698
Pt, 90 Pt-10 Rh.....	Zn, —, Cu.....	$E = 3.115t^{1.188}$	1536	1717
	—, Sb, Cu.....	$E = 3.115t^{1.188}$	1536	1717
W ₆ Heraeus.....	Zn, Sb, Cu.....	$E = -565 + 16.207t + 0.0001301t^2$	1525	1710
Pt, 90 Pt-10 Ir.....	Zn, —, Cu.....	$E = 10.210t^{1.088}$	1516	1697
	—, Sb, Cu.....	$E = 10.967t^{1.088}$	1522	1704
W ₃ Carpentier.....	Zn, Sb, Cu.....	$E = -538 + 14.80t + 0.001911t^2$	1528	1705
Pt, 90 Pt-10 Ir.....	Zn, —, Cu.....	$E = 6.095t^{1.141}$	1541	1728
	—, Sb, Cu.....	$E = 6.006t^{1.148}$	1541	1728
W ₈ Heraeus.....	Zn, Sb, Cu.....	$E = -170 + 0.2125t + 0.002031t^2$	1507	1687
90 Pt-10 Rh, 80 Pt-20 Rh	800°, 1000°, 1200°...	$E = -800 + 1.525t + 0.001375t^2$	1531	1734
	800°, —, 1200°...	$E = 0.001266t^{2.071}$	1497	1671
W ₉ Heraeus.....	Zn, Sb, Cu.....	$E = -182 + 0.2850t + 0.001925t^2$	1507	1710
90 Pt-10 Rh, 80 Pt-20 Rh	800°, 1000°, 1200°...	$E = -758 + 1.485t + 0.001325t^2$	1530	1755
	800°, —, 1200°...	$E = 0.001448t^{2.048}$	1498	1693
R ₁ Heraeus.....	600°, 900°, 1200°...	$E = +308 + 3.312t - 0.000050t^2$	1551	1738
Ir, 90 Ir-10 Ru.....	600°, —, 1200°...	$E = 7.836t^{0.8887}$	1565	1757
	600°, —, 1200°...	$E = +344 + 3.222t$	1547	1726
R ₂ Heraeus.....	600°, 900°, 1200°...	$E = -223 + 3.486t - 0.000072t^2$	1533	(1704)?
Ir, 90 Ir-10 Ru.....	600°, —, 1200°...	$E = 1.997t^{1.086}$	1517	(1676)?
	600°, —, 1200°...	$E = -172 + 3.358t$	1526	(1690)?

The two formulæ most frequently used to express the relation between temperature and electromotive force for the usual platinum-rhodium, and platinum-iridium thermocouples, are

$$(a) \quad E = a + bt + ct^2 \quad (\text{Avenarius})$$

$$(b) \quad E = mt^n \quad (\text{Holman})$$

where t is the temperature of the hot junction, and the cold junctions are at 0°

The melting points of platinum and palladium have been computed (Table XVII) by both types of equation, and for the iridium, iridium-ruthenium couples a linear extrapolation is also given.

It will at once be seen that widely different results are obtained, depending on the type of equation and the calibration data used, although within the range 400° to 1100° these two equations differ nowhere by more than about 2° for the usual 10 per cent rhodium or iridium couples. For the couples W_8 and W_9 the curvature is so great that the values obtained by equations (a) and (b) differ greatly even within the calibrated interval.

The mean values for the melting points of palladium and platinum, as given by the 10 per cent rhodium and iridium couples P_1 , P_2 , P_3 , P_4 , S_1 , S_2 , W_1 , W_6 , W_8 , are as follows:

Metal	Equation (a) Zn, Sb, Cu	Equation (b)	
		Zn, Cu	Sb, Cu
Palladium	1530°	1544°	1542°
Platinum	1706°	1732°	1730°

The values obtained for the melting points of palladium and platinum by the application of equation (a) to the thermocouple observations are considerably lower than the values obtained by the optical observations and Wien's equation. For the 10 per cent rhodium and iridium couples, equation (b) gives values which are much closer to the optical values. If by the "true temperature" we mean the temperature by the thermodynamic or ideal gas scale, Wien's equation, with the constant c_2 determined from observations within the range of practical gas thermometry, gives at present the closest convenient representation of the true temperature.

V. GENERAL DISCUSSION AND CONCLUSIONS.

The precautions necessary in the construction and use of an experimental black-body to give ideal black-body radiation have been described at length. It was possible to construct a black-body in which the temperature distribution at 1300° was constant to 1° throughout the greater part of the radiating inclosure.

Five pyrometer lamps were calibrated by comparison with thermocouples up to 1300°, using this uniformly heated black-body as radiating source. The relation between the black-body temperature and the current through the lamps is represented by an equation of the form $I = a + bt + ct^2$, with an average deviation of less than 1°.5.

In order to extend the optical scale to higher temperatures, the absorption coefficients of absorbing mirrors and glasses were determined for red and green light. The effect of lack of monochromatism of so-called monochromatic glasses on temperature measurements was investigated, as well as the effect of the polarization of light emitted by palladium and platinum on the absorption coefficient of the mirrors.

The relation of the radiation of palladium and platinum to that of a black-body was investigated, experimentally, for red and green light, up to 1250°. Three methods of estimating the actual temperature of the radiating surface were used.

Table XVIII gives the results obtained where

t = true temperature of the radiating surface, and

s = corresponding black-body temperature.

TABLE XVIII.**Radiation from Platinum and Palladium.**

s	700°	800°	900°	1000°	1100°	1200°	1300°	1400°	1500°	
PLATINUM.										
$\begin{matrix} t-s \\ (\lambda=0.66 \mu) \end{matrix}$	}	46	58	72	90	110	132	155	(178)	(200)
$\begin{matrix} t-s \\ (\lambda=0.55 \mu) \end{matrix}$				53	68	86	103	121	(139)	(157)
PALLADIUM.										
$\begin{matrix} t-s \\ (\lambda=0.66 \mu) \end{matrix}$	}	46	58	72	90	110	129	(143)		

The melting points of palladium and platinum were measured by three methods:

(1) The *black-body method* in which the temperature of melting within an electrically heated iridium furnace was measured with an optical pyrometer.

(2) The *surface-radiation method* in which the black-body temperature of electrically heated ribbons of these metals was measured at the instant of melting.

(3) The *thermoelectric method* based on extrapolation by several formulæ. Ten thermocouples of the following types were used: Pt, 90 Pt-10 Rh; Pt, 90 Pt-10 Ir; 90 Pt-10 Rh, 80 Pt-20 Rh; Ir, 90 Ir-10 Ru.

The results of the experiments by the black-body method are summarized in Table XIX, where

$t_1 = T_1 - 273^\circ$ = observed black-body temperature,

K = absorption coefficient of the screen used,

λ = wave length of light,

c_2 = constant of the equation, $\log K = \frac{c_2}{\lambda} \log e \left(\frac{1}{T_1} - \frac{1}{T_2} \right)$

$t_2 = T_2 - 273^\circ$ = calculated temperature $^\circ\text{C.}$ of melting point.

The highest temperatures measured with the sector disk were 1242° in the experiments with red, and 1365° in experiments with blue light. At these temperatures, the scale defined by the pyrometer lamps, which is here practically in agreement with the extrapolated thermoelectric scale, is certainly not much in error; while the temperatures measured by the absorbing mirrors, 1080° for the red and 1155° for the green, are within the range of known temperatures; and the results obtained by the use of mirrors are in most satisfactory agreement with those obtained with the sector disk.

The surface radiation method, which is not capable of the same order of accuracy as the preceding, and which is regarded only as corroborative, gives for the melting point of platinum 1750° , and for palladium 1563° on the basis of a linear extrapolation of the relation between s° and $t^\circ - s^\circ$. On account of the changes in the surface of palladium at high temperatures, this method gives a

higher value for that metal than the black-body method, as is to be expected.

TABLE XIX.
Melting Points by Black-body Method.

PLATINUM.

Group	t_1 obs.	t_2 comp.	K	λ	c_2	No. of obs.	Remarks	Melt- ing Point
a	1242°	1745°	35.5	0.668	14500	10	Sector disk and Wein's equation	1745°
b	1310	1740	"	.547	"	6	" "	
c	1365	1741	"	.462	"	4	" "	
d	1080	1746	199	.668	"	16	Absorbing mirrors and Wien's equation	
e	1155	1744	228	.547	"	7	" "	

PALLADIUM.

a'	985°	1540°	200	0.666	14500	8	Absorbing mirrors and Wein's equation	1540°
b'	984.5	1539	"	"	"	7	" "	
c'	1124.5	1540	35.5	"	"	11	Sector disk and Wein's equation	
d'	1547	-----	1	"	-----	7	Lamp equation extrapolation	

The temperatures estimated by thermoelectric extrapolation differ very considerably depending upon the formula and calibration data used. The two types of formula

$$(a) \quad E = a + bt + ct^2$$

$$(b) \quad E = mT^n$$

which agree to within 2° up to 1200° for the usual 10 per cent platinum-rhodium and platinum-iridium thermocouples, were compared at the melting points of palladium and platinum, for the four types of couple mentioned above. The usual 10 per cent platinum-rhodium and platinum-iridium thermocouples give, by formula (a), 1530° and 1706°, and by formula (b), 1543° and 1731°, for the melting points of palladium and platinum, respectively. The other types

of thermocouples give results differing 25° or more from these values. Harker²² using formula (a) and several types of thermocouples found 1710° for platinum. Holborn and Henning²³ have published the values 1535° and 1710° for the melting points of palladium and platinum, presumably by this same method. The temperature scale obtained by extrapolation based on the use of formula (a) and the usual 10 per cent platinum-rhodium and platinum-iridium thermocouples, differs from the optical scale resulting from this investigation by the amounts given in Table XX.

TABLE XX.
Thermoelectric and Optical Scales.

Temperatures on Thermoelectric Scale	1200°	1300°	1400°	1500°	1600°	1700°
Optical-Thermoelectric.....	0	2	6	14	25	43

Nernst and Wartenberg²⁴ using the black-body method and a Wanner optical pyrometer adjusted for yellow light, found for palladium 1541° and for platinum 1745° , where the constant of the Wien equation was taken as $c_2 = 14600$, and the melting point of gold as 1064° . On the basis of $c_2 = 14500$, as used in the present investigation, Nernst and Wartenberg's values become 1545° for palladium and 1750° for platinum.

Since the completion of the experiments described in this paper, Holborn and Valentiner,²⁵ in an elaborate intercomparison of the optical temperature scale with the scale of the gas thermometer to 1600° , have found for the constant of the Wien equation, $c_2 = 14200$, and for the melting points of palladium and platinum, 1582° and 1789° , respectively.

The generally accepted value, $c_2 = 14500$, used in the present investigation, is that based on the experiments of Lummer and Pringsheim, of Paschen, and of Wanner. It becomes of interest to discuss the effect on the results of the present investigation of changing the value of c_2 . We have therefore in Table XXI given

²²Harker, Chemical News, 91, p. 262; 1905.

²³Holborn and Henning, Sitzber. Berlin Akad., 12, p. 331; 1905.

²⁴Nernst and Wartenberg, Verh. Deutsch Phys. Ges., 8, p. 48; 1906.

²⁵Holborn and Valentiner, Ann. der Physik., 22, p. 1; 1907.

these results recomputed on the basis of $c_2=14600$ and $c_2=14200$ and also the effect of expressing the observed temperatures (t_1) on the Holborn-Valentiner gas scale. The key to the table is the same as for Table XIX. The letters a, b, c, etc., under column headed "Group" refer to observations in the corresponding line of Table XIX.

TABLE XXI.

Melting Points of Platinum and Palladium Reduced by Several Methods.

PLATINUM.

Group	t_1 obs.	t_2 comp.	K	γ	c_2	Remarks	Melting points (comp.)	
a	1242°	1740°	35.5	0.663	14600	{ Groups b-e give same Melting Point as Group a. Value of c_2 used by Nernst and Warten- berg.....	1740°	I
a	1242	1759	"	.668	14200	{ Value of c_2 found by Hol- born and Valentiner....	1756	II
b	1310	1750	"	.547	"	" " "		
c	1365	1750	"	.462	"	" " "		
a	1246	1766	"	.668	"	{ t_1 expressed on H.-V. gas scale	1766	III
b	1319	1765	"	.547	"	" " "		
c	1377	1768	"	.462	"	" " "		
d	1080	1760	192	.668	"	{ K computed on H.-V. gas scale	1758	IV
e	1155	1757	219	.547	"	" " "		
d	1080	1748	179	.668	"	{ K computed for $c_2=14200$. " " "	1746	V
e	1155	1743	201	.547	"	" " "		

PALLADIUM.

c ¹	1124°5	1536°	35.5	0.666	14600	{ Value of c_2 used by Nernst and Wartenberg.....	1536°	I ¹
c ¹	1124.5	1551	"	"	14200	{ Value of c_2 found by Hol- born and Valentiner....	1551	II ¹
a ¹	985	1551	192	"	"	{ K computed on H.-V. gas scale	1551	IV ¹
a ¹	985	1540	171	"	"	{ K computed for $c_2=14200$. " " "	1540	V ¹

From Table XXI (*I* and *I'*) it will be seen that the observations when reduced for $c_s = 14600$, would give 1740° for platinum and 1536° for palladium. The results of the sector-disk experiments (*II* and *II'*), when reduced for $c_s = 14200$, give 1756° and 1551° . The same observations when the observed temperature t_1 is expressed on the Holborn-Valentiner gas scale give (*III*) for platinum, 1766° . The value for palladium is unchanged because the observed value of t_1 ($= 1124.5$) is within the range of the Holborn-Day gas scale with which the Holborn-Valentiner scale was made to coincide at this temperature.

With the absorbing mirrors, assuming $c_s = 14200$ and computing K (experiments on p. 172) on the Holborn-Valentiner gas scale (*IV* and *IV'*), the platinum point would be 1758° and the palladium point 1551° . If, in the computation of K , temperatures are expressed on the scale defined by the pyrometer lamps and c_s is taken as 14200 (*V* and *V'*), the melting points would be 1746° and 1540° , respectively.

Using $c_s = 14200$ and expressing the observed temperatures t_1 on the Holborn-Valentiner scale, the mean of the experiments with the mirrors and with the sector disk gives 1762° for platinum and 1551° for palladium. These seem to be the maximum temperatures that can be assigned from our experiments. Furthermore, when this method of reduction is used, the observations with the sector disk and with the absorbing mirrors are not in agreement.

As a further illustration of the effects on our results of reduction by the two methods, namely, on the one hand, using $c_s = 14500$ and the temperature scale given by the pyrometer lamps, and, on the other hand, using $c_s = 14200$ and expressing temperatures on the Holborn-Valentiner scale, we have computed the absorption factor of the sector disk by both methods, as shown in Table XXII.

The observed value of K is that obtained by measurements with a circular dividing engine.

The difference in K , as computed by the two methods, can not be accounted for by differences in t_1 , as at 1250° for example the two temperature scales differ by less than 5° , but is due to the different values assumed as the melting points of platinum and palladium.

TABLE XXII.

Computation of Absorption Factor of Sector Disk.

Group	t_1 obs	t_2 as- sumed	λ	c_2	Remarks	K com- puted	K ob- served
a	1243°	1745°	.668	14500	Pyrometer lamp scale, Wien equation	35.6	35.5
b	1310	1745	.547	14500	Pyrometer lamp scale, Wien equation	36.9	
c ¹	1124.5	1540	.666	14500	Pyrometer lamp scale, Wien equation	35.5	

a	1246	1789	.668	14200	Holborn-Valentiner gas scale.	38.8	
b	1319	1789	.547	14200	“ “	41.1	
c ¹	1124.5	1582	.666	14200	“ “	43.1	

		t_2 obs					
p —	950.5	1253	.660	14500	Pyrometer lamp scale, Wien equation	35.2	
“	950.5	1257	.660	14200	Holborn-Valentiner scale, Wien equation	35.0	

In view of the above discussion, the most probable values of the melting points of palladium and platinum, resulting from the experiments of this investigation, are 1540° and 1745°, respectively, on the basis of Wien's equation. A further correction has to be applied for the lack of blackness of the iridium furnace (p. 185), so that our final values are 1546° for the melting point of palladium and 1753° for the melting point of platinum.

The wide fluctuations in the values of these melting points as obtained by various observers within the past two years by the use of different methods, makes further experimental work necessary. On account of the more satisfactory theoretical basis of the Stefan-Boltzmann law, which gives temperatures on the thermodynamic scale, it is planned to attack the problem from this standpoint.

WASHINGTON, March 4, 1907.

THE MUTUAL INDUCTANCE OF A CIRCLE AND A COAXIAL SINGLE LAYER COIL—THE LORENZ APPARATUS AND THE AYRTON-JONES ABSOLUTE ELECTRODYNAMOMETER.

By Edward B. Rosa.

In the Lorenz experiment for determining resistance in absolute measure a circular disk is rotated in a magnetic field produced by an electric current in coaxial coils or in a coaxial helix, and the electromotive force induced in the disk is balanced against the difference of potential at the terminals of the resistance to be measured. This induced electromotive force is proportional to the speed and to the number of lines of magnetic force passing through the disk due to the current in the surrounding coils or helix. The latter usually consists of a single layer winding on a carefully ground cylindrical surface or in a very true screw thread cut in such a surface. In order to know the number of lines of magnetic force passing through the disk it is necessary to calculate the mutual inductance of the cylindrical winding and the circle forming the boundary of the disk.

1. LORENZ'S FORMULA.

This problem of finding the mutual inductance of a circle and a coaxial single layer winding was first solved by Lorenz.¹ Assuming that the current was uniformly distributed over the surface of the cylinder forming a current sheet he integrated over the length of the cylinder the expression for the mutual inductance of a circular element and the given circle. This expression is an elliptic integral. Lorenz reduced the integrated form to a series and gave the following formula for the mutual inductance of the disk and solenoid of what is now called the Lorenz apparatus. He called it, however,

¹ Wied. Annalen, 25, p. 1; 1885.

the constant of the apparatus instead of mutual inductance and denoted it by C ; it is of course the whole number of lines of magnetic force passing through the disk due to unit current in the surrounding solenoid.

$$C = \frac{\pi q r^2}{d} [Q(a_1) + Q(a_2)] \quad (1)$$

$$Q(a) = 2\pi q \sqrt{\frac{a-1}{a}} \left[1 + \frac{3}{8} \frac{q^2}{a^2} + \frac{5}{16} \frac{q^4}{a^4} \left(\frac{7}{4} - a \right) + \frac{35}{128} \frac{q^6}{a^6} \left(\frac{33}{8} - \frac{9}{2} a + a^2 \right) + \dots \right] \quad (2)$$

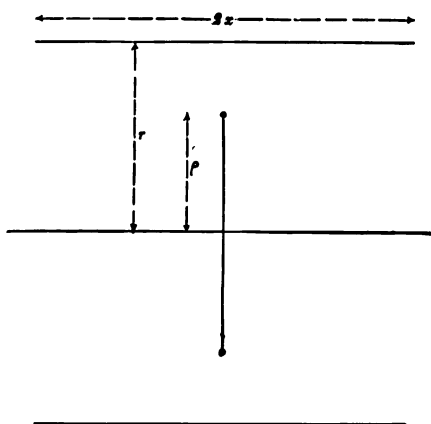


Fig. 1.

ρ = radius of the disk, Fig. 1.
 r = radius of the solenoid.
 $2x$ = length of winding of solenoid.
 $q = \rho/r$ = ratio of the two radii.
 $d = \frac{2x}{n}$ = distance between centers of successive turns of wire.
 $a = \frac{x^2 + r^2}{r^2}$

If the disk be not exactly in the mean plane of the solenoid, and x_1 be the distance from the plane of the disk to one end

of the solenoid and x_2 to the other,

$$a_1 = \frac{x_1^2 + r^2}{r^2} \quad a_2 = \frac{x_2^2 + r^2}{r^2}$$

Then $Q(a_1)$ is found by substituting the values of a_1 in equation (2) above, and $Q(a_2)$ by using the value of a_2 for a in the same equation.

The sum of these two quantities multiplied by $\frac{\pi q r^2}{d}$ gives the constant C , that is, the mutual inductance sought.

As Lorenz gave the expression for the general term of (2), his equation can be extended. The following is the general term:

$$Q(a) = 2\pi \sum_{m=0}^{\infty} q^{2m+1} \frac{1 \cdot 3 \dots 2m-1}{2 \cdot 4 \dots 2m} \cdot \frac{1}{1 \cdot 2 \dots (m+1)} \cdot \frac{d^m}{da^m} \left(\frac{a-1}{a} \right)^{m+1} (2a)$$

2. JONES'S FORMULÆ.

Two solutions were given by Jones,² both in terms of elliptic integrals. The current was considered to flow not in a current sheet, but along a spiral winding or helix. The first solution was in the form of a series, convergent only when AO_1 , Fig. 2, is less than the difference in the radii of inner and outer coils; that is, when AO_1 is less than $A-a$. As this is a serious limitation, and the formula is a laborious one to use, I shall not give it. The second solution is exact and general, and is in terms of elliptic integrals of all three kinds. The second formula is as follows:

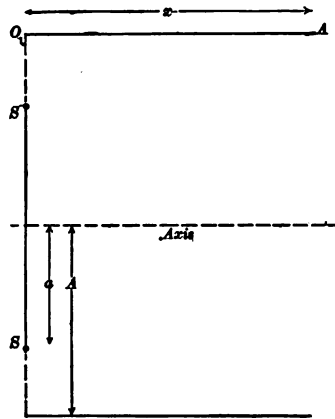


Fig. 2.

$$M_\theta = \Theta (A+a) ck \left\{ \frac{F-E}{k^2} + \frac{c'^2}{c^2} (F-\Pi) \right\} \quad (3)$$

M_θ = mutual inductance of helix O_1A , Fig. 2, on the disk S in the plane of one end.

$\Theta = 2\pi n$, $1/n$ = pitch of winding, Θ = whole angle of winding.

F and K are complete elliptic integrals to modulus k , where

$$k^2 = \frac{4Aa}{(A+a)^2 + x^2} \quad c^2 = \frac{4Aa}{(A+a)^2} \quad c'^2 = 1 - c^2.$$

Π = complete elliptic integral of the third kind, to modulus k .

The elliptic integral Π of the third kind can be expressed in terms of incomplete integrals of the first and second kinds, and the value of M_θ can then be calculated by the help of Legendre's tables. The calculation is, however, extremely tedious, especially when the value is to be determined with high precision.

² Proc. Roy. Soc., 68, p. 198; 1898.

If we find the part of this due to the end AP (shown dotted and extending to infinity) and subtract it, the remainder is the part due to O,A, which is the quantity sought.

If OA, Fig. 4, is a disk of radius a and density n its potential V at P is

$$V_P = 2\pi n \left\{ (a^2 + r^2)^{\frac{1}{2}} - r \right\} = 2\pi n \left\{ \frac{a^2}{2r} - \frac{1.1}{2.4} \frac{a^4}{r^3} + \frac{1.1.3}{2.4.6} \frac{a^6}{r^5} - \frac{1.1.3.5}{2.4.6.8} \frac{a^8}{r^7} + \dots \right\} \quad (4)$$

For a point Q off the axis we must insert in (4) the zonal harmonics corresponding to the angle θ . Thus

$$V_Q = 2\pi n \left\{ \frac{a^2}{2r} P_0 - \frac{1}{8} \frac{a^4}{r^3} P_2 + \frac{1}{16} \frac{a^6}{r^5} P_4 - \frac{5}{128} \frac{a^8}{r^7} P_6 + \frac{35}{1280} \frac{a^{10}}{r^9} P_8 - \dots \right\} \quad (5)$$

V_Q will be the magnetic potential at any point Q if the disk is covered with a layer of positive magnetism of density n , or if it is the end of an infinite solenoid wound with n turns of wire per centimeter and carrying unit current. The expression is convergent when $r > a$. Differentiating V_Q with respect to r we have the magnetic force at Q in the direction of the vector r . Thus,

$$-\frac{dV_Q}{dr} = 2\pi n \left\{ \frac{a^2}{2r^2} P_0 - \frac{3}{8} \frac{a^4}{r^4} P_2 + \frac{5}{16} \frac{a^6}{r^6} P_4 - \frac{35}{128} \frac{a^8}{r^8} P_6 + \frac{315}{1280} \frac{a^{10}}{r^{10}} P_8 - \dots \right\} \quad (6)$$

The values of the zonal harmonics are as follows ($\mu = \cos \theta$):

$$P_0 = 1$$

$$P_2 = \frac{1}{2}(3\mu^2 - 1)$$

$$P_4 = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$$

$$P_6 = \frac{1}{16}(231\mu^6 - 315\mu^4 + 105\mu^2 - 5)$$

$$P_8 = \frac{1}{128}(6435\mu^8 - 12012\mu^6 + 6930\mu^4 - 1260\mu^2 + 35)$$

$$P_{10} = \frac{1}{256}(46189\mu^{10} - 109395\mu^8 + 90090\mu^6 - 30030\mu^4 + 3465\mu^2 - 63)$$

$$P_{12} = \frac{1}{1024}(52003 \times 13\mu^{12} - 176358 \times 11\mu^{10} + 230945 \times 9\mu^8 - 145860 \times 7\mu^6 + 45045 \times 5\mu^4 - 6006 \times 3\mu^2 + 231)$$

Substituting the above values of the zonal harmonics in (6) would give the value of the magnetic force at any point Q in terms of a , r , and μ .

Let R , Fig. 5, be a circle coaxial with S and lying on the spherical surface through Q with center O . A zone at Q on the spherical surface of angular width $d\theta$ and radius $r \sin \theta$ has an

area $2\pi r^2 \sin \theta d\theta$ and through it pass dN lines of magnetic force.

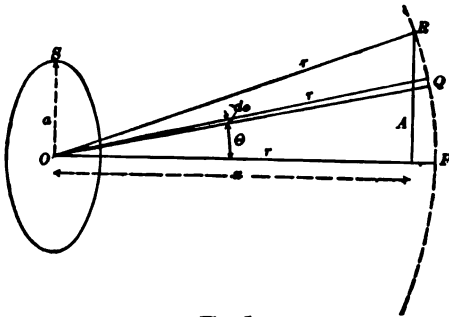


Fig. 5.

$$dN = 2\pi r^2 \sin \theta d\theta \left(\frac{dV_Q}{dr} \right)$$

$$\text{Since } \mu = \cos \theta, \\ d\mu = -\sin \theta d\theta$$

$$\therefore dN = 2\pi r^2 d\mu \left(-\frac{dV_Q}{dr} \right)$$

$$\text{Or, } dN = 4\pi^2 n d\mu \left\{ \frac{a^2}{2} - \frac{3}{16} \frac{a^4}{r^2} (3\mu^2 - 1) + \frac{5}{128} \frac{a^6}{r^4} (35\mu^4 - 30\mu^2 + 3) \right. \\ \left. - \frac{35}{2048} \frac{a^8}{r^6} (231\mu^6 - 315\mu^4 + 105\mu^2 - 5) + \dots \right\} \quad (7)$$

The whole number of lines of magnetic force N passing through the circle R of radius A and distance x from the magnetic disk S ($x = \sqrt{r^2 - A^2}$) is found by integrating (7) with respect to μ from

$$\mu = \frac{x}{r} = \frac{x}{\sqrt{A^2 + x^2}} \text{ to } \mu = 1$$

Thus,

$$N = 2\pi^2 a^2 n \int_{\mu=\frac{x}{r}}^{\mu=1} \left\{ 1 - \frac{3}{8} \frac{a^2}{r^2} (3\mu^2 - 1) + \frac{5}{64} \frac{a^4}{r^4} (35\mu^4 - 30\mu^2 + 3) \right. \\ \left. - \frac{35}{1024} \frac{a^6}{r^6} (231\mu^6 - 315\mu^4 + 105\mu^2 - 5) + \dots \right\} d\mu \quad (8)$$

or

$$N = 2\pi^2 a^2 n \left[\left(1 - \frac{x}{(x^2 + A^2)^{1/2}} \right) + \frac{3}{8} \frac{a^2}{(x^2 + A^2)^{3/2}} \left(\frac{x^3}{(x^2 + A^2)^{1/2}} - \frac{x}{(x^2 + A^2)^{3/2}} \right) \right. \\ \left. - \frac{5}{64} \frac{a^4}{(x^2 + A^2)^{5/2}} \left(\frac{7x^5}{(x^2 + A^2)^{1/2}} - \frac{10x^3}{(x^2 + A^2)^{3/2}} + \frac{3x}{(x^2 + A^2)^{5/2}} \right) \right. \\ \left. + \frac{35}{128} \frac{a^6}{(x^2 + A^2)^{7/2}} \left(\quad \right) + \dots \right] \quad (9)$$

N is the number of lines of magnetic force due to the disk S (of radius a) passing through the coaxial circle R (of radius A) at distance x , Fig. 5. It is also the number of lines due to a larger disk of radius A , Fig. 6, passing through a circle of radius a at a

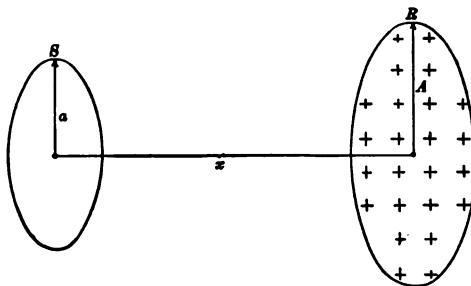


Fig. 6.

distance x ; or the number of lines due to the semi-infinite solenoid AP , Fig. 3, wound with n turns per cm and carrying unit current, passing through the circle S . N is therefore the mutual inductance M_{AP} of the semi-infinite end AP , Fig. 3, which we are to subtract from $\frac{1}{2}M_{\infty}$ to give M the mutual inductance of the solenoid O_1A with respect to the circle S . Putting $\sqrt{x^2 + A^2} = d$, equation (9) above reduces to

$$M_{AP} = 2\pi^2 a^2 n \left[1 - \frac{x}{d} - \frac{3}{8} \frac{a^2}{d^3} \frac{xA^2}{d^2} - \frac{5}{64} \frac{a^4}{d^5} \left(\frac{3xA^4 - 4x^3A^2}{d^2} \right) - \dots \right] \quad (10)$$

$$\therefore M_{O_1A} = \frac{1}{2}M_{\infty} - M_{AP} = \frac{2\pi^2 a^2 n x}{d} \left[1 + \frac{3}{8} \frac{a^2 A^2}{d^2} + \frac{5}{64} \frac{a^4 A^4}{d^4} \left(3 - \frac{4x^2}{A^2} \right) \right. \\ \left. + \frac{35}{128} \frac{a^6 A^6}{d^6} \left(\quad \right) + \dots \right] \quad (11)$$

This is the value of the mutual inductance of half the solenoid (O_1A) on the circle S sought. In order to obtain an expression for M that will give accurate numerical values when the circle S is relatively large it is necessary to carry out the series to a larger number of terms than have been given above. Substituting in (6) the values of all the zonal harmonics to P_{12} , integrating and reducing as is done above for the first terms, the following expression is obtained, which is amply accurate for the most refined experimental work. ($N=2n_1x$, the whole number of turns of wire on the solenoid in the length $OA=2x$).

$$M_{OA} = \frac{2\pi^2 a^2 N}{d} \left[1 + \frac{3}{8} \frac{a^2 A^2}{d^2} + \frac{5}{64} \frac{a^4 A^4}{d^4} X_2 + \frac{35}{512} \frac{a^6 A^6}{d^6} X_4 + \frac{63}{1024} \frac{a^8 A^8}{d^8} X_6 \right. \\ \left. + \frac{231}{4096} \frac{a^{10} A^{10}}{d^{10}} X_8 + \frac{429}{16384} \frac{a^{12} A^{12}}{d^{12}} X_{10} + \dots \right] \quad (12)$$

$$\left. \begin{aligned} X_2 &= 3 - 4 \frac{x^2}{A^2} \\ X_4 &= \frac{5}{2} - 10 \frac{x^2}{A^2} + 4 \frac{x^4}{A^4} \\ X_6 &= \frac{35}{16} - \frac{35}{2} \frac{x^2}{A^2} + 21 \frac{x^4}{A^4} - 4 \frac{x^6}{A^6} \\ X_8 &= \frac{63}{32} - \frac{105}{4} \frac{x^2}{A^2} + 63 \frac{x^4}{A^4} - 36 \frac{x^6}{A^6} + 4 \frac{x^8}{A^8} \\ X_{10} &= \frac{231}{128} - \frac{1155}{32} \frac{x^2}{A^2} + \frac{1155}{8} \frac{x^4}{A^4} - 165 \frac{x^6}{A^6} + 55 \frac{x^8}{A^8} - 4 \frac{x^{10}}{A^{10}} \end{aligned} \right\} \quad (13)$$

a = radius of disk or circle S , Fig. 2.

A = radius of the solenoid.

x = length O_1A of one end of the solenoid.

$d = \sqrt{x^2 + A^2}$ = half the diagonal of the solenoid.

N is the whole number of turns of wire in the length x .

This formula is very easy to use in numerical calculation, notwithstanding it looks somewhat elaborate. The logarithm of $\frac{a^2 A^2}{d^2}$, multiplied by 2, 3, 4, etc., gives the logarithm of the corre-

sponding factor in each of the other terms. Similarly, the various terms X_2, X_4 , etc., contain only powers of $\frac{x^2}{A^2}$ besides the numerical coefficients. It is hence a far simpler matter to compute M with high precision by this formula than by Jones's formula, the latter containing as it does elliptic integrals of all three kinds and involving the tedious interpolations for incomplete elliptic integrals.

4. TESTS OF THE NEW FORMULA.

EXAMPLE 1.

As an example take the case given by Jones³, to calculate the mutual inductance of the disk and solenoid of the Lorenz apparatus built for McGill University.

$$\begin{aligned} A &= 10.513365 \text{ inches.} \\ a &= 6.509985 \quad " \\ x &= 2.51240 \quad " \\ N &= 201 \text{ turns.} \end{aligned}$$

The mutual inductance was computed by Mr. Rhodes under the direction of Professor Ayrton by Jones's first method. The dimensions in inches were used and the result then converted into centimeters. The value of M found was as follows:

$$M = 2M_0 = 18056.36 \text{ inches} = 45862.33 \text{ cm}$$

After rewinding the coil and regrinding the disk the dimensions were as follows:

$$\begin{aligned} A &= 10.512295 \text{ inches.} \\ a &= 6.507495 \quad " \\ x &= 2.51240 \quad " \\ N &= 201 \text{ turns.} \end{aligned}$$

The mutual inductance was calculated from these dimensions by Professors Ayrton and Jones by the second formula of Jones given above. The result was as follows:

$$M = 2M_0 = 18042.52 \text{ inches} = 45827.18 \text{ cm}$$

³ Proc. Roy. Soc., 68, p. 196; 1898.

We may now calculate M for these two cases by formula (12) above.

	1st Case.	2d Case.
A	10.513365 inches.	10.512295 inches.
a	6.509985	6.307495
x	2.51240	2.51240
A^2	110.53087	110.50837
x^2	6.31215	6.31215
N	201.	201.
$d^2 = A^2 + x^2$	116.84302	116.82052
$\log d^2$	2.0676027	2.0675191
$\log \frac{a^2 A^2}{d^2} =$	$\bar{1}.5354382$	$\bar{1}.5351848$
$\log x^2/A^2 =$	$\bar{2}.7566940$	$\bar{2}.7567824$
X_2	2.77157	2.77152
X_4	1.94197	1.94185
X_6	1.2559	1.2557
X_8	0.6685	0.6683
X_{10}	0.184	0.184
1st term	1.0000000	1.0000000
2d "	.1286677	.1285926
3d "	.0254913	.0254611
4th "	.0053500	.0053526
5th "	.0010709	.0010682
6th "	.0001793	.0001787
7th "	.0000079	.0000078
Sum, $S =$	1.1607671	1.1606610
$\log S =$	0.0647451	0.0647054
" $2\pi^2 =$	1.2953298	1.2953298
" $a^2 =$	1.6271600	1.6268278
" $N (= 201) =$	2.3031961	2.3031961
	5.2904310	5.2900591
$\log d =$	1.0338013	1.0337595
$\log M =$	4.2566297	4.2562996
$\therefore M =$	18,056.34 inches.	18,042.62 inches.
Value by Jones's formula	18,056.36 "	18,042.52 "
Difference	-.02	+.10

The difference in the results obtained by formula (12) from the value obtained by Jones's formula amounts in the first case to about one part in a million, and in the second case to about five parts in a million. These differences are wholly negligible in the most refined experimental work. In using formula (12) one can judge the degree of approximation by the convergence of the terms, and so tell when enough terms have been calculated for the particular case. In the above example the breadth of the coil $2x$ is exceptionally small, and formula (12) is not as convergent as for wider coils.

EXAMPLE 2.

Take as a second example the case given by Jones⁴ to illustrate his first formula.

$$A = 10 \text{ inches} \quad a = 5 \text{ inches} \quad x = 2 \text{ inches}$$

$$d^2 = 104 \quad \frac{a^2 A^2}{d^4} = \frac{2500}{10816} \quad \log \frac{a^2 A^2}{d^4} = \bar{1}.3638733$$

$$X_1 = 2.8400$$

$$X_4 = 2.1064$$

$$X_6 = 1.5208$$

$$X_8 = 1.0173$$

$$X_{10} = 0.5815$$

$$\text{1st term} = 1.0000000$$

$$2 \quad " = .0866771$$

$$3 \quad " = .0118537$$

$$4 \quad " = .0017781$$

$$5 \quad " = .0002670$$

$$6 \quad " = .0000379$$

$$7 \quad " = .0000060$$

$$\text{Sum} = 1.1006198$$

$$\frac{2\pi^2 a^2}{d} = 48.38972$$

$$\therefore M = 53.25868 N, N \text{ being the number of turns of wire on the coil.}$$

Jones gives $M = 53.25879 N$.

The difference between these values is 2 parts in a million.

⁴Phil. Mag., 27, p. 61; 1889. In this example, P_0 should be 0.654870 instead of .54870, as printed in Jones's article.

5. SECOND DERIVATION OF EQUATION 12.

Gray^a gives an expression for the mutual kinetic energy of two solenoidal coils from which equation (12) can be derived. Gray's expression is general; the coils may or may not be concentric and their axes may be at any angle with one another. The most important cases in practice is when the coils are concentric, and their axes are either in the same straight line or at right angles. Making the coils concentric causes half the terms in Gray's series to disappear. Making them coaxial reduces the zonal harmonic factor of each term to unity. When the coils have their axes at right angles the mutual energy becomes zero, but its derivative gives the torque between the coils, and we thus obtain the formula for the Gray absolute electro-dynamometer. Putting the current in each coil equal to unity the mutual energy becomes the mutual inductance M .

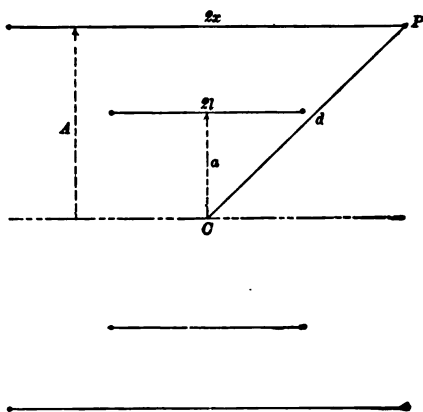


Fig. 7.

Put $2x$ = length of outer coil,
Fig. 7.

$2l$ = length of inner coil,
Fig. 7.

A = radius of outer coil,
Fig. 7.

a = radius of inner coil,
Fig. 7.

d = diagonal CP.

n_1, n_2 = number of turns per
cm in the outer and
inner windings respec-
tively.

W = mutual kinetic energy.

With these changes Gray's expression for the mutual energy becomes:

$$W = \pi^2 a^2 A^2 n_1 n_2 i_1 i_2 (K_1 k_1 Z_1 + K_2 k_2 Z_2 + K_3 k_3 Z_3 + \dots) \quad (14)$$

K_1, K_2 , etc., are functions of the length and radius of the outer coil, k_1, k_2 , etc., are functions of the length and radius of the inner coil, and Z_1, Z_2 , etc., are zonal harmonic functions of the angle ϕ between the axes. When $\phi = 0$ the mutual inductance is

^a Absolute Measurements, 2, Part I, p. 274.

$$M = \pi^2 a^2 A^2 n_1 n_2 (K_1 k_1 + K_2 k_2 + K_3 k_3 + K_4 k_4 + K_5 k_5 + \dots) \quad (15)$$

where

$$K_1 = \frac{4x}{A^2 d}$$

$$K_2 = \frac{x}{d^3}$$

$$K_3 = \frac{x}{4d^5} (4x^2 - 3A^2)$$

$$K_4 = \frac{x}{4d^{13}} (4x^4 - 10x^2 A^2 + \frac{5}{2} A^4)$$

$$K_5 = \frac{x}{4d^{17}} (4x^6 - 21x^4 A^2 + \frac{35}{2} x^2 A^4 - \frac{35}{16} A^6)$$

$$k_1 = 2l$$

$$k_2 = l(4l^2 - 3a^2)$$

$$k_3 = l(4l^4 - 10l^2 a^2 + \frac{5}{2} a^4)$$

$$k_4 = l(4l^6 - 21l^4 a^2 + \frac{35}{2} l^2 a^4 - \frac{35}{16} a^6)$$

$$k_5 = l(4l^8 - 36l^6 a^2 + 63l^4 a^4 - \frac{105}{4} l^2 a^6 + \frac{63}{32} a^8)$$

Substituting these values in (15), and putting N_1 for $2\pi n_1$, and N_2 for $2\pi n_2$, N_1 and N_2 being the whole number of turns of wire on the outer and inner solenoids, respectively,

$$\begin{aligned} M = \frac{2\pi^2 a^2 N_1 N_2}{d} \left\{ 1 + \frac{A^2 a^2}{8d^4} \left(3 - 4 \frac{l^2}{a^2} \right) \right. \\ + \frac{A^4 a^4}{32d^8} \left(3 - 4 \frac{x^2}{A^2} \right) \left(\frac{5}{2} - 10 \frac{l^2}{a^2} + 4 \frac{l^4}{a^4} \right) \\ + \frac{A^6 a^6}{32d^{12}} \left(\frac{5}{2} - 10 \frac{x^2}{A^2} + 4 \frac{x^4}{A^4} \right) \left(\frac{35}{16} - \frac{35}{2} \frac{l^2}{a^2} + 21 \frac{l^4}{a^4} - 4 \frac{l^6}{a^6} \right) \\ + \frac{A^8 a^8}{32d^{16}} \left(\frac{35}{16} - \frac{35}{2} \frac{x^2}{A^2} + 21 \frac{x^4}{A^4} - 4 \frac{x^6}{A^6} \right) \\ \left. \left(\frac{63}{32} - \frac{105}{4} \frac{l^2}{a^2} + 63 \frac{l^4}{a^4} - 36 \frac{l^6}{a^6} + 4 \frac{l^8}{a^8} \right) \dots \right\} \quad (16) \end{aligned}$$

If we put $l=0$ and $N_2=1$, we have the case of the inner coil reduced to a circle at the center of the outer solenoid. In this case equation (16) reduces to (12) except that it is not carried out to

as many terms. Searle⁶ and Airey have given an independent derivation of equation (16). Since deriving equation (12) I have found that Lorenz's equations (1) and (2) combined can be put into the same form, the four terms of (2) giving the first four terms of (12). I have verified the next two terms also by expanding the general term (2a). This operation is, however, somewhat tedious for the higher terms, as the m th term involves the m th derivative of $\left(\frac{a-1}{a}\right)^{m+1}$

6. EFFECT OF THICKNESS OF DISK.

The disk of a Lorenz apparatus is several millimeters in thickness at its edge, and it is important to know whether in any given case the mutual inductance is appreciably less than it would be for the ideal case of a disk of infinitesimal thickness assumed in all the formulæ. We can calculate the effect of the thickness by means of formula (16). Still keeping $N_2 = 1$, let l have a value equal to half the thickness of the disk. This effect will be greater with short coils than with long ones, but in any case the change would appear mainly in the first two terms after the unity term. Suppose, for example, that the disk of the McGill, Lorenz apparatus were 5 mm thick. Then l would be 0.25 cm, a is about 16.5 cm and $l/a = 1/66$, $l^2/a^2 = 1/4356$. Neglecting terms in l^3/a^3 and higher powers, equation (16) may be written, putting $N_2 = 1$,

$$M = \frac{2\pi^2 a^2 N_1}{d} \left\{ 1 + \frac{A^2 a^2}{8d^2} \left(3 - 4 \frac{l^2}{a^2} \right) + \frac{A^4 a^4}{32d^4} \left(3 - 4 \frac{x^2}{A^2} \right) \left(\frac{5}{2} - \frac{10l^2}{a^2} \right) + \frac{A^6 a^6}{32d^6} \left(\frac{5}{2} - 10 \frac{x^2}{A^2} + 4 \frac{x^4}{A^4} \right) \left(\frac{35}{16} - \frac{35}{2} \frac{l^2}{a^2} \right) + \frac{A^8 a^8}{32d^8} \left(\frac{35}{16} - \frac{35}{2} \frac{x^2}{A^2} + 21 \frac{x^4}{A^4} - 4 \frac{x^6}{A^6} \right) \left(\frac{63}{32} - \frac{105}{4} \frac{l^2}{a^2} \right) + \dots \right\} \quad (17)$$

Evidently each term after the first is made a little smaller by reason of the finite value of l . If we multiply the relative change of each term by the value of the term given on page 218, we shall have the change in M due to the thickness of the disk.

⁶The Electrician; Dec. 8, 1905.

Thus, since $l^2/a^2 = .000230$,

$$\begin{aligned} .000230 \times \frac{4}{3} &= .000306 = \text{relative change in second term.} \\ .000230 \times 4 &= .000920 = \text{relative change in third term.} \\ .000230 \times 8 &= .001840 = \text{relative change in fourth term.} \\ .000230 \times 13\frac{1}{3} &= .003067 = \text{relative change in fifth term.} \\ .1287 \times .000306 &= .0000394 = \text{change in } S \text{ due to second term.} \\ .0255 \times .000920 &= .0000235 = \text{change in } S \text{ due to third term.} \\ .0053 \times .001840 &= .0000097 = \text{change in } S \text{ due to fourth term.} \\ .0011 \times .003067 &= .0000034 = \text{change in } S \text{ due to fifth term.} \end{aligned}$$

Total corrections = $.0000760 = 1$ part in 15000 of M .

This reduction of the mean value of the mutual inductance of the coil and disk, due to the thickness of the disk, is small but very appreciable. If the thickness at the edge were reduced to 3 mm the change would be only .000027, or 1 in 43000, a quantity scarcely to be neglected, however, in the most exact work. As already suggested, this effect of the thickness of the disk would be reduced by making the coil longer, which is advantageous for other reasons. A longer coil will give a larger value of M , that is, a stronger field, which is very important, and also will reduce the variation in M due to the disk not being exactly centered. Displacement of the disk from the exact center along the axis reduces M ; displacements of its center along the radius of the coil increases M . Obviously, such displacement would make no change in M if the coil were infinitely long and the field therefore uniform. But the longer the coil the more nearly uniform is the field, and hence the less is the error due to lack of exact centering.

The magnetic force at the center of the disk is $\frac{2\pi N_1}{d}$ the area of the disk is πa^2 , hence $\frac{2\pi^2 a^2 N_1}{d}$ is the whole number of lines of force which would pass through the disk if the field were uniform and had throughout the value it has at the center. This is the first term in the expression for M , equation (12). Hence the values of the additional terms are a measure of the variation of the

field radially. For example, in the McGill apparatus, M is 16 per cent greater than it would be if the field were uniform and had the value it has at the center. If the coil were shorter this excess would be greater; if it were longer it would be less, and this is of course desirable.

7. EQUATION FOR THE GRAY ELECTRODYNAMOMETER.

When the axes of the two coils are at right angles to one another their mutual inductance is zero; but the moment of the force T tending to turn either coil about the center in the plane of their axes is a maximum and equal to $dW/d\phi$. Gray gives the general equation for this torque, which in the case of concentric coils becomes

$$T = \pi^2 a^2 A^2 n_1 n_2 i_1 i_2 \sin \phi (K_1 k_1 Z'_1 + K_3 k_3 Z'_3 + K_5 k_5 Z'_5 + \dots) \quad (18)$$

where ϕ is the angle between the axes of the coils. When $\phi = 90^\circ$, Z'_1 , Z'_3 , etc., have the following values:

$$\begin{aligned} Z'_1 &= 1 & Z'_7 &= -\frac{35}{16} \\ Z'_3 &= -\frac{3}{2} & Z'_9 &= +\frac{315}{128} \\ Z'_5 &= +\frac{15}{8} & Z'_{11} &= -\frac{693}{256} \end{aligned}$$

Substituting these values in (18) as well as the values of K_1 , K_3 , k_1 , k_3 , given above, we obtain for the maximum value of the torque, for $\phi = 90^\circ$,

$$\begin{aligned} T_m = \frac{2\pi^2 a^2 N_1 N_2 i_1 i_2}{d} \left\{ 1 + \frac{3}{16} \frac{A^2 a^2}{d^2} L_2 - \frac{15}{256} \frac{A^4 a^4}{d^4} X_2 L_4 + \frac{35}{512} \frac{A^6 a^6}{d^6} X_4 L_6 \right. \\ \left. - \frac{315}{4096} \frac{A^8 a^8}{d^8} X_6 L_8 + \dots \right\} \quad (19) \end{aligned}$$

In this equation X_2 , X_4 , X_6 , X_8 are functions of x and A and have the values given above (p. 216), and L_2 , L_4 , L_6 , etc., are the same functions of l and a .

If the coils are so proportioned that the length of the winding is to the radius of the coil as $\sqrt{3}$ is to 1, for each coil, i. e., $4l^2 = 3a^2$ and $4x^2 = 3A^2$, L_2 and X_2 are zero and the expression for the torque becomes

$$T_m = \frac{2\pi^2 a^3 N_1 N_2 i_1 i_2}{d} \left\{ 1 + \frac{35}{512} \frac{A^6 a^6}{d^{12}} X_1 L_6 - \frac{315}{4096} \frac{A^8 a^8}{d^{16}} X_6 L_8 + \dots \right\} \quad (20)$$

But in this case, since $d^2 = \frac{7}{4} A^2$,

$$\frac{A^6 a^6}{d^{12}} = \left(\frac{4}{7}\right)^{12} \frac{a^6}{A^6} \quad X_1 = -\frac{11}{4}$$

$$\frac{A^8 a^8}{d^{16}} = \left(\frac{4}{7}\right)^{16} \frac{a^8}{A^8} \quad X_6 = L_6 = -\frac{13}{16}$$

$$X_8 = L_8 = +\frac{243}{64}$$

$$\therefore T_m = \frac{2\pi^2 a^3 N_1 N_2 i_1 i_2}{d} \left\{ 1 + .0001851 \frac{a^6}{A^6} + .0000307 \frac{a^8}{A^8} + \dots \right\} \quad (21)$$

If $a = A/2$, the correction terms together amount to only a few parts in a million and may be neglected. Hence, if the relations $4x^2 = 3A^2$ and $4l^2 = 3a^2$ are exactly realized, the first term is enough to use even in the most refined experimental work and with the largest moving coil that it is practicable to use. If, however, this relation is not quite realized (as it probably never would be exactly) the slight correction to be made can be calculated from the second and third terms of (19).

8. THE AYRTON-JONES ELECTRODYNAMOMETER.

The Ayrton-Jones absolute electro-dynamometer⁷ consists essentially of a cylinder wound with a single layer of wire suspended from the arm of a balance inside of and coaxial with a larger fixed cylinder, the latter being also wound with a single layer of wire,

⁷Journal Inst. E. E., 35, p. 11; 1904-5.

Fig. 8. The winding of the fixed cylinder is, however, divided into two parts, the current flowing in opposite directions in the two halves. Thus, if the effect of the current in the lower half BC is to draw the inner cylinder ED down, the upper half of the cylinder AB will (because the current flows in opposite direction) repel the suspended cylinder and so urge it down, with a force equal to the force of BC, if the inner cylinder is suspended symmetrically with respect to the other two. Since the whole force is twice the force acting between AB and ED, we may now disregard the lower cylinder in finding the value of the force, simply doubling the force calculated for AB on ED. Jones⁸ has shown that the force

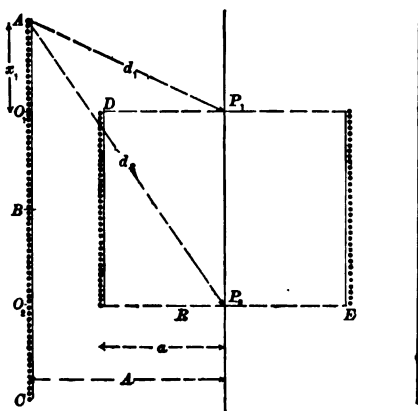


Fig. 8.

between AB and ED is proportional to the difference in the mutual inductances M_1 and M_2 of the coil AB on the two circles S and R at the ends of the inner cylinder. If the currents in the two coils are i_1 , i_2 , and n_2 is the number of turns of wire per cm on ED, the force in dynes on ED is

$$F = i_1 i_2 n_2 (M_1 - M_2)$$

Professor Jones gave two proofs of this formula, the second⁹ of which, because of its simplicity and importance, I shall reproduce here.

⁸ Proc. Roy. Soc., 68, p. 198; 1898.

⁹ This proof was suggested to Professor Jones by Prof. Andrew Gray.

If dx is an element of the suspended coil at the end S, Fig. 10, and M_1 is the mutual inductance of a circle at S with respect to the outer coil, the potential energy of this element is

$$W_1 = i_1 i_2 M_1 n_s dx,$$

where i_1 and i_2 are the currents in the two coils, respectively, n_s is the number of turns of wire per cm on the suspended coil, and dx is the length of the element. $M_1 i_1$ is the energy of unit current in one turn at S; the number of current turns is $n_s dx i_1$. If this element be at R its potential energy is

$$W_2 = i_1 i_2 M_2 n_s dx,$$

where M_2 is the mutual inductance of a circle at R with respect to the outer winding. If the element be carried from S to R, work is done equal to

$$W_1 - W_2 = i_1 i_2 (M_1 - M_2) n_s dx.$$

This is, however, equivalent to moving the whole cylinder vertically through a distance dx . Therefore, if F be the force acting on the suspended cylinder,

$$F dx = i_1 i_2 n_s (M_1 - M_2) dx$$

or

$$F = i_1 i_2 n_s (M_1 - M_2), \quad \text{as stated above.}$$

It is hence only necessary to calculate M_1 and M_2 by formula (1), (3), or (12) in order to be able to calculate the force due to unit current in the balance; or, conversely, knowing the force F by weighing, the absolute value of the current flowing in the coils can be calculated.

If the inner coil has the same length as the coil AB and is symmetrically situated with respect to the two coils, then the two end circles S and R will lie in planes passing through the middle of AB and BC, respectively.

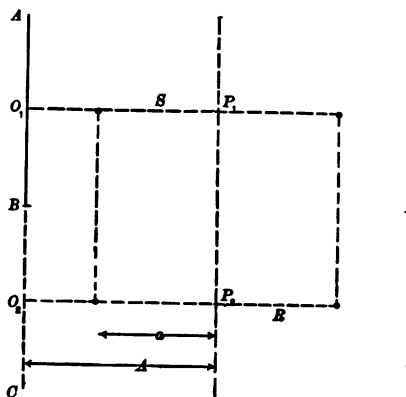


Fig. 9.

The mutual inductance of AB on S, Fig. 9, is twice the mutual inductance of AO_1 on S. Similarly, the mutual inductance of AB on R is the mutual inductance of AO_2 on the circle R minus that of BO_2 on R. Hence the same formula can be used in obtaining M_2 that is used for M_1 , merely varying the value of x , the distance from the plane of the circle to the end of the winding and of the diagonal d ; that is, the three values of x will be O_1A , O_2A , and O_2B in the three cases, respectively, and the corresponding values of d will be D_1A , D_2A , and D_2B .

9. CALCULATION OF ELECTRODYNAMOMETER CONSTANT.

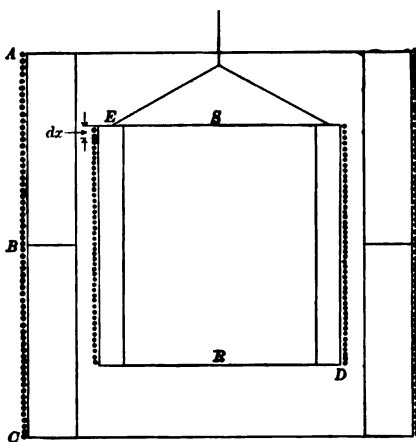


Fig. 10.

As a further test of the formulæ let us calculate the constant of an electro-dynamometer of the Ayrton-Jones type, of which AB, Fig. 8, is the upper fixed coil and ED is the moving coil, the circle S at the upper end lying in the plane through the middle of AB and the circle R at the lower end of ED lying in the middle plane of the lower fixed coil BC.

Assume the dimensions as follows:

$A = 16$ cm = radius of fixed coil, Fig. 10.

$a = 10$ cm = radius of moving coil.

$x_1 = 8$ cm = half length of AB = O_1A

$x_2 = 24$ cm = 1.5 times AB = O_2A

$n_1 = 10$ = number of turns per cm

$N_1 = 80$ = number of turns in distance $O_1A = x_1$, Fig. 9.

$N_2 = 240$ = number of turns in distance $O_2A = x_2$

$d_1 = \sqrt{A^2 + x_1^2} = 8\sqrt{5}$ = diagonal AP_1 , Fig. 9.

$d_2 = \sqrt{A^2 + x_2^2} = 8\sqrt{13}$ = diagonal AP_2

We have to determine two mutual inductances, namely, M_S between the coil O_1A of 80 turns on the circle S, and M_R between the coil

O₁A of 240 turns on the circle R. In each case the circle is in the plane passing through the lower end of the coil.

Formula (12) will be used, taking N_1 , x_1 , and d_1 in the first case and N_2 , x_2 , and d_2 in the second case.

	<i>For M_s</i>	<i>For M_R</i>
<i>A</i>	16 cm	16 cm
<i>a</i>	10	10
<i>x</i>	8	24
<i>A</i> ²	256	256
<i>x</i> ²	64	576
<i>N</i> = <i>nx</i>	80	240
<i>a</i> ²	320	832
log <i>a</i> ²	2.5051500	2.9201233
$\frac{a^2 A^2}{d^4}$	1.3979400	2.5679934
$\frac{x^2}{A^2}$	1.3979400	0.1760913
<i>X₁</i>	+2.000	— 6.00
<i>X₂</i>	+0.250	+ 0.25
<i>X₃</i>	—0.9375	+23.5
<i>X₄</i>	—1.203	—45.7
<i>X₁₀</i>	—0.562	—49.0
1st term	1.0000000	1.0000000
2d "	+ .0937500	+ .0138683
3d "	+ .0097656	— .0006411
4th "	+ .0002670	+ .0000009
5th "	— .0002253	+ .0000027
6th "	— .0000662	— .0000002
7th "	— .0000036	.0000000
<i>Sum</i> = <i>S</i>	1.1034875	1.0132306
log <i>S₁</i> =	0.0427674	log <i>S₂</i> = 0.0057083
" $2\pi^2$ =	1.2953298	" $2\pi^2$ = 1.2953298
" a^2 (= 100) =	2.0000000	" a^2 (= 100) = 2.0000000
" N_1 (= 80) =	1.9030900	" N_2 (= 240) = 2.3802112
	5.2411872	5.6812493
" d_1 =	1.2525750	" d_2 = 1.4600616
log <i>M_s</i> =	3.9886122	log <i>M_R</i> = 4.2211877
∴ <i>M_s</i> =	9741.19	<i>M_R</i> = 16641.32

10. EXAMPLE 2 BY JONES'S FORMULA.

We will now calculate M_s and M_R by Jones's second formula given above, using also the following equation to find $F - \Pi$:

$$\frac{k'^2 \sin \beta \cos \beta (F - \Pi)}{c} = F(k)E(k', \beta) + E(k)F(k', \beta) - F(k)F(k', \beta) - \frac{\pi}{2}$$

	For M_s	For M_R
A	16 cm	16 cm
a	10	10
x	8	24
$\Theta = 2\pi N$	160 π	480 π
$c = \frac{2\sqrt{Aa}}{A+a}$	0.9730085	0.9730085
$c' = \sqrt{1 - c^2}$	0.2307692	0.2307692
$k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + x^2}}$	0.9299812	0.7149701
$k' = \sqrt{1 - k^2}$	0.3676073	0.6991550
$\log \sin \beta \left(\sin \beta = \frac{c'}{k'} \right)$	9.7977938	9.5186043
$F(k)$	2.4373371	1.8636661
$E(k)$	1.1323456	1.3449927
$\frac{F-E}{k^2}$	1.5088957	1.0146546
$F(k', \beta)$	0.6852557	0.3394833
$E(k', \beta)$	0.6721988	0.3333201
$\frac{k'^2 \sin \beta \cos \beta (F - \Pi)}{c}$	-0.8266738	-1.1256799
$\frac{c'^2}{c^2} (F - \Pi)$	-0.6851799	-0.4045298
$\log \left\{ \frac{F-E}{k^2} + \frac{c'^2}{c^2} (F - \Pi) \right\}$	1.9157773	1.7854187
$\log (\Theta(A+a)ck)$	4.0728340	4.4357689
$\log M$	3.9886113	4.2211876
	$M_s = 9741.17$ cm	$M_R = 16641.32$

M_s differs from the value obtained by formula (12) by 2 parts in a million, M_R is identical.

M_8 is the mutual inductance of the winding O_1A on S . The inductance M_1 of the whole coil AB on S is twice as much, that is

$$M_1 = 19482.34$$

The inductance of AB on R is M_R above, minus the inductance of O_2B on R which is the same as that of O_1A on S , that is, M_8 . Therefore.

$$M_2 = 16641.32 - 9741.17 = 6900.15$$

Hence $M_1 - M_2 = 12582.19$ cm.

The force of attraction of the one winding AB in dynes is

$$\frac{1}{2}f = i_1 i_2 n_2 (M_1 - M_2).$$

The force due to the second winding BC is equal to this. Suppose $i_1 = i_2 = 1$ ampere = 0.1 c.g.s. unit of current and $n_2 = 10$ turns per cm. Then

$$i_1 i_2 n_2 = 0.10$$

$$\therefore f = 0.20 \times 12582.19 \text{ dynes}$$

$$= 2516.438 \text{ dynes}$$

$$2f = 5032.876 \text{ dynes} = \text{change of force on reversal of current}$$

$$= 5.1356 \text{ gms where } g = 980.$$

If there are two sets of coils, one on each side of the balance, as in the ampere balance built for the National Physical Laboratory, the force would be doubled again.

In calculating the mutual inductance of the disk and surrounding solenoid in the Lorenz apparatus the series (12) will be more convergent when the winding is long, and of course more convergent when the disk is not of too great diameter.

11. EQUIVALENCE OF SPIRAL WINDING AND CURRENT SHEET.

The derivation of Lorenz's formula is given in his collected works.¹⁰ Its exact equivalence to equation (12) might have been anticipated from the fact that both are based on the same hypothesis, namely, that the current is uniformly distributed over the solenoid in a cur-

¹⁰ Oeuvres Scientifiques de L. Lorenz, t. 2—1. Copenhagen, 1899.

rent sheet, as is Jones's first formula. Jones's second formula, on the other hand, assumes the current flowing along the axis of a wire wound spirally around the solenoid, making some integral number of turns but having any pitch whatever. The exact equivalence of this formula to (1) and (12) is at first surprising, but Lord Rayleigh has shown¹¹ from simple physical considerations that this must be true when as in this case the disk is circular and coaxial with the solenoid.

In an actual instrument, however, the current flows neither in a current sheet nor along the *axis* of a spiral wire, but is distributed

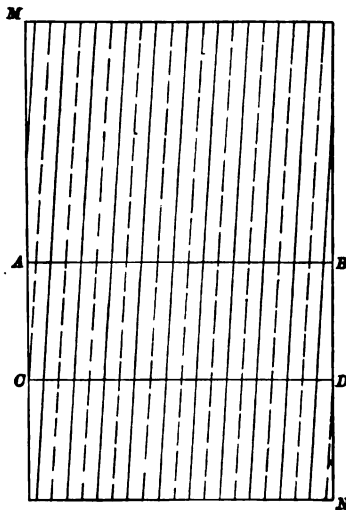


Fig. 11.

throughout the entire section of the spirally wound wire. It is therefore worth while to inquire whether the departure of an absolute electro-dynamometer from the ideal conditions assumed in the theory of the instrument can be the source of an appreciable error. The case is similar to the one I have considered elsewhere¹² in which the winding was assumed circular. It was shown that no appreciable error is due to the finite cross section of the wire. Let us now consider a spiral of fine wire wound on a short cylinder. Let MN, Fig. 11, be the surface of the cylinder developed in a plane, and AB a

¹¹ Report Brit. Asso., p. 241; 1899.

¹² This Bulletin, 2, p. 71; 1906.

winding of n turns of uniform pitch, beginning and ending in the two bounding planes of the cylinder. The magnetic force on the axis at the center of the cylinder due to unit current in this wire is independent of the particular points A, B, where the winding begins and ends. A second winding C, D, indicated by the dotted line, will produce exactly the same magnetic force at the center as AB, or exactly the same number of lines of force through any circle coaxial with the cylinder. Hence it is immaterial whether the current flows through one of these wires or through both in parallel or through any number of similar windings all in parallel, the total

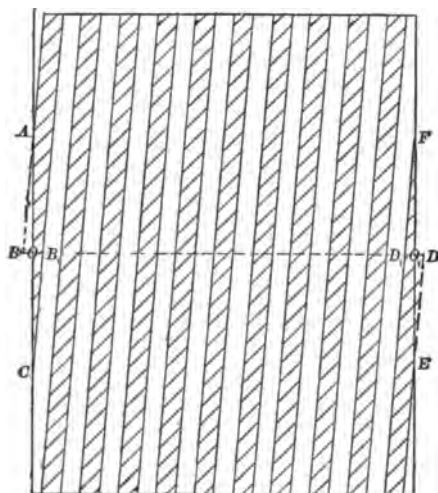


Fig. 12.

current being of course the same in each case. Hence a winding of flat tape, Fig. 12, of any width having the same pitch and number of turns will have the same mutual inductance with respect to any coaxial circle as a winding of very fine wire, *provided the current begins and ends in the bounding planes of the cylinder*. If the distance between the separate turns of tape is reduced to zero we have a current sheet, which is thus seen to be equivalent to a winding of fine wire.

If the tape be wound on edge, Fig. 11, the winding may begin at any point in the bounding plane M provided it ends at a corresponding point in the other bounding plane N. By the same reason-

ing as before a winding of n turns of thin tape on edge has the same magnetic force at any point on the axis and the same mutual inductance with respect to any coaxial circle as a winding of square or rectangular wire, the depth of the latter being equal to the depth of the tape. When the breadth of the rectangular wire is equal to the pitch of the winding, the wire covers the entire surface and we have a thick current sheet. It can easily be proved¹⁸ that for the dimensions of wire and cylinder likely to be employed in any absolute electro-dynamometer, the effect of a thick current sheet equivalent to the winding will not be appreciably different from that of a thin current sheet having a radius equal to the mean radius of the thick sheet.

12. EFFECT OF LEAD WIRES.

We have assumed in what precedes that the current enters and leaves the winding in the bounding planes MN of the short cylinder on which the wire is wound. Thus, if the winding consists of thin or thick tape, the breadth of which equals half the pitch, the bounding planes will cut off the conductor diagonally, leaving the wedge-shaped terminals AC and EF (Fig. 12), and the mathematical conditions would require that the current be introduced along the entire distance CA and leave along EF, so that the magnitude of the current at any point would be proportional to the breadth of the tape. As CA is half the circumference of the cylinder, this evidently is an impracticable method of introducing the current. Let us therefore inquire whether it will make any appreciable difference if the conductor is cut off abruptly at BB₁, DD₁ by a plane passing through the axis of the cylinder, and the current be introduced at BB₁ and withdrawn at DD₁. We have the triangular portion of the conductor COB₁ omitted and in its place the portion AOB; likewise EO₁D replaces FO₁D₁. The current in COB₁ averages one-fourth of the total current and the distance CO is one-fourth of one circumference. Hence the magnetic effect of COB₁ is practically equivalent to one-sixteenth of one turn at the end, or one-twenty-eighth of the average turn in the Gray electro-dynamometer, such as that discussed in the article to which reference was

¹⁸This Bulletin, 2, p. 71; 1906.

made above. That instrument had altogether 872 turns, and hence the effect of the triangular portion of the terminal COB_1 is about one-twenty-five thousandth of the total. This is practically negligible, seeing that it would produce an error of only 1 in 50000 in the current. But it is compensated almost exactly by the portion ABO , which is a little further from the center of the coil than COB_1 . In the dynamometer in question there were 20 turns per centimeter, and hence BB_1 = one-fourtieth cm. The difference in the average distances of the two elements from the central plane of the instrument would therefore be less than one-eightieth cm, and the difference in their magnetic effects would therefore be less than one-one thousand five hundredth of either. Hence the error produced by introducing the total current at the end BB_1 of the wire or tape would be less than $\frac{1}{1500} \times \frac{1}{25000}$, or less than 1 part in 37,500,000, or for the two ends together 1 in 18,750,000. If the coil had a smaller radius or coarser winding the error would be greater; but it probably never would amount to as much as one part in a million for any instrument designed to be used as an absolute instrument.

We can therefore be sure that the formulæ derived for the Gray, Ayrton-Jones, and other electrodynometers having large coils, or for the Lorenz apparatus, on the assumption that the current is distributed over the surface of the cylinder as a current sheet can be safely employed provided the leads are properly twisted together and the return current is brought from DD_1 , back to BB_1 along an element of the cylinder. The length of cylinder to be employed in the formulæ is, as already explained¹⁴, *the over-all length of the winding including the insulation on the first and last wires*, when the winding consists of a single layer of insulated wire wound with the adjacent turns in contact. For a winding *in a screw thread it is n times the pitch*, or *the length from center to center of $n+1$ turns*, n being the whole number of turns of wire on the cylinder.

I have here discussed these questions of the equivalence of a winding of wire to a current sheet and of the length of the equivalent current sheet, because I have recently received letters of inquiry from persons engaged in absolute measurements, who were not

¹⁴ This Bulletin, 2, p. 77; 1906.

quite satisfied that the conclusion reached in my former paper applied to a spiral winding. I hope that the above discussion makes it clear that it does. A spiral winding approaches a circular winding as the pitch decreases, and nothing is assumed in the preceding as to the magnitude of the pitch.

WASHINGTON, March 1, 1907.

ON THE ESTABLISHMENT OF THE THERMODYNAMIC SCALE OF TEMPERATURE BY MEANS OF THE CONSTANT-PRESSURE THERMOMETER.

By Edgar Buckingham.

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1. INTRODUCTION.

Two principal methods of gas thermometry are in use. In the constant-volume thermometer a mass of gas is kept at constant volume and its pressure observed at the melting point of ice, at the

condensing point of steam¹, and at the temperature to be determined. If these three pressures be denoted by p_0 , p_{100} , and p , the centigrade temperature according to the scale of this thermometer is by definition

$$t_v = 100 \frac{p - p_0}{p_{100} - p_0} \quad (1)$$

The numerical value thus assigned to a given temperature depends slightly on the initial pressure and is somewhat different for different gases. There are thus various "constant-volume scales," all agreeing at 0° and 100°C but differing at other points, so that the value assigned to any given fixed temperature other than the two standard temperatures is slightly different on different scales.

In the constant-pressure thermometer a mass of gas is kept at constant pressure, and its volume observed at the two standard temperatures and at the temperature to which a numerical value is to be assigned. If these volumes be v_0 , v_{100} , and v , the centigrade temperature according to the scale of this thermometer is by definition

$$t_p = 100 \frac{v - v_0}{v_{100} - v_0} \quad (2)$$

The resulting value depends somewhat on the magnitude of the constant pressure and on the nature of the gas used, so that there are various "constant-pressure scales."

If the thermometric gases employed be always far from condensation so that they follow Boyle's law approximately, all these scales are nearly identical; the numerical value assigned to any given temperature is nearly the same, no matter which scale is used. The adoption of any particular scale is arbitrary and, except in work of the highest precision, one is as good as another.

In exact thermometry it is customary to refer all temperatures to the international "normal scale" of the constant-volume hydrogen thermometer in which p_0 = the pressure of 1 meter of mercury. This thermometer can be used down to very low temperatures, though for practical reasons it is convenient to modify the scale by using a larger initial pressure p_0 . Helium also is used for very low temperatures and its scale is nearly identical with the hydrogen scale.

¹ In what follows it is always to be understood that in making these observations the ice and the steam are at standard atmospheric pressure.

At moderately high temperatures the materials which have been used for thermometer bulbs appear to become permeable to or to react with hydrogen, so that exact work with the hydrogen thermometer becomes impossible at temperatures above 300°C . Helium exhibits some of the same phenomena as hydrogen. For higher temperatures nitrogen is found satisfactory. Air has also been used, and for medium temperature ranges is entirely satisfactory. These four gases are the only ones that have been in common use.

The constant-pressure method has fallen somewhat into disrepute and has not been used much in recent years. It has, however, certain practical advantages at high temperatures where there is danger of softening the bulb, since it obviates the necessity of applying a variable compensating pressure to the outside of the bulb, and it has recently^{*} been recommended for high temperatures by Barus.

There are thus in practical use a number of different gas-thermometer scales using different gases, different initial pressures, and two different principles. In precision measurements, the differences between these scales become very sensible, and it is important that we should know the relations subsisting among them so as to be able to reduce measurements made in terms of any one to some standard scale.

The normal constant-volume hydrogen scale, as maintained at the International Bureau of Weights and Measures, is the standard usually adopted, and temperatures are stated in terms of that scale; but, as has been said, the hydrogen thermometer can not be used at even moderately high temperatures, so that direct comparison is possible only for what must now be regarded as a very limited range. Nitrogen, on the other hand, though suitable for work at high temperatures, will not do for very low temperatures, because its critical temperature is too high. There thus appears to be no gas known which is satisfactory throughout the entire range of temperatures accessible to gas thermometry. Instead, therefore, of attempting to refer all measurements to a single gas scale (a process which it seems must inevitably involve a considerable extrapolation at one end or the other), it is preferable to refer all measurements to some more general scale—one which is valid for all temperatures.

^{*} *Rapports Congr. Int. de Physique*, Paris, 1900; 1, p. 148.

By means of the two laws of thermodynamics any desired number of such scales might be defined, but there is one particular one, Lord Kelvin's thermodynamic scale, which has a practical advantage over all others in that it may be made to agree with all the gas scales in use as closely as they agree with one another, so that for most purposes its adoption does not complicate matters by adding a new scale to the number of those already in use. Lord Kelvin's scale may be defined as follows: Let an ideal reversible thermal engine work between two reservoirs of heat. Let the quantities of heat taken in or given out by the engine in the parts of its cycle when it is in contact with the two reservoirs be Q_1 and Q_2 . Then the ratio θ_1/θ_2 of the temperatures of the two reservoirs is by definition the same as the ratio of Q_1 to Q_2 , or

$$\frac{\theta_1}{\theta_2} = \frac{Q_1}{Q_2} \quad (3)$$

This equation defines only the ratio of two temperatures and not their numerical values. If the further condition be imposed that the temperatures of the melting point of ice and the condensing point of steam shall differ by 100° , the scale is completely determined. By Carnot's theorem, the ratio Q_1/Q_2 depends only on the temperatures and not on the particular nature of the heat engine; hence the thermometric scale thus defined is independent of the properties of any particular substance. When the scale of temperature is defined in this way, the temperature of the melting point of ice, or shortly "the ice point," is approximately 273° , and therefore that of the "steam point" is 373° . If we denote by θ_0 the exact value of the thermodynamic temperature of the ice point, we may define the "centigrade thermodynamic temperature" t_θ of any other point, by the equation

$$t_\theta = \theta - \theta_0 \quad (4)$$

This amounts to saying that the centigrade thermodynamic temperature is the number of thermodynamic centigrade degrees from the ice point to the given temperature to be numbered. It is this centigrade thermodynamic scale which is nearly identical with the centigrade scales of the various gas thermometers.

The coefficient of pressure, which is always nearly $\frac{1}{273}$, is defined by the equation

$$\beta = \frac{p_{100} - p_0}{100 p_0}$$

If we let $\frac{1}{\beta} = T'_0$ and $T' = T'_0 + t_v$, we then have the simple equation

$$\frac{T'}{T'_0} = \frac{p}{p_0} \quad (v = \text{constant}) \quad (5)$$

The quantity T' may evidently be regarded as the temperature measured, not from the ice point but from $-T'_0$. It is called the "absolute temperature" by the scale of the thermometer in question.

The definition contained in equation (5) is analogous to that of the absolute thermodynamic temperature θ_0 given by equation (3), and is therefore incomplete until the further condition is added that the temperatures of the ice and steam points shall differ by 100° . The value of T'_0 is then always approximately 273.

The centigrade temperature on the constant-volume scale may now, when convenient, be defined by the equation

$$t_v = T' - T'_0 \quad (6)$$

which is analogous to equation (4) and, under the condition that the ice and steam points shall be 100° apart, equivalent to equation (1), the original definition of t_v .

The coefficient of expansion, which is also always nearly $\frac{1}{273}$, is defined by the equation

$$\alpha = \frac{v_{100} - v_0}{100 v_0}$$

If we let $\frac{1}{\alpha} = T_0$ and $T = T_0 + t_p$, equation (2) gives us

$$\frac{T}{T_0} = \frac{v}{v_0} \quad (p = \text{constant}). \quad (7)$$

The quantity T may be regarded as the temperature measured on the constant-pressure scale from $-T_0$, and it is known as the "abso-

lute temperature" by the constant-pressure scale. Here again, to complete the definition we add the condition that the ice and steam points shall be 100° apart, and the value of T_0 is then always approximately 273.

The centigrade temperature by the constant-pressure scale may now be considered as defined by the equation

$$t_p = T - T_0 \quad (8)$$

which is analogous to equations (4) and (6) and equivalent to equation (2), the original definition of t_p .

Let T' , T , and θ be the numerical values of a given temperature on the three absolute scales. Then it may readily be shown by elementary thermodynamics that if the thermometer is filled with an ideal gas,

$$\frac{T'}{T_0} = \frac{T}{T_0} = \frac{\theta}{\theta_0} \quad (9)$$

or that the three scales are identical. The ideal gas is defined as one which follows Boyle's law and in which a free expansion, with no external work, would cause no change in the temperature. These conditions may be written in the form

$$(\rho v)_t = \text{constant} \quad (10)$$

$$\lambda = 0. \quad (11)$$

No real gas satisfies these conditions exactly, but all the common thermometric gases, as they are used in gas thermometers, do satisfy them approximately. Hence it is that the ordinary gas scales and the thermodynamic scale are all approximately the same, and the problem of finding the mutual relations of the various scales is reduced to the investigation of the departures of the actual gases from the ideal state and the computation of corrections for the departures.

Stated in this way the problem looks simple, but two difficulties arise at once: first, the variations of ρv at constant temperature are small and are not known with sufficient completeness; and second, the quantity λ , which may be defined as the amount of heat that would have to be added to a unit mass of gas during a free expansion

in which its volume increased by unity to keep the temperature from changing, is so small that it has not even been observed, still less measured. The difficulty of getting the experimental data necessitates a different mode of attacking the problem.

When Joule's experiments on free expansion had failed to determine the value of λ further than to show that it was very small, Lord Kelvin devised the well known porous-plug experiment which he then carried out in coöperation with Joule. In this experiment a single quantity is determined which depends on the departure of the gas from both of the conditions of ideality at once. If these departures could be determined separately the theory of the constant-volume thermometer would be quite simple. As it is, it is impossible to give a thoroughly satisfactory thermodynamic theory of the constant-volume thermometer on the basis of existing data, but a knowledge of the Joule-Thomson effect makes it possible to give the theory of the constant-pressure thermometer.

2. THEORY OF THE POROUS-PLUG EXPERIMENT.

In the porous-plug experiment the gas is forced to flow steadily through a porous plug. The plug is so insulated that no heat can enter or leave it by conduction. The pressure and temperature of the gas are observed on both sides of the plug. It was found by Joule and Lord Kelvin that hydrogen became warmer, while nitrogen, air, oxygen, and carbonic acid became colder in traversing the plug.

Let p_1 , v_1 , ϵ_1 represent the pressure, the specific volume, and the specific internal energy of the gas on the high-pressure side of the plug, and let p_2 , v_2 , ϵ_2 be the corresponding values on the low-pressure side. Since the gas enters the plug uniformly at the constant pressure p_1 , the work done on it by the pump is $p_1 v_1$ per gram. The plug having sufficient resistance, the gas issues in a steady stream at the lower pressure p_2 , and in issuing from the plug does a quantity of work $p_2 v_2$ per gram in pushing forward the gas ahead of it. The total work done on one gram of the gas is therefore $p_1 v_1 - p_2 v_2$. If no heat enters or leaves the gas by conduction, and if the flow is slow enough that the change in kinetic energy is negligible, the work done on the gas must be equal to the increase of its internal energy, so that we have

$$\epsilon_2 - \epsilon_1 = p_1 v_1 - p_2 v_2 \quad (12)$$

This equation is rigorously true under the conditions imposed, no matter what the pressures p_1 and p_2 are. If the fall of pressure be infinitesimal, equation (12) reduces to

$$\Delta\epsilon = -\Delta(pv) = -p\Delta v - v\Delta p \quad (13)$$

In Fig. 1 let the initial state of the unit mass of gas be represented by the point A with the coordinates p, v, t , the temperature t being measured on any convenient scale.

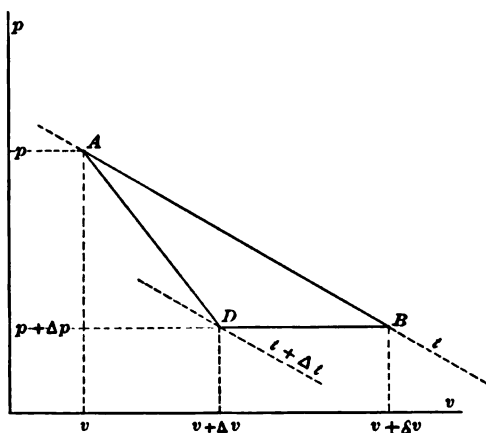


Fig. 1.

Let the state of the gas after passing the plug be represented by D with the coordinates $p + \Delta p, v + \Delta v, t + \Delta t$; Δp and Δt are negative while Δv is positive. Let the point B , with the coordinates $p + \Delta p, v + \delta v, t$, represent the state the gas would have been in if, during the same fall of pressure, its temperature had been kept constant.

After the gas, in passing through the porous plug, has reached the state D , let it be heated at constant pressure till it attains the state B . Then the excess of its internal energy at B over that at A is

$$\delta\epsilon = \Delta\epsilon + Q + W \quad (14)$$

where $\Delta\epsilon$ is the excess of the internal energy at D over that at A , and Q and W are, respectively, the heat added to the gas and the work done on it during the isopiestic change DB which brings it back to its original temperature t .

Let μ be the limiting value of the ratio of the fall of temperature to the fall of pressure, i. e., $\mu = \Delta t / \Delta p$. Then we have $\Delta t = \mu \Delta p$ and

$$Q = -\mu C_p \Delta p \quad (15)$$

where C_p is the specific heat at constant pressure, in ergs. The work done on the gas is

$$W = -p (\delta v - \Delta v) \quad (16)$$

and furthermore, since the change of volume from $v + \Delta v$ to $v + \delta v$ is caused by heating at constant pressure over an interval $-\Delta t$, we have

$$\begin{aligned} \delta v - \Delta v &= (-\Delta t) \left(\frac{\partial v}{\partial t} \right)_p \\ &= -\mu \left(\frac{\partial v}{\partial t} \right)_p \Delta p \end{aligned} \quad (17)$$

whence

$$\Delta v = \delta v + \mu \left(\frac{\partial v}{\partial t} \right)_p \Delta p \quad (18)$$

Substituting in equation (14) the values given by equations (13) to (18), we have

$$\delta \epsilon = -p \delta v - v \Delta p - \mu C_p \Delta p \quad (19)$$

and since the differentials in this equation now all relate to the isothermal change AB , it may be written

$$\left(\frac{\partial \epsilon}{\partial v} \right)_t = -p - (v + \mu C_p) \left(\frac{\partial p}{\partial v} \right)_t \quad (A)$$

Now it is known that in any reversible change of state of a system with only two degrees of freedom and acted on by no external forces except a uniform normal pressure,

$$\delta \epsilon = \theta \delta \eta - p \delta v \quad (20)$$

where η is the entropy and θ the thermodynamic temperature. Equation (20) is merely a statement of the two laws of thermodynamics for such a system. If the process is isothermal, equation (20) may be written

$$\left(\frac{\partial \epsilon}{\partial v}\right)_\theta = \theta \left(\frac{\partial \eta}{\partial v}\right)_\theta - p \quad (21)$$

To give this a physical meaning the entropy η must be eliminated, as follows: Subtracting $\delta(\theta\eta)$ from equation (20) gives

$$\delta(\epsilon - \theta\eta) = -\eta\delta\theta - p\delta v \quad (22)$$

and since the first member is a complete differential the second is also, so that

$$\left(\frac{\partial \eta}{\partial v}\right)_\theta = \left(\frac{\partial p}{\partial \theta}\right)_v \quad (23)$$

Using this relation and noting that if θ is constant, t also is constant, we may put equation (21) into the form

$$\left(\frac{\partial \epsilon}{\partial v}\right)_t = \theta \left(\frac{\partial p}{\partial \theta}\right)_v - p \quad (24)$$

whence by comparison with equation (A) we have

$$\theta \left(\frac{\partial p}{\partial \theta}\right)_v = -(v + \mu C_p) \left(\frac{\partial p}{\partial v}\right)_t \quad (B)$$

an equation which may serve as the starting point of the theory of the gas thermometer when treated by the use of the Joule-Thomson effect.

This equation (B) is rigorously true, having been deduced solely from the two laws of thermodynamics with no further hypotheses. The quantity μC_p evidently has the dimensions of a volume; it is therefore independent of the scale of temperature, provided that the same scale be used for μ as for C_p . In applying the equation to the practical question of finding the corrections of the gas thermometer, one further assumption must be made. The quantity μ can not be measured directly since it is defined as the ratio of two infinitesimals. But the experiments appear to have shown that the ratio of the change of temperature to the difference of pressure on the two sides of the plug is, for small pressures at all events, constant and independent of both the fall of pressure and the absolute value of the pressure. We assume that this constant ratio found at any given mean temperature is the same as the true value of μ , and take the observed values of the ratio as values of μ for use in the equations.

3. THEORY OF THE CONSTANT-VOLUME THERMOMETER.

If in equation (B), $v\left(\frac{\partial p}{\partial v}\right)_t$ be replaced by its equivalent $\left[\frac{\partial}{\partial v}(pv)\right]_t - p$, the equation takes the form

$$\theta\left(\frac{\partial p}{\partial \theta}\right)_v = p - \left[\frac{\partial}{\partial v}(pv)\right]_t - \mu C_p\left(\frac{\partial p}{\partial v}\right)_t \quad (25)$$

Inverting, multiplying by dp , and integrating at constant volume between any two temperatures, θ_0 and θ , at which the pressures at the constant volume $v = \phi$ are p_0 and p , gives

$$\log \frac{\theta}{\theta_0} = \int_{p_0}^p \frac{dp}{p - \left[\frac{\partial}{\partial v}(pv)\right]_t - \mu C_p\left(\frac{\partial p}{\partial v}\right)_t} \quad (v = \phi) \quad (26)$$

If T' be the absolute temperature as measured by a constant-volume thermometer filled with the gas in question,

$$p = p_0 \frac{T'}{T'_0} \quad (27)$$

by definition, and upon eliminating p from equation (26) we have

$$\log \frac{\theta}{\theta_0} = \int_{T'_0}^{T'} \frac{dT'}{T' - \frac{T'_0}{p_0} \left[\frac{\partial}{\partial v}(pv)\right]_{T'_0} - \frac{T'_0}{p_0} \mu C_p \left(\frac{\partial p}{\partial v}\right)_{T'_0}} \quad (C)$$

The integration of this equation demands not only a knowledge of the value of μC_p at all temperatures between θ_0 and θ , but also a complete knowledge of the isothermal compressibility of the gas within the same limits.

A slightly different method of integration may be adopted. Divide equation (25) through by θ^2 , rearrange, substitute

$$\frac{1}{\theta} \frac{\partial p}{\partial \theta} - \frac{p}{\theta^2} = \frac{\partial}{\partial \theta} \left(\frac{p}{\theta} \right) \quad (28)$$

and integrate between θ_0 and θ ; the result is

$$\frac{p}{\theta} - \frac{p_0}{\theta_0} = - \int_{\theta_0}^{\theta} \frac{1}{\theta^2} \left[\frac{\partial}{\partial v}(pv) + \mu C_p \frac{\partial p}{\partial v} \right] d\theta \quad (D)$$

This equation may serve the same purpose as (C), but the data on compressibility and on the values of μC_p must be given, not as before in terms of the constant-volume gas scale T' , but in terms of the thermodynamic scale θ which is to be established. This point is, however, of little importance, for since the whole second member is obviously only a small correction term, no large error will be introduced by identifying T' with θ in computing this correction, and at the worst, a result may be obtained by successive approximations.

The fundamental difficulty is the necessity of having the information about the departure of the gas from Boyle's law, in addition to the specific heat, the Joule-Thomson effect, and the coefficient of pressure which determines T_0' . This difficulty is inherent in the theory of the constant-volume thermometer, so far as based on the Joule-Thomson effect. Various treatments of the subject have been given³, differing in substance only in the different methods of

handling the terms $\frac{\partial}{\partial v}(pv)$ and $\frac{\partial p}{\partial v}$. From the imperfect experimental knowledge we have of the general form of the equation of state

$$f(p, v, t) = 0 \quad (29)$$

particular equations are set up which shall give, in various ways, the values of the correction terms in equations (C) and (D) with sufficient accuracy for the purpose in hand. The corrections to be determined are small and the final results are doubtless nearly correct, but all hypotheses regarding the compressibility are dispensed with in the theory of the constant-pressure thermometer. This superior simplicity in the theory of the constant-pressure thermometer is not due to any superiority in the constant-pressure method itself, but to the nature of the data available. The results of the porous-plug experiment are, in fact, not well adapted for use in finding the corrections of the constant-volume scale. If the free-expansion experiments of Gay-Lussac and Joule could be carried out with sufficient accuracy, they would give the values of the quantity λ , defined above as the amount of energy that would have

³ Rose-Innes *Phil. Mag.* (5) 45, p. 227; 1898; 50, p. 251; 1900; (6) 2, p. 130; 1901: Callendar, *Phil. Mag.* (6) 5, p. 48; 1903.

to be added to each gram of gas in an isothermal expansion in which the specific volume increased by unity and the gas did no external work. In such an expansion the internal energy of the gas would increase simply by the amount λ , and we should have the equation

$$\left(\frac{\partial \epsilon}{\partial v}\right)_t = \lambda \quad (30)$$

Comparison of this equation with (24) gives

$$\lambda = \theta \left(\frac{\partial p}{\partial \theta}\right)_v - p \quad (31)$$

which may easily be put into either of the two forms

$$\log \frac{\theta}{\theta_0} = \int_{T_0}^{T'} \frac{dT'}{T' + \frac{T'_0 \lambda}{p_0}} \quad (32)$$

$$\frac{p}{\theta} - \frac{p_0}{\theta_0} = \int_{\theta_0}^{\theta} \frac{\lambda d\theta}{\theta^2} \quad (33)$$

These equations correspond to equations (C) and (D), but do not, for integration, necessitate any information regarding the departure of the gas from Boyle's law.

If the manner in which the gas departs from Boyle's law is known, it is a simple matter to find the relation of the constant-volume and constant-pressure scales. Hence if the relation of either to the thermodynamic scale is known, the relation of the other can be found.

4. THEORY OF THE CONSTANT-PRESSURE THERMOMETER.

Returning to equation (B), namely,

$$\theta \left(\frac{\partial p}{\partial \theta}\right)_v = -(v + \mu C_p) \left(\frac{\partial p}{\partial v}\right)_t \quad (B)$$

let us make use of the relation

$$\left(\frac{\partial p}{\partial t}\right)_v \left(\frac{\partial t}{\partial v}\right)_p \left(\frac{\partial v}{\partial p}\right)_t = -1 \quad (34)$$

which is rigorously true, whatever be the scale in which t is measured. If $t = \theta$ and equation (B) be multiplied by $\left(\frac{\partial v}{\partial \theta}\right)_p / \left(\frac{\partial p}{\partial \theta}\right)_v$, the result is

$$\theta \left(\frac{\partial v}{\partial \theta} \right)_p = v + \mu C_p \quad (\text{E})$$

which is the form in which the fundamental equation of the porous-plug experiment is most frequently given. In its integral form the equation is

$$\log \frac{\theta}{\theta_0} = \int_{v_0}^v \frac{dv}{v + \mu C_p} \quad (35)$$

If the absolute temperature by the constant-pressure thermometer be defined as usual by the equation

$$\frac{v}{v_0} = \frac{T}{T_0} \quad (36)$$

equation (35) takes the form

$$\log \frac{\theta}{\theta_0} = \int_{T_0}^T \frac{dT}{T + \frac{T_0 C_p}{v_0}} \quad (\text{F})$$

By a slightly different method of treatment, similar to one already used in the last section, equation (E) may also be thrown into the form

$$\frac{v}{\theta} - \frac{v_0}{\theta_0} = \int_{\theta_0}^{\theta} \frac{\mu C_p d\theta}{\theta^2} \quad (\text{G})$$

Equations (F) and (G) play the same part in the theory of the constant-pressure thermometer as equations (32) and (33) would play in that of the constant-volume thermometer, if we had experimental data on λ such as we actually have for μC_p . If they be compared with equations (C) and (D), it is obvious that the theory of the constant-pressure thermometer, in so far as it is based on the Joule-Thomson effect, is decidedly simpler than that of the constant-volume thermometer, and especially that it does not necessitate making any hypotheses to help out the imperfection of the experimental data on compressibility. The theory of the constant-pressure thermometer

has thus one less weak point than that of the constant-volume thermometer.

It may be remarked that if the pressure of the gas were infinitesimal so that v_0 became infinite, while μC_p remained finite, the correction term in (F) would vanish and the integration would give

$$\frac{\theta}{\theta_0} = \frac{T}{T_0}$$

If, therefore, it is possible to find by extrapolation, the limit of the rate of thermal expansion of a gas as its density approaches zero, the thermodynamic scale is established directly and with no further knowledge of the value of μC_p than that it does not become infinite.

By means of equation (36), equation (G) may be put in the form

$$\frac{T}{T_0} - \frac{\theta}{\theta_0} = \frac{\theta}{v_0} \int_{\theta_0}^{\theta} \frac{\mu C_p d\theta}{\theta^2} \quad (37)$$

from which we see again that if the gas is rarified so that p approaches zero, i. e., $v_0 = \infty$, while μC_p remains finite, the constant-pressure scale approaches identity with the thermodynamic scale.

Now μ has already, in accordance with the indications of experiment, been assumed to be sensibly independent of the pressure, so long as the pressure is small; and furthermore, it is known that C_p varies only very slightly with the pressure. Hence for any given temperature θ , the value of the correction term

$$\frac{\theta}{v_0} \int_{\theta_0}^{\theta} \frac{\mu C_p d\theta}{\theta^2}$$

is nearly proportional to $\frac{1}{v_0}$. But since the gas follows Boyle's law approximately, $\frac{1}{v_0}$ is nearly proportional to the constant pressure π to which the gas in the thermometer is subject, so that for any given temperature we have

$$\frac{T}{T_0} - \frac{\theta}{\theta_0} = A\pi$$

where A is nearly constant. If we write this in the form

$$\frac{T - T_0}{T_0} - \frac{\theta - \theta_0}{\theta_0} = A\pi$$

substitute $T - T_0 = t_p$ and $\theta - \theta_0 = t_\theta$, and note that T_0 and θ_0 are both approximately 273, we get the equation

$$t_p - t_\theta = 273 A\pi \quad (38)$$

We have therefore the important result that the difference between the numerical values of any given temperature on the centigrade constant-pressure scale and on the centigrade thermodynamic scale, i. e., the thermodynamic correction of the centigrade constant-pressure scale at the given temperature, is very nearly proportional to the constant pressure π at which the gas is kept.

By introducing the approximation $p\theta = R\theta$ and treating equation (D) as equation (G) has just been treated, we may reach a similar result for the constant-volume thermometer, namely, that the thermodynamic corrections of the centigrade constant-volume scale are approximately proportional to the initial pressure p_0 at the ice point.

These two propositions are very useful, for they enable us, after finding the corrections for any gas thermometer of either type, to compute at once and with quite sufficient exactness the corrections for the same gas at any other pressure.

5. NUMERICAL DATA ON v_0 , C_p , AND T_0 .

The numerical data needed for finding the relations of the constant-pressure scales of air, nitrogen, and hydrogen to the thermodynamic scale will next be considered. These data, some of the latest of which do not appear to have been used for this purpose, will then be utilized in computing the thermodynamic temperature of the ice point, and in computing corrections at various temperatures.

In order to make practical use of equation (F) we must know the values of the constants v_0 and T_0 , and the value of μC_p as a function of T throughout the range of temperature T_0 to T . The ice point will be taken for the initial temperature T_0 ; its numerical value on the constant-pressure scale is the reciprocal of the mean coefficient

of expansion between the ice and steam points. The specific volume is to be taken at the pressure for which the coefficient of expansion is determined. The specific heat C_p is nearly constant, and since it occurs only in a small correction term, it is sufficient to treat it as a constant and use its mean value for the given interval. The Joule-Thomson effect will be considered separately and at length.

(a) *The Specific Volumes.*—Of the four quantities v_0 , T_0 , C_p , and μ , which appear in the correction term, the specific volume and the absolute temperature of the ice point are the most exactly known. The temperature of the ice point, which also appears as the lower limit of the integration, must be known with the highest attainable exactness, but in the correction term the uncertainties of v_0 and T_0 are of much less importance than those of C_p and especially of μ . The best data available have, however, been used for the specific volume v_0 , but it has not been thought necessary to refer to all the original papers, the figures having been taken from an article by D. Berthelot⁴, which contains a very complete and useful discussion of the densities and compressibilities of a number of the more permanent gases.

The data needed are the density ρ_0 at the ice point and under a pressure of one atmosphere, and the mean value of the coefficient K in the equation

$$\rho_2 v_2 = \rho_1 v_1 [1 + K(\rho_2 - \rho_1)] \quad (39)$$

within the small range of the reduction here needed. In Table I are given for air, nitrogen, and hydrogen the values of: (1) the density according to the experiments of Leduc, Morley, and Rayleigh; (2) the coefficient K according to the experiments of Chappuis, Leduc and Sacerdote, and Rayleigh (the value for air is taken from Regnault); (3) the pressures π for which the specific volumes at the ice point are to be found; (4) the specific volumes for these pressures as computed by means of equation (39) from the values of ρ_0 and K just given. It seems highly improbable that any of these values of v_0 is in error by as much as 1/1000, and such an error is of small importance compared with the uncertainties in the values of C_p and μ . The values of K assume one atmosphere as the unit of pressure.

⁴ Zeitschr. El. Chem., 10, p. 621; 1904.

TABLE I.
Specific Volumes.

	ρ_0	K_0	π	v_0
Air	0.00129278	-0.00191	1000.8 mm	587.015
Nitrogen	0.00125034	-0.000559	1001.9	606.626
Hydrogen	0.000069846	-0.000772	1000.5	8461.24

(b) *The Specific Heats.*—A collection of data on the specific heats may be found on page 406 of the third edition of Landolt and Börnstein's tables.

For air, the values given by Regnault, Witkowski, and Lussana for various temperatures between -20° and $+200^\circ$ lie between 0.2370 and 0.2378 with no indication of systematic variation with the temperature. E. Wiedemann obtained the value 0.2389, and Holborn and Austin⁵ have recently obtained, for the mean value between 20° and 440° , 0.2366 observed directly, and 0.2377 computed from measurements on oxygen and nitrogen separately. These values are all for a pressure of one atmosphere. The variation of the specific heat with pressure was investigated by Lussana⁶, who found for air

$$C_p = 0.23707 + 0.001498 (p - 1)$$

where p is given in atmospheres. Lussana's value will be used. The range of temperature in his experiments was from 94° to the mean temperature of the calorimeter, which was 28° . Taking for the mechanical equivalent of heat at 28° , the value $J = 4.173 \times 10^7$ ergs, we get for the specific heat of air at the constant pressure of 1001 mm

$$C_p = 9.913 \times 10^6 \text{ ergs.}$$

For nitrogen, the only experimental data available are those recently given by Holborn and Austin.⁵ Their observations may be represented by the equation

$$[C_p]_0^t = 0.2322 + 0.00002155 t$$

⁵Phys. Rev., **21**, p. 209; 1905.

⁶Nuovo Cimento (3), **86**, p. 134; 1894.

where $[C_p]_0^t$ is the mean specific heat, at atmospheric pressure, between 0° and t° . Assuming that the specific heat varies with the pressure in the same way that Lussana found for air, the value becomes, for a pressure of 1002 mm,

$$[C_p]_0^t = 0.2327 + 0.0000216 t$$

The calorimeter temperature was 18° , for which we may set $J = 4.182 \times 10^7$ and so get, finally, the formula

$$[C_p]_0^t = (9.731 + 0.000903 t) \times 10^6 \text{ ergs.}$$

For hydrogen, the values of the specific heat at a pressure of one atmosphere and at various temperatures between -30° and $+200^\circ$, as given by Regnault, E. Wiedemann, and Lussana, fall between 3.3996 and 3.410 with no indication of systematic variation with the temperature. Lussana's⁶ value will be used, namely,

$$C_p = 3.4025 + 0.0133 (p - 1)$$

The mean temperature of the calorimeter was 28° to 29° , so that J may be given the same value as before, and the result for a pressure of 1000 mm is

$$C_p = 1.4218 \times 10^8 \text{ ergs.}$$

(c) *The Coefficients of Expansion.*—For the coefficient of expansion α , of which T_0 is the reciprocal, the most recent data by Chapuis⁷ have been used. These values, together with the values of v_0 , and of $[C_p]_0^{100}$ already given, are collected in Table II.

TABLE II.

Data on the Specific Volumes, Specific Heats, and Coefficients of Expansion of Air, Nitrogen, and Hydrogen.

	Air	Nitrogen	Hydrogen
π	1000.778	1001.855	1000.460
α	0.00367282	0.00367315	0.00366004
T_0	272.270	272.246	273.221
v_0	587.01	606.63	8461.2
C_p	9.913×10^6	10.091×10^6	142.18×10^6

⁷ Trav. et Mém. Bur. Int., XIII, 1903.

6. EXPERIMENTAL DATA ON THE JOULE-THOMSON EFFECT.

There must, finally, be considered the numerical data on the Joule-Thomson effect upon which there are available: (1) the experiments of Joule and Lord Kelvin⁸ on hydrogen, nitrogen, air, oxygen, and carbonic acid; (2) the experiments of E. Natanson⁹ on carbonic acid, which will not be used because they relate only to room temperatures; (3) the experiments of Kester¹⁰ on carbonic acid. The experimental results must be examined somewhat in detail.

There are, to start with, twenty-two experiments on air. The authors group the results about the temperatures 7°1, 39°5 and 92°8, and represent them by an equation of the form $\mu = \frac{a}{T^2}$. If, however, all the twenty-two values of μ be plotted separately against the temperatures, it is at once evident that the experimental errors are so large as to make the deduction of anything but a linear equation between μ and T from these observations alone, a somewhat visionary refinement. Taking the values of the cooling effect in centigrade degrees for a fall of pressure at the plug of 100 inches of mercury, as given in the tables, the best straight line that can be drawn to represent the results has approximately the equation

$$\mu = 0.908 - 0.00434 \, t$$

where t is the centigrade temperature.

Upon nitrogen, three experiments were made, one at 7°2, one at 91°4, and one at 92°, the nitrogen having been diluted with oxygen to the extent of 7.9 per cent, 2.2 per cent, and 12.5 per cent respectively, in the three cases. The method of finding the value of μ for pure nitrogen does not appear to be stated specifically, but it was doubtless by means of the mixture rule. The best that can be done with these observations, *by themselves*, is to determine a straight line from them; its equation is

$$\mu = 1.068 - 0.00475 \, t$$

The two results at the higher temperature differ by nearly 20 per cent.

⁸ Kelvin, Math. and Phys. Papers, 1, pp. 419-424.

⁹ Wied. Ann., 81, p. 502; 1888.

¹⁰ Phys. Rev., 21, p. 260; 1905.

With hydrogen, seventeen experiments were made, in some of which the hydrogen formed only a small impurity in a mass of air. The value of μ at 0° is only about one-ninth as large for hydrogen as for air, so that the correction needed in getting the value for pure hydrogen is large and therefore uncertain. It seems by all means best to disregard entirely the results obtained when the amount of impurity was large. There then remain nine experiments in which the impurity in the hydrogen was less than 6.1 per cent. Four of these, at a mean temperature of $6^\circ.8$, gave values of μ from -0.075 to -0.126 with a mean of -0.0991 , while the remaining five, at a mean temperature of $90^\circ.1$, gave values from -0.098 to -0.234 with a mean of -0.1483 . The straight line thus determined has the equation

$$\mu = -0.0951 - 0.00059 \, t$$

Upon oxygen, six experiments were carried out. In three of these the percentages of nitrogen mixed with the oxygen were 54.6, 22.4, and 51.0, respectively, while in the others the percentages were 5.1, 3.6, and 4.0. Throwing out the first three, for obvious reasons, we have remaining one experiment at $8^\circ.7$, one at $89^\circ.5$, and one at $95^\circ.5$, of which the last two give values of μ differing by over one-third of their mean.

The seventeen values obtained by Joule and Lord Kelvin in their experiments on carbonic acid are grouped about five temperatures. When the mean values of μ are plotted against the temperature, the curve $\mu = f(t)$ has a distinct curvature with its convexity toward the origin. The gas was always somewhat impure and the results for pure carbonic acid were computed by a modification of the mixture rule. Kester's experiments, in which the values for the pure gas were found by graphical extrapolation from the results obtained with varying amounts of impurity, appear more reliable and his results will be used in what follows. They give a curve similar to that obtained by Joule and Lord Kelvin, but the values of μ are somewhat larger and the agreement of the separate experimental values is better.

From this brief summary of the available data on the Joule-Thomson effect it may be seen that our quantitative knowledge of this phenomenon is very imperfect. Of the three thermometric

gases, air, nitrogen, and hydrogen, air, the least important, is the only one upon which any extended series of observations at different temperatures has been made, and even in this case the observations leave much to be desired. Furthermore, all the observations are confined between the limits 0° and 100° , and they are too evidently inaccurate to give any feeling of security in using an empirical equation deduced from them for the purpose of extrapolation outside these limits. The observations will therefore be examined from another point of view, and an attempt made to improve the situation with regard to the Joule-Thomson effect.

7. APPLICATION OF THE LAW OF CORRESPONDING STATES.

In its most general form, the law of corresponding states affirms that if for each particular substance the pressure, volume, and temperature be measured in the proper units and from the proper zero points, there exists a single surface,

$$f(p, v, t) = 0$$

common to all substances¹¹. In the more special form in which it was first announced by van der Waals¹², it says that if the pressures, volumes, and absolute temperatures be measured for each substance in terms of their values at the critical point, the equation of state is the same for all substances and contains no quantities dependent on the special properties of particular substances. In this more specific form the law is not generally true, but it appears to be very approximately true within particular groups of substances. It will now be applied to the five gases for which the Joule-Thomson effect has been measured, namely, hydrogen, nitrogen, air, oxygen, and carbonic acid. This seems justifiable a priori for air and the three diatomic gases. For carbonic acid we have the fact that its isothermals, when reduced to the proper scale, are congruent with those of air, as has been shown by Amagat¹³, though they are not represented by van der Waals's equation as shown by Raveau¹⁴ from Amagat's,

¹¹ See K. Meyer, *Zeitschr. Phys. Chem.*, **32**, p. 1; 1900.

¹² Über die Continuität des gasförmigen und flüssigen Zustandes.

¹³ *Journ. de Phys.* (3), **6**, p. 5; 1897.

¹⁴ *Journ. de Phys.* (3), **6**, p. 432; 1897.

experimental data. It will be assumed, then, that for these five gases "corresponding" states are states in which the pressures, volumes, and temperatures are the same fractions of the critical pressures, volumes, and temperatures, the temperatures being counted from -273° , and that the properties of the gases are expressible in terms of these "reduced" units by a single equation of state.

It follows that if the gases be taken in corresponding states and subjected to the porous-plug experiment, a fall of pressure at the plug by a given fraction of the critical pressure will cause a cooling effect which is for all the gases the same fraction of the critical temperature. Now experiment has shown that for the small pressures used in gas thermometry, the thermal effect per unit fall of pressure at the plug is sensibly independent not only of the fall of pressure but of the initial value of the pressure, and consequently of the initial specific volume. Hence for present purposes, so long as the pressures are only a few atmospheres, the gases may be considered to be in corresponding states if only they are at the same reduced temperature, i. e., each at the same fraction or multiple of its critical temperature measured from -273° C. The proposition under consideration may then be stated as follows: "At equal reduced temperatures, the reduced Joule-Thomson effect is the same," meaning by the "reduced Joule-Thomson effect" the ratio of the change in the reduced temperature to the fall in the reduced pressure at the plug.

But the Joule-Thomson effect changes with the temperature; the cooling effect decreases as the mean temperature of the experiment rises, changing finally into a heating effect, for hydrogen at all events; and it is the nature of this change, or the form of the relation $\mu=f(\tau)^{16}$, as well as the absolute value of μ , that is needed in integrating the equations of the constant-pressure thermometer. So far as the law of corresponding states is valid, this function must be the same in general form for all the gases, and if μ and τ are expressed in reduced units by μ' and τ' there must be an equation

$$\mu' = f(\tau')$$

which is the same for all the gases, not merely in general form but also as regards numerical values.

¹⁶ The symbol τ is used here to signify $273+t$.

To put this in practical shape let us write

$$\mu' = \mu \frac{p_c}{\tau_c}$$

where p_c and τ_c are the critical pressure and critical temperature of the gas in question, and μ is the cooling observed, in centigrade degrees, when the pressures on the two sides of the porous plug differ by 100 inches of mercury. Reduced to practical terms our proposition then is that if all the values of μ' for all five gases be plotted as ordinates against the reduced temperatures $\tau' = \frac{\tau}{\tau_c}$ as abscissas, the points will all lie along a single smooth curve. If it is found that this is the case, within the apparent limits of error of the observations, we shall be justified in using this curve found from *all* the experiments as a more probable representation of the actual facts than could be obtained from the few observations available for each particular gas. If the curve can be represented by any empirical equation, then by changing back from reduced to ordinary units, a separate equation

$$\mu = f(\tau)$$

may be found for each gas, the same in general form for all but with different values of the constants.

In Table III are given, for the five gases mentioned above: (1) the critical temperatures measured from -273°C ; (2) the critical pres-

TABLE III.
Critical Constants.

Gas	τ_c	p_c
Carbonic acid	304.4	72.9
Oxygen	154.2	50.8
Air	133	39.3
Nitrogen	128	33.6
Hydrogen	32	19.4

ures in atmospheres. These figures are taken from Landolt and Börnstein (3d. ed., pp 182-186) and from Mathias (*Le Point Critique*

des Corps Purs, Paris, 1904). The value $p_c = 13$ atmospheres is also given for hydrogen, and if the equation of Clausius be assumed, it agrees better with Chappuis' values of the compressibility of hydrogen than does the higher value, $p_c = 19.4$ atmospheres.

These values of the critical constants have been used in computing the reduced values of μ , namely,

$$\mu' = \mu \frac{p_c}{\tau_c}$$

which, with the reduced temperatures, are shown in Table IV. In this table are given: (1) the observed cooling effect in centigrade degrees for a fall of pressure of 100 inches of mercury, according to Kester for carbonic acid, and according to Joule and Kelvin for the other gases; (2) the centigrade temperatures at which the values were observed; (3) the number of separate experiments on which each value is based; (4) the reduced temperatures of experiment, i. e., $\tau' = \tau/\tau_c$; (5) the reduced cooling effects, $\mu' = \mu p_c/\tau_c$; (6) the values of μ' computed by the equation

$$\mu'_1 = \frac{0.555}{\tau' - 0.42} - 0.008 (\tau' - 0.42) - 0.054 \quad (40)$$

(7) the values of $\mu' - \mu'_1$; (8) the values of μ' computed by the equation

$$\mu'_2 = \frac{0.7}{\tau' - 0.32} - 0.123 \quad (41)$$

(9) the values of $\mu' - \mu'_2$.

On Plate I the separate values of μ' are shown together with the curve plotted from equation (40). The values for hydrogen were obtained by using $p_c = 19.4$ atm. The point at $\tau' = 6.02$ is the inversion point of hydrogen found by Olszewski¹⁶ at -80.5°C . Since the excess of pressure in Olszewski's experiments was over 100 atmospheres, it could hardly have been expected that this point would fall in so well with the others¹⁷.

¹⁶ Phil. Mag. (6), 8, p. 535; 1902; Ann. Phys., 7, p. 818, 1902.

¹⁷ See A. W. Porter, Phil. Mag. (6), 11, p. 554; 1906.

TABLE IV.
Joule-Thomson Effect in Reduced Units.

Gas	1	2	3	4	5	6	7	8	9
	Cooling observed	Temp. of Expt.	Num- ber of Expts.	Re- duced tem- per- ature	Re- duced cooling				
	μ	$t^\circ C$		τ'	μ'	μ'_1	$\mu' - \mu'_1$	μ'_2	$\mu' - \mu'_2$
CO ₂	4.82	- 0.4	8	0.896	1.154	1.081	+0.073	1.092	+0.062
"	3.92	20.4	11	0.964	0.986	0.962	+0.024	0.964	+0.022
"	3.47	39.5	3	1.027	0.830	0.855	-0.025	0.867	-0.037
"	3.17	59.5	3	1.092	0.759	0.767	-0.008	0.784	-0.025
"	2.90	79.5	1	1.158	0.695	0.692	+0.003	0.712	-0.017
"	2.63	96.5	2	1.214	0.629	0.639	-0.010	0.660	-0.031
O ₂	1.07	8.7	1	1.827	0.352	0.329	+0.023	0.341	+0.011
"	0.80	89.5	1	2.351	0.264	0.218	+0.046	0.220	+0.044
"	0.57	95.5	1	2.390	0.188	0.212	-0.024	0.215	-0.027
Air	0.88	7.1	8	2.106	0.260	0.260	± 0.000	0.269	-0.009
"	0.75	39.5	8	2.350	0.222	0.218	+0.004	0.222	± 0.000
"	0.51	92.8	6	2.750	0.151	0.165	-0.014	0.165	-0.014
N ₂	1.03	7.2	1	2.189	0.271	0.246	+0.025	0.252	+0.019
"	0.58	91.4	1	2.847	0.151	0.155	-0.004	0.154	-0.003
"	0.69	92.0	1	2.852	0.201	0.155	+0.046	0.154	+0.047
H ₂	-0.099	6.8	4	8.74	-0.060	-0.054	-0.006		
$p_c=19.4$	-0.148	90.1	5	11.34	-0.090	-0.091	+0.001		
"	± 0.000	-80.5		6.02	± 0.000	-0.001	+0.001		
H ₂					-0.040			-0.040	± 0.000
$p_c=13$					-0.060			-0.059	+0.001
"					± 0.000			± 0.000	± 0.000

Examination of Table IV and Plate I will show that equation (40) represents the observations, well within their apparent experimental errors. Equation (41) represents all the observations about as well as equation (40) except in the case of carbonic acid.

We have thus found that the points representing the reduced observations do lie along one smooth curve as nearly as we can tell. It therefore seems justifiable to assume that this curve, based on all the observations, gives a better representation of the facts than can be obtained from the few observations on each gas separately, and this assumption will be made.

By reducing back to temperatures in centigrade degrees and expressing μ as the fall of temperature for a fall of pressure at the plug of one dyne per square centimeter, we may obtain an unreduced equation for each gas having the form

$$\mu = \frac{a}{\tau - b} - c(\tau - b) - d \quad (42)$$

if equation (40) be used, or

$$\mu = \frac{a_1}{\tau - b_1} - d_1 \quad (43)$$

if equation (41) be used. The constants a , b , c , d , or a_1 , b_1 , d_1 , will, of course, have a different set of values for each gas. It appears at first sight that these values will be considerably affected by errors in the critical data. The coefficients of the reduced equation (40) or (41) will be affected somewhat; but these errors are in the main eliminated by using the same critical constants in reducing back to ordinary units, and can become of importance only in the case of a wide extrapolation beyond the limits of temperature within which the experiments were made. The only large errors likely to occur are in the case of hydrogen, for which the values of μ are also small and therefore uncertain.

The values of the constants of equations (42) and (43) are shown in Table V. The values for hydrogen are based on $p_c = 19.4$ for the constants of equation (42), and on $p_c = 13$ for the constants of equation (43).

TABLE V.

Values of the Constants of Equations (42) and (43).

	Air	Nitrogen	Hydrogen
$\log_{10} a$	5.86687	5.90264	6.93706
b	55.86	53.76	13.44
$\log_{10} c$	11.77897	11.84702	10.08556
$\log_{10} d$	8.73212	8.78353	8.42001
$\log_{10} a_1$	5.96752	4.00229	5.21057
b_1	42.56	40.96	10.24
$\log_{10} d_1$	7.08848	7.13989	8.95023

8. REMARKS ON THE EQUATION $\mu = f(\tau)$.

The general form of the reduced equation (40) was decided on by inspection of a smooth curve drawn so as to represent the observations as well as possible, and the constants were adjusted by trial. This curve has its lower asymptote inclined, and the observations can not be well reproduced by making the asymptote horizontal, if the value $p_c = 19.4$ be adopted for hydrogen. On the other hand, the omission of the term $c(\tau - b)$ makes it impossible to represent the values for carbonic acid so well, though the simplified equation reproduces the hydrogen values a little better, on the assumption that the critical pressure of hydrogen is 13 atmospheres instead of 19.4. The values of the constants might be changed slightly, but no great improvement could be obtained. The values given in Table V are those that have been used in the computations in the remainder of the work.

Equations (42) and (43) are quite similar in form to those proposed by Rose-Innes¹⁸, namely,

$$\mu = \frac{A}{T} - B \quad (44)$$

and by D. Berthelot¹⁹, namely,

$$\mu = \frac{A}{\theta^2} - B \quad (45)$$

and having four and three arbitrary constants, respectively, can naturally be made to fit the observations somewhat better than the simpler equations with only two constants. It is obvious, however, that unless the observations on carbonic acid are very erroneous, the curve is asymptotic to a line at about $\tau' = 0.4$ and not to the vertical axis. Joule and Kelvin's observations on carbonic acid, which are also plotted in Plate I, though they were not considered in making up the equations, point even more strongly in the same direction. It will be noticed upon examination of the separate points, that in deciding upon the equation of the curve, the observations on air, for which the experiments were most numerous, have

¹⁸ Phil. Mag. (5), 45, p. 227; 1898.

¹⁹ Trav. et Mém. Bur. Int., XIII; 1903.

been given more weight than those on oxygen and nitrogen; also, that of the two observations on nitrogen at the higher temperature, the one in which the gas had the least impurity (2.2 per cent) lies much closer to the curve than the other in which the gas was more impure (12.5 per cent). It thus seems probable that if our use of the law of corresponding states is justifiable at all, the middle part of the curve has about the right position. To fit either equation (44) or equation (45) to the observations, we must either raise the middle part of the curve, thus spoiling the agreement for air, or else content ourselves with very rough approximation in the case of carbonic acid.

Regarding equation (45) it should be stated that M. Berthelot deduces the equation

$$\mu MC_p = \frac{A}{\theta^2} - B$$

where M is the molecular weight, from theoretical considerations on the form of the equation of state by the aid of the law of corresponding states. If MC_p is constant this reduces to equation (45). In the case of carbonic acid, C_p decreases sensibly as the temperature falls; hence μ would have to increase more rapidly with falling temperature than is indicated by equation (45). In other words, if this variation of specific heat were taken into account, the carbonic acid points on Plate I would all be raised and representation of all the observations by an equation with the vertical axis as an asymptote would still be imperfect.

The main result of the last two sections is the establishment of the fact that the observations on the Joule-Thomson effect may all be represented by a single empirical equation. Equation (42) is the one that has been used in most of the following computations, though results have in some cases also been obtained by the use of equation (43) for the sake of comparison. It is assumed that this equation (42) represents the true values of the Joule-Thomson effect from about the critical temperature to some twelve times the critical temperature. The equation thus affords a reasonable means of extrapolating down to the critical temperature of hydrogen and up to twelve times the critical temperature of nitrogen or about 1200° C. Within this range, the use of the equation is in a certain

sense only an interpolation by an empirical formula adjusted to the observations. Outside the range $\tau' = 1$ to $\tau' = 12$, the use of the equation is in every sense an extrapolation and therefore hazardous.

9. INTEGRATION OF THE EQUATIONS OF THE CONSTANT-PRESSURE THERMOMETER.

The first fundamental equation for the constant-pressure thermometer is

$$\log \frac{\theta}{\theta_0} = \int_{\tau_0}^{\tau} \frac{dT}{T + \frac{T_0 C_p \mu}{v_0}} \quad (p = \pi) \quad (\text{F})$$

In this equation T_0 and v_0 are constants, and C_p varies so little that it may safely be treated as a constant. The cooling effect μ has been shown to be capable of representation by an equation of the form

$$\mu = \frac{a}{\tau - b} - c(\tau - b) - d \quad (46)$$

The difference between τ , or $t + 273$, and T is so small that we shall not commit any serious error by replacing τ by T in equation (46). If this expression for μ be substituted in equation (F) the result of the integration is too complicated for convenient use in computation.

Now our first problem is the determination of the value, in centigrade degrees, of the thermodynamic temperature of the ice point, and for this purpose no serious extrapolation outside the range of the observations on μ is needed. Hence it is permissible, in the integration, to substitute for the general expression (46) any other which gives a curve nearly coincident with that given by equation (46) between 0° and 100° C. We do not in this way throw away the work done in obtaining the more general equation, for the absolute values of μ will be based on the numerical values of the constants in the general reduced equation, i. e., on *all* the observations and not merely on the few for each gas. If two such simpler expressions for μ be used, one of which represents a curve much closer to that of the general equation than the other, and if the computed values of the temperature of the ice point be sensibly the

same, it is certain that neither of them differs sensibly from the result that would have been obtained from the general expression itself.

The simplest equation is that of a straight line

$$\mu = a - bT \quad (47)$$

If this be substituted in (F) the result of the integration is

$$\frac{\theta}{\theta_0} = \left[\frac{T + A/B}{T_0 + A/B} \right]^{v_0/B} \quad (48)$$

where

$$\left. \begin{aligned} A &= aT_0C_p \\ B &= v_0 - bT_0C_p \end{aligned} \right\} \quad (49)$$

This equation has been used to compute the value of θ_0 , the thermodynamic temperature of the ice point, by setting $\theta = \theta_0 + 100$ and $T = T_0 + 100$. The values of a and b in equation (47) which determine the constants A and B of equations (48) and (49) were found: first, directly from the observations, i. e., from the straight line equations given in § 6; and second, by computing the values they would have for straight lines, intersecting the curves defined by equation (46) at 20°C and 80°C . The values thus found for θ_0 will be tabulated in the next section.

A second method is to use the equation

$$\mu = \frac{a}{T^2} \quad (50)$$

given by Joule and Lord Kelvin. This does not represent the facts at all for hydrogen; but for oxygen, nitrogen, and air between 0° and 100° the reduced equation

$$\mu' = \frac{1.2135}{T'^2}$$

represents the observations within their experimental errors. From this equation may be obtained the values $\log_{10} a = 2.33047$ for air, and $\log_{10} a = 2.34860$ for nitrogen. Using the value of μ given by equation (50) the integral of equation (F) becomes

$$\frac{\theta}{\theta_0} = \left[\frac{T^3 + aT_0C_p/v_0}{T_0^3 + aT_0C_p/v_0} \right]^{\frac{1}{3}} \quad (51)$$

which looks simpler than (48) but is not so easy to use in extended computations. The values of θ_0 found in this way from the data on air and nitrogen will be given in the next section.

Rose-Innes's equation

$$\mu = \frac{A}{T} - B$$

which is a much better approximation than either (47) or (50) might also be used, but the result of integrating equation (F) on this basis is nearly as complicated and unsuitable for computation as those obtained from the more general equations (42) and (43).

As a check on the foregoing methods, the general expressions have also been used in equation (G), namely,

$$\frac{v}{\theta} - \frac{v_0}{\theta_0} = \int_{\theta_0}^{\theta} \frac{\mu C_p d\theta}{\theta^2} \quad (G)$$

The values of the thermodynamic temperature θ between the ice and steam points are almost certainly within 0.3 of τ , i. e., of the ordinary centigrade temperatures by the normal scale, plus 273°. Since the whole second member of equation (G) is merely a small correction term, no serious error will be caused by writing

$$\mu = \frac{a}{\theta - b} - c(\theta - b) - d \quad (52)$$

or

$$\mu = \frac{a_1}{\theta - b_1} - d_1 \quad (53)$$

i. e., by identifying τ with θ in this correction term, and using the numerical values of the constants already given in Table V. The results of making these substitutions and integrating equation (G) are

$$\begin{aligned} \frac{v}{\theta} - \frac{v_0}{\theta_0} = C_p \left[\left(\frac{a}{b} - cb + d \right) \left(\frac{1}{\theta} - \frac{1}{\theta_0} \right) + \frac{a}{b^2} \log_{\theta_0} \frac{\theta_0(\theta - b)}{\theta(\theta_0 - b)} \right. \\ \left. - c \log_{\theta_0} \frac{\theta}{\theta_0} \right] = \phi \end{aligned} \quad (54)$$

and

$$\frac{v}{\theta} - \frac{v_0}{\theta_0} = C_p \left[\left(\frac{a_1}{b_1} + d_1 \right) \left(\frac{1}{\theta} - \frac{1}{\theta_0} \right) + \frac{a_1}{b_1} \text{lcg.} \frac{\theta_0(\theta - b_1)}{\theta(\theta_0 - b_1)} \right] = \phi_1 \quad (55)$$

Since by definition

$$\frac{v}{v_0} = \frac{T}{T_0} \quad (56)$$

it follows from (54) that

$$\frac{\theta}{\theta_0} = \frac{T}{T_0} - \frac{\theta\phi}{v_0} \quad (57)$$

or from (55) that

$$\frac{\theta}{\theta_0} = \frac{T}{T_0} - \frac{\theta\phi_1}{v_0} \quad (58)$$

To use either of these equations for finding θ_0 , we set $\theta = \theta_0 + 100$ and $T = T_0 + 100$. The value of T_0 is known; the value of θ_0 to be used in finding the value of the correction term was taken to be $273^\circ.14$ in a first approximation, and in a second approximation the value of θ_0 found in the first approximation was used.

10. VALUES FOUND FOR THE THERMODYNAMIC TEMPERATURE OF THE ICE POINT.

The values found for θ_0 by the several methods described in § 9 are collected in Table VI.

For carbonic acid we find $\theta_0 = 273^\circ.131$ from Chappuis' value of the coefficient of expansion at a pressure of 1377 mm, and $\theta_0 = 273^\circ.112$ from his value at a pressure of 998 mm. The mean specific heat was taken from Lussana's data corrected to 50° by Regnault's data. From Chappuis' value of the coefficient of expansion at a pressure of 518 mm, the resulting value is $\theta_0 = 272^\circ.700$. The experimental results at this pressure were less concordant, and it has seemed justifiable to disregard this value in taking the mean. Since the value $\theta_0 = 273^\circ.14$ assumed in making the first approximation is so close to the resulting value, a second approximation is unnecessary. No computation was made by the second form of equation for μ , since it does not fit the observations well for carbonic acid.

In considering the values given in Table VI we may at the start confine our attention to the values obtained from the reduced equations for μ , the other values having been computed and included in the table merely to illustrate the magnitude of the differences caused by the use of different methods of treating the same experimental data. A detailed examination of the experimental data does not

TABLE VI.
Temperature of the Ice Point.

Method	Air	N ₂	H ₂	CO ₂
$\mu = a - bT$				
<i>a</i> and <i>b</i> direct from observations.....	273.246		273.056	
$\mu = a - bT$				
<i>a</i> and <i>b</i> from equation (42) at 20° and 80° C..	273.269	273.279	273.056	
$\mu = a/T^2$				
<i>a</i> from equation (42).....	273.275	273.275		
$\mu = \frac{a}{\theta - b} - c(\theta - b) - d$				
first approximation.....	273.263	273.244	273.060	273.121
$\mu = \frac{a}{\theta - b} - c(\theta - b) - d$				
second approximation.....	273.266	273.245	273.061	
$\mu = \frac{a_1}{\theta - b_1} - d_1$				
based on last value of θ_0	273.276	273.249	273.055	

enable us to say whether the values for the different gases should be assigned equal weights or not. Using the means of the last two values for air, nitrogen, and hydrogen, the arithmetic mean is

$$\theta_0 = 273.^\circ 192$$

while if the value for carbonic acid be included, the mean is

$$\theta_0 = 273.^\circ 174$$

Another point of view is possible, however. If the discrepancies be assumed to be due, not to errors in the determination of the coefficients of expansion, but to errors in the elements which enter into the computation of the correction term which reduces T_0 to θ_0 , notably to errors in μC_p , it may be argued that the uncertainty of this correction is proportional to its total magnitude, and a weight may be assigned to each value proportional to the reciprocal of $\theta_0 - T_0$. These differences $\theta_0 - T_0$ are, for the four gases, approximately as 6 : 6 : 1 : 42. If weights be assigned in this way, the mean obtained is $\theta_0 = 273^\circ.106$, while if the weights are made proportional to the square roots of these reciprocals, the weighted mean is $\theta_0 = 273^\circ.131$. Since Chappuis' experiments show that hydrogen does not give so consistent results in the gas thermometer as nitrogen, it seems probable that these methods of weighting give too much importance to hydrogen.

All we can conclude from these values alone is that the value of θ_0 is probably between $273^\circ.1$ and $273^\circ.2$, but it is worth while to compare them with values obtained by other methods. The most interesting computation of this sort is that of D. Berthelot contained in Volume XIII of the *Travaux et Mémoires* of the International Bureau at Sèvres.

As was stated in §1, the problem of finding the relation of any gas scale to the thermodynamic scale—including the determination of θ_0 —is essentially the problem of investigating the departures of the gas from the ideal state and the computation of corrections for these departures. The usual method of doing this is to use the values of the Joule-Thomson effect. This is the method which has been pursued in the present paper, the only innovations introduced being in the treatment of the value of the Joule-Thomson effect where Kester's results on carbonic acid have become available, and in the use of Holborn and Austin's recently published results on the specific heat of nitrogen. But an entirely different mode of attack is possible. As was remarked in §4, the equations show that if the gas became infinitely rare, while μC_p remained finite, the gas scale would become identical with the thermodynamic scale. Hence, if it be possible to find, by extrapolation from finite to infinitesimal pressures, the limiting values of the coefficients of expansion α , and

of pressure β , these values should be identical with each other and with γ , the coefficient of expansion of the ideal gas, so that the thermodynamic scale would become known at once in its relation to the gas scales at ordinary pressures without any consideration of either the Joule-Thomson effect or the specific heat. The general application of this method presupposes a knowledge of the equation of state of the gas for low pressures.

M. Berthelot has used this method with great success. For finding the value of θ_0 , the data needed are the values of α and β for any given pressure, and the compressibility at 0° . These quantities being known for nitrogen and hydrogen, from Chappuis' experiments, the assumption that the departures from Boyle's law remain uniform down to zero pressure makes it possible to compute immediately the value of γ , the mean coefficient of expansion of the ideal gas between the ice and steam points. The reciprocal of this is the desired value of θ_0 .

The mean of four independent and very consistent values obtained in this way for nitrogen is $\theta_0 = 273^\circ.088$. The mean of three independent and somewhat less consistent values for hydrogen is $\theta_0 = 273^\circ.069$. The mean of all seven values is

$$\theta_0 = 273^\circ.079$$

which M. Berthelot considers to be probably a little low but correct within $0^\circ.02$.

M. Berthelot also computes the value of θ_0 , by the use of the Joule-Thomson effect, with the following results:

Gas	θ_0
Hydrogen	273.05
Air	273.19
Carbonic acid	273.10
Mean.....	273.11

The values for hydrogen and carbonic acid are nearly the same as those here obtained, namely, $273^\circ.058$ and $273^\circ.121$, while the value for air is sensibly lower than the value $273^\circ.271$ obtained in the present work. The difference is probably due to the different assumptions regarding the value of μ . It has been pointed out (§ 8)

that the equations used in the present work appear to represent the actual observations on the Joule-Thomson effect somewhat better than M. Berthelot's equation. He gives no value for nitrogen, doubtless because at the time when his paper was published no direct measurements on the specific heat of nitrogen were available. The value for nitrogen based on the new data is nearly as high as that for air, namely, $273^{\circ}25$, as compared with $273^{\circ}27$. Assuming that the value for nitrogen, if computed by M. Berthelot's method, would also be $0^{\circ}02$ lower than that for the air, the mean for the four gases computed by his method would be $\theta_0 = 273^{\circ}13$. There does not seem, on the whole, to be any sufficient reason for considering the value for nitrogen to be less reliable than the others.

We may then collect the results as follows:

Method	Value of θ_0	Gases	Author
Extrapolation to $p=0$	273.079	H ₂ and N ₂	Berthelot
Thermodynamic: $\mu = \frac{A}{\theta^2} - B$	273.13	H ₂ , N ₂ , CO ₂ , Air	"
Thermodynamic: $\mu = \frac{a_1}{\theta - b_1} - d_1$ } and $\mu = \frac{a}{\theta - b} - c(\theta - b) - d$ }	273.174	H ₂ , N ₂ , CO ₂ , Air	Buckingham

The mean of these,

$$\theta_0 = 273^{\circ}13$$

is probably not far from the true value. It remains for future experiments on the value of μC_p to decide whether the consistently high values obtained by the thermodynamic method of computation for air and nitrogen are erroneous, or whether there is some as yet undetected error in the experimental data from which M. Berthelot has obtained his beautifully consistent but much lower results for hydrogen and nitrogen.

11. CORRECTIONS OF THE CONSTANT-PRESSURE HYDROGEN AND NITROGEN THERMOMETERS BETWEEN THE ICE AND STEAM POINTS.

By the "correction of the gas scale to the thermodynamic scale" is meant, not the difference between the absolute temperatures by the two scales, a quantity which would be uncertain by the whole amount of the uncertainty in θ_0 , but the difference between the centigrade scale of the gas thermometer and the centigrade thermodynamic scale as defined in §1. These corrections are small between 0° and 100° and they may be found with an accuracy exceeding that attainable in the use of the gas thermometer by the simplest of the methods described in §9. From the assumption that the Joule-Thomson effect might be treated as a linear function of the temperature, or that

$$\mu = a - bT \quad (59)$$

we obtained the equation

$$\frac{\theta}{\theta_0} = \left[\frac{T + A/B}{T_0 + A/B} \right]^{v_0/B} \quad (60)$$

where

$$\left. \begin{aligned} A &= aT_0C_p \\ B &= v_0 - bT_0C_p \end{aligned} \right\} \quad (61)$$

If the values of a and b be found from the general equation (40) by making the straight line intersect the curve at 20° and 80° , the corrections may easily be computed.

The resulting values for the constant-pressure hydrogen thermometer are given in Table VII, in comparison with values computed by other writers. Column 1 gives the temperature; column 2, the corrections computed by Rose-Innes²⁰; columns 3 and 4, the corrections computed by Callendar²¹ by two methods; column 5, the values according to D. Berthelot²²; column 6, the values computed by the method just described; column 7, the values given by the same method when a and b are found directly from the equations given in § 6. The values computed by Rose-Innes and Callendar are given in their

²⁰ Phil. Mag. (6), 2, p. 130; 1901.

²¹ Phil. Mag. (6), 5, p. 48; 1903.

²² Trav. et Mém. Bur. Int. XIII, 1903.

papers for a pressure of 760 mm; they have therefore been multiplied by $\frac{4}{3}$, so that as given in Table VII all the corrections refer to a pressure of 1000 mm.

The figures in columns 2, 3, 6, and 7, obtained from the consideration of the Joule-Thomson effect, are all sensibly in agreement. The values in columns 4 and 5, obtained without using the Joule-Thom-

TABLE VII.

Corrections of the Constant-Pressure Hydrogen Scale, $\pi=1000$ mm.

[The corrections are all to be *subtracted*.]

1	2	3	4	5	6	7
$t^{\circ}\text{C}$	Rose-Innes 1901	Callendar, I 1903	Callendar, II 1903	D. Berthelot 1903	E. B., I	E. B., II
10	0.0015	0.0015	0.0007		0.0015	0.0013
20	0.0024	0.0025	0.0013	0.0008	0.0026	0.0023
30	0.0031	0.0032	0.0016		0.0033	0.0029
40	0.0033	0.0035	0.0018	0.0010	0.0037	0.0032
50	0.0033	0.0035	0.0018		0.0039	0.0034
60	0.0031	0.0033	0.0016	0.0009	0.0037	0.0032
70	0.0025	0.0027	0.0014		0.0031	0.0028
80	0.0019	0.0020	0.0010	0.0005	0.0024	0.0022
90	0.0011	0.0011	0.0005		0.0013	0.0013

son effect, by general considerations on the equation of state, are, respectively, about $\frac{1}{2}$ and $\frac{1}{4}$ of the mean of the others. The absolute values are thus in doubt, but as the largest of all the values is less than 0.004 the corrections may be considered negligible.

In Table VIII are given similar values for the constant-pressure nitrogen scale. As before, all the corrections are given for a pressure of 1000 mm. The figures in column 3 are for air, but the corrections computed by the same method for nitrogen would doubtless be very nearly the same.

The agreement among these sets of value is relatively much closer than was the case for hydrogen, though since the values are larger, the absolute magnitude of the divergence is also a trifle larger, the greatest difference being about 0.008. The mean of columns 2,

4, 5, and 6, given in column 7, is everywhere within 0.0001 of Berthelot's value and may be regarded as correct within the present limits of experimental precision in gas thermometry.

TABLE VIII.

Corrections of the Constant-Pressure Nitrogen Scale, $\pi = 1000$ mm.

[The corrections are all to be subtracted.]

1	2	3	4	5	6	7
$t^\circ\text{C}$	Rose-Innes 1901	Callendar, I 1903 (Air)	Callendar, II 1903	D. Berthelot 1903	E. B.	Mean (omitting col. 3)
10	0.0120	0.0097	0.0109	0.010	0.0078	0.010
20	0.0205	0.0163	0.0188	0.017	0.0137	0.017
30	0.0261	0.0207	0.0236	0.022	0.0179	0.022
40	0.0288	0.0225	0.0260	0.024	0.0203	0.025
50	0.0289	0.0226	0.0260	0.024	0.0209	0.025
60	0.0269	0.0207	0.0240	0.022	0.0198	0.023
70	0.0228	0.0176	0.0204	0.019	0.0172	0.020
80	0.0168	0.0131	0.0151	0.014	0.0129	0.015
90	0.0092	0.0069	0.0081	0.007	0.0071	0.008

12. COMPUTATION OF THE CORRECTIONS FOR TEMPERATURES OUTSIDE THE FUNDAMENTAL INTERVAL.

Since the methods of computing used in this paper involve the use of the Joule-Thomson effect, and since, with the exception of Olszewski's value for the inversion point of hydrogen, the data on this quantity have been obtained from experiments at temperatures between 0°C and 100°C , any computation of the relations of the gas scales to the thermodynamic scale at temperatures outside this interval must involve an extrapolation. The general reduced equation (40) or (41) affords the means of making this extrapolation with a considerable degree of probability. It embraces observations at temperatures all the way from a little below the critical temperature to somewhat over eleven times the critical temperature. For the purpose of extrapolation it will now be assumed that the corresponding unreduced equation for each gas is valid, not merely between 0° and 100° , but from the critical temperature of that gas to at

least eleven times the critical temperature. On this basis corrections will be computed for the hydrogen thermometer down to -240°C , for the nitrogen thermometer up to 1200°C , and for the air thermometer at 445° . In these computations equations (54) and (57) will be used, and these must now be considered more closely.

In the equation

$$\mu = \frac{a}{\tau - b} - c(\tau - b) - d \quad (62)$$

the temperature τ is not the temperature by either the constant-volume or the constant-pressure scale of the gas in question, but the centigrade temperature, as stated by various observers, plus 273° . It may safely be assumed that the tabulated values for the temperatures of the observations on μ , as well as for the critical temperatures, are given in terms of the normal constant-volume hydrogen scale, and that this scale agrees with the centigrade thermodynamic scale, both within the errors of the observations. In the computation by means of equations (54) and (57) this temperature τ is identified with the absolute thermodynamic temperature θ . But it now appears that the absolute thermodynamic temperature of the ice point is somewhat above 273° , and the value $273^{\circ}.13$ appears most probable. Strictly speaking, therefore, a second approximation ought now to be made. Instead of adding 273° to the observed values in getting the reduced temperatures and cooling effects, the computation should be revised, using $273^{\circ}.13$ instead of 273° . The coefficients of the reduced equation $\mu' = f(\tau')$ ought to be redetermined, and from these, the values of the coefficients in the unreduced equation for each gas. But it is quite certain that the accuracy of the experimental data does not warrant this and that the improvement in accuracy would be illusory.

All the corrections between 0° and 100° are so small that it is of little importance what method is used for computing them, but when it is a question of wide extrapolation, more caution is needed. The one essential condition is that the corrections shall vanish at 0° and 100° . Now, when equations (54) and (57) were used to find the value of the ice point for nitrogen they gave $\theta_0 = 273.244$. It is evident therefore that if the same values of a, b, c, d be adopted, but θ_0 be arbitrarily set equal to 273.13 or any other value than 273.244 , the

correction will not vanish at 100° . Hence the values of a , b , c , and d already given for the three gases do not, when taken in conjunction with the single value $\theta_0 = 273.13$, satisfy the essential condition of making the corrections vanish at 100° .

Two methods of procedure suggest themselves. We may take the values of the constants already determined and use them in connection with the value of θ_0 found for each particular gas from the condition that the correction shall vanish at 100° as well as at 0° . Or we might adopt the value $\theta_0 = 273.13$ for all the gases and adjust the constants for each gas so as to fulfill the condition of giving no correction at 100° . In view of the fact that the values of μ and therefore of the constants a , b , c , d are somewhat in doubt, and that there certainly is one definite true value of θ_0 , even if it be not precisely 273.13 , the second method seems more logical. But since there is no way of telling just what the errors are, and therefore no satisfactory way of adjusting the constants, the safest method is to keep strictly to the results of experiment and use for each gas the separate value of θ_0 found for it. This method has therefore been pursued.

In Table IX are given the values of the coefficients of equation (54) which were used in computing the temperature of the ice point as described in §9. They were obtained from the values of a , b , c , d (see Table V) found for each gas from the reduced equation (40) as described in §7. The corresponding values of θ_0 are also given, so that the figures for each gas form a consistent system of values which may be used in the first of the two methods of extrapolation just considered.

TABLE IX.

Values of the Constants of Equation (54).

	Air	Nitrogen	Hydrogen
$\log_{10} \left(\frac{a}{b} - cb + d \right)$	6.13710	6.18852	7.82499
$\log_{10} \frac{a}{b^2}$	8.37367	8.44172	8.68026
$\log_{10} c$	11.77897	11.84702	10.08556
θ_0	273.266	273.245	273.061

13. CONSTANT-PRESSURE CORRECTIONS FOR HYDROGEN AND NITROGEN OUTSIDE THE FUNDAMENTAL INTERVAL.

The computation of the corrections between 0° and 100° may be made by very simple methods. The values tabulated in § 11 are all small and the results from different methods agree closely. Equations (54) and (57), together with the constants given in Table IX, have also been used to compute the corrections of the constant-pressure hydrogen and nitrogen thermometers for several points. The same corrections have been computed by Callendar by two methods, and by Berthelot. The four sets of values for hydrogen are collected in Table X, Callendar's values for the pressure $\pi=760$ mm having been multiplied by $\frac{4}{5}$ to make them comparable with the others. The two values given in the column headed "E. B., II" were obtained by equations (55) and (58), which are based on the value $p_c=13$ atmospheres for hydrogen, and were computed for the sake of finding how much they differed from the values given by equations (54) and (57) based on $p_c=19.4$ atmospheres for hydrogen.

TABLE X.

Constant-Pressure Corrections for Hydrogen, $\pi=1000$ mm.

$t^{\circ}\text{C}$	Callendar, I	Callendar, II	D. Berthelot	E. B., I	E. B., II
-240			+1.368	+0.833	
-200	+0.781	+0.268	+0.250	+0.271	+0.349
-150	+0.211	+0.085	+0.060	+0.111	
-100	+0.066	+0.029	+0.021	+0.046	
+ 50	-0.0035	-0.0018	-0.0011	-0.0050	
+200	+0.017	+0.009	+0.005	+0.024	
+300	+0.041	+0.022		+0.064	
+400			+0.021	+0.110	+0.068
+450	+0.081	+0.045			

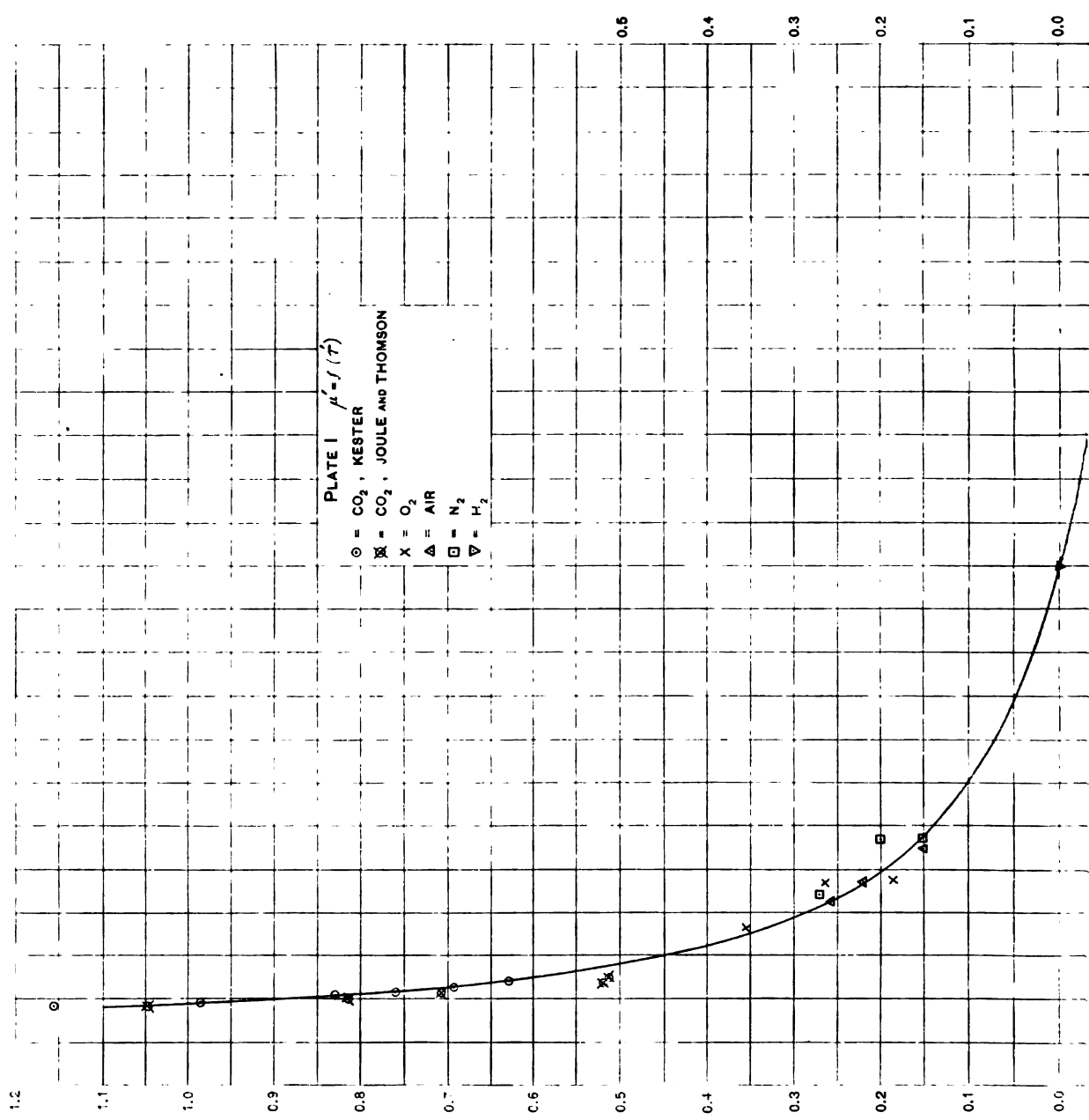
At first sight there appears to be hardly any agreement between the various values in this table except in the respect that most if not all of the corrections are within the experimental errors unavoidable in gas thermometry at the present time. A closer inspection shows that there is a fair agreement, for the low temperatures,

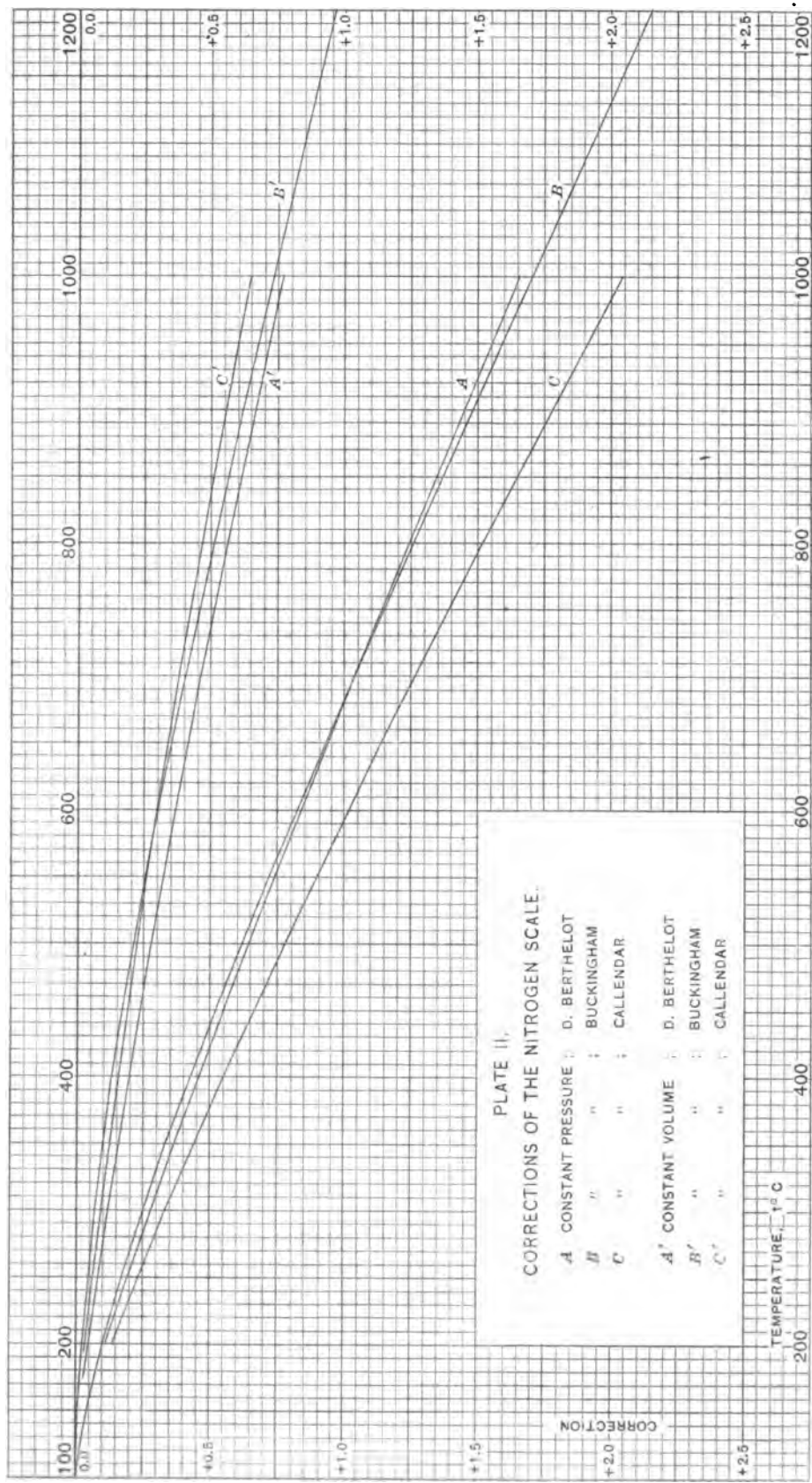
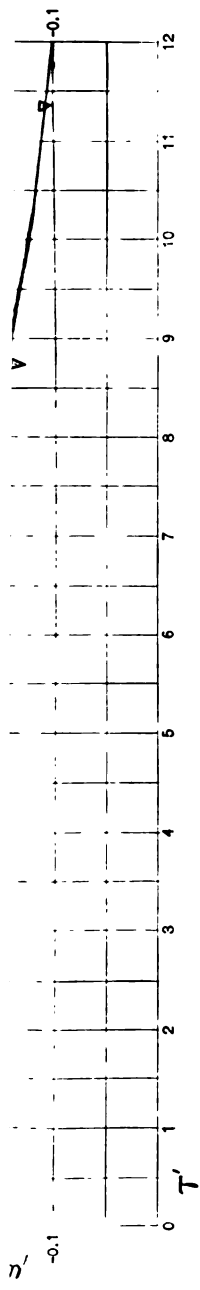
between the columns headed "Callendar, II," "D. Berthelot," and "E. B., I," and the mean of these sets of values is probably the best indication we have at present of the true values. At the higher temperatures the divergence is much greater. As regards the values computed by equations (54) and (57), it may be said that they are based on an extrapolation in every sense, since with hydrogen, which has the lowest critical temperature of all the gases considered, there is no possibility of using the law of corresponding states for checking the values at temperatures above 100° by comparison with other gases. Somewhat similar remarks are applicable to Berthelot's results, although since his computations are based on an apparently very satisfactory form of the equation of state and not on a confessedly empirical equation for μ , they appear to deserve more confidence than the values computed by the present writer. The curves plotted from the various sets of figures have a general similarity of form but do not lie close together. It is obvious that there are inconsistencies in the experimental data on the behavior of hydrogen, which it remains for future experiment, particularly on the Joule-Thomson effect and on the possible influence of adsorption on the coefficients of expansion and pressure, to clear up.

The same method of computation has been pursued for nitrogen, and the results, reduced by $1/500$ so as to make them applicable to a pressure of 1000 mm., together with those of Callendar and Berthelot, are collected in Table XI.

TABLE XI.
Constant-Pressure Corrections for Nitrogen, $\pi=1000$ mm.

$t^{\circ}\text{C}$	Callendar, II	D. Berthelot	E. B., I	E. B., II
-100	+0.437	+0.469	+0.326	+0.348
+ 50	-0.025	-0.024	-0.0190	
200	+0.135	+0.113	+0.105	
400		+0.457	+0.433	+0.456
450	+0.660			
600		+0.844	+0.827	
800		+1.248	+1.262	
1000	+2.047	+1.654	+1.706	+1.721
1200			+2.157	+2.170





Since the temperatures considered in this table all lie within the limits of the curve on Plate I, i. e., with the exception of 1200° , within the limits where the law of corresponding states affords us the support of comparison between different gases, the results would appear a priori to deserve more confidence than those for hydrogen. We find accordingly that the agreement is satisfactory, especially between Berthelot's values and those of the present paper. The two methods here pursued give nearly the same results, although the equations, namely (40) and (41), are based on two quite different values of the critical pressure of hydrogen, which probably include the true value between them. The greatest discrepancy between M. Berthelot's results and my own is at the lowest temperature. This was to have been expected. For the equation of state, upon the consideration of which M. Berthelot's computations are founded, leads to a value for μ which differs from the ones here used more and more as the temperature is lower and lower. Fortunately, this divergence of opinion as to the better form of equation for μ is of little importance, because there is little likelihood that the nitrogen thermometer will be used for low temperatures. Callendar's results are somewhat larger, but the difference of 0.4 at 1000° is small compared with the experimental errors at this temperature. On Plate II the three sets of results are plotted. Curve (A) represents Berthelot's values; curve (B), those of the present paper computed by the first method; and curve (C), Callendar's values. It seems probable that the mean of the three gives the true values with a precision not likely to be soon surpassed in experimental work.

14. THE BOILING POINT OF SULPHUR.

If the values given in Table IX be used in equations (54) and (57) to find the correction of the constant-pressure air thermometer at $t_{\theta}=445^{\circ}$, the result is $t_p=444.503$, hence the correction of the constant-pressure air thermometer for which $\pi=1001$ mm is, at this temperature,

$$t_{\theta}-t_p=+0.497$$

The value of $[C_p]_0$ used in the computation was found by taking Lussana's value for air at a pressure of 1001 mm and assuming that the specific heat of air varies with the temperature in the same way

as Holborn and Austin found for nitrogen. Increasing the specific heat by 1 per cent increases the correction by 0.01 . An independent computation, based on $\theta_0 = 273.2613$, gave $+0.504$ for the correction, or sensibly the same.

Callendar and Griffiths²³, using a constant-pressure air thermometer at atmospheric pressure, found for the boiling point of sulphur the value 444.53 . At this pressure the correction as just given becomes $\frac{3}{4} \times 0.497 = 0.37$; hence, according to this determination, the boiling point of sulphur on the centigrade thermodynamic scale is

$$(S. B. P.)_0 = 444.90$$

Chappuis and Harker²⁴, using a constant-volume nitrogen thermometer with an initial pressure of 528 mm, found for the same point the value 445.27 . Somewhat later, Chappuis²⁵, after the publication of Holborn and Day's results on the expansion of porcelain, concludes that this value should be lowered by about 0.5 , which would bring it to 444.77 . In §17 it will be shown that the correction for this thermometer is $+0.09$, which gives for the value of the boiling point of sulphur on the centigrade thermodynamic scale, according to the determination of Chappuis and Harker,

$$(S. B. P.)_0 = 444.86$$

The agreement of these two values, which is much better than could have been anticipated, in view of the experimental difficulties, is the best test at present possible of our methods of computing the corrections in question.

15. THE RELATION OF THE CONSTANT-VOLUME AND CONSTANT-PRESSURE SCALES.

It has already been pointed out (§3) that to treat the theory of the constant-volume thermometer by means of the Joule-Thomson effect we must have a complete knowledge of the compressibility of the gas throughout the range of temperature under consideration, although since the terms involving the compressibility are small, a merely approximate knowledge is sufficient. But if we have this

²³ Phil. Trans. 1891, A, p. 119.

²⁴ Phil. Trans. 1900, A, p. 102.

²⁵ Phil. Mag. (6), 8, p. 246; 1902.

knowledge of the compressibility, it is possible to find the relation of the constant-volume and constant-pressure scales; and since we have already, by a plausible method, found the thermodynamic corrections of the constant-pressure scale, we may deduce the corrections of the constant-volume scale from them by a process which is much simpler than integrating the general equations of the constant-volume thermometer given in §3. In either case, some hypothesis as to the equation of state must be made, and the correctness of the results obtained depends on how near the particular hypothesis comes to the truth. Since the corrections of the constant-volume thermometer have been computed by other writers by other methods, it seems worth while to use the simple method just suggested and compare the results with theirs. This will now be done.

If t_v be the centigrade temperature by the constant-volume gas scale, the gas being at the constant specific volume $v = \phi$ and the pressure at the ice point being $p_0 = \pi$, we have, by definition,

$$p = \pi (1 + \beta t_v) \quad (63)$$

If t_p be the centigrade temperature by the constant-pressure gas scale, the gas being under the constant pressure $p = \pi$ and the specific volume at the ice point being $v_0 = \phi$, we have, by definition,

$$v = \phi (1 + \alpha t_p) \quad (64)$$

Let the departures of the gas from Boyle's law be represented as usual by the equation

$$p_2 v_2 = p_1 v_1 [1 + K(p_2 - p_1)] \quad (65)$$

in which K is the average value, between p_2 and p_1 , of $\frac{1}{pv} \frac{\partial}{\partial p}(pv)$ and is some function of the temperature. Let the gas be at the temperature represented on the two scales by t_v and t_p , respectively. Then, if we let $p_2 = p$ and $v_1 = \phi$, we have, by substituting in equation (65) from equations (63) and (64),

$$\pi(1 + \beta t_v)\phi = \pi\phi(1 + \alpha t_p)(1 + K\pi\beta t_v)$$

whence

$$t_p = t_v \frac{\beta}{\alpha} \frac{1 - K\pi}{1 + K\pi\beta t_v} \quad (66)$$

If, therefore, we have the values of the coefficient of expansion α , the coefficient of pressure β , and K as a function of t_v , we may at once find the value t_p on the constant-pressure scale, of any temperature which on the constant-volume scale is denoted by t_v . Equation (66) is entirely rigorous, being a direct consequence of the definitions of t_v , t_p , and K contained in equations (63), (64), and (65). To make use of it we have to make some hypothesis about the variation of K with t_v , since experiments on compressibility at the pressures involved in gas thermometry have been made at only a very few temperatures.

16. ASSUMPTIONS REGARDING THE VARIATION OF K WITH TEMPERATURE.

If in equation (65) we let $p_1 = p$ and $p_2 = p + \delta p$ we have

$$pv + \delta(pv) = pv + Kpv\delta p$$

whence

$$K = \frac{1}{p} \frac{\partial}{\partial p} (pv) \quad (67)$$

We also have

$$\frac{\partial}{\partial p} (pv) = v + p \frac{\partial v}{\partial p} \quad (68)$$

The first step is to assume some form of equation of state as valid throughout the small range of pressures under consideration.

Van der Waals's equation,

$$p = \frac{R\theta}{v-b} - \frac{a}{v^2} \quad (69)$$

would indicate that the centigrade constant-volume scale was identical with the centigrade thermodynamic scale. Since the constant-volume scales of different gases do not agree precisely, as shown by Chappuis' experiments, van der Waals's equation must be considered inadequate.

The first equation of Clausius, namely,

$$\left[p + \frac{a}{\theta(v+c)^2} \right] (v-b) = R\theta, \quad (70)$$

may be made to represent the properties of gases more closely than that of van der Waals. From this we obtain

$$\frac{\partial}{\partial p}(pv) = \frac{a(v-c)(v-b)^2 - R\theta^2(v+c)^2b}{2a(v-b)^2 - R\theta^2(v+c)^2} \quad (71)$$

If, while θ remains finite, the pressure approaches zero, so that v becomes infinite, equation (71) approaches the form

$$\frac{\partial}{\partial p}(pv) = b - \frac{a}{R} \frac{1}{\theta^2} \quad (72)$$

We next assume that the value of $\frac{\partial}{\partial p}(pv)$ is sensibly constant for pressures between zero and the highest pressure to be considered, which is only about 5.5 meters of mercury. This amounts to assuming that within this small range of pressures the isothermal curves $pv=f(p)$ are sensibly straight lines. Experiment shows them to be very nearly straight for the more nearly ideal gases, such as hydrogen and nitrogen, so far as the experiments have gone, so that this assumption is a legitimate expression of an experimental approximation. We may also safely assume that whatever be the real form of the equation of state, the gas approaches the ideal state as a limit when the pressure approaches zero. Since the absolute thermodynamic scale is identical with the scale of an ideal gas, we may write, for $p=0$,

$$\lim_{p \rightarrow 0} \frac{(pv)}{(pv)_0} = \frac{\theta}{\theta_0}$$

and if we then take as our unit of pv the limiting value of pv at the ice point θ_0 , we have for any other temperature θ ,

$$\lim_{p \rightarrow 0} (pv) = \frac{\theta}{\theta_0} \quad (73)$$

Since K is itself a very small quantity, pv does not vary much along each isothermal within the given range of pressures, and we may use the value given by equation (73) in equation (67). We then have finally, for the value of K obtained from equations (67) and (70),

$$K_\theta = \frac{\theta_0}{\theta} \left(b - \frac{a}{R} \frac{1}{\theta^2} \right) \quad (74)$$

If we have experimental data on the values of K for two different temperatures, of which the values θ_0 and θ are known approximately, the constants $\frac{a}{R}$ and b may be determined and an equation set up for computing the value of the coefficient K at all other temperatures.

A large part of the paper of Berthelot³⁶, which has so frequently been referred to, is devoted to the establishment and verification of an equation of state. This equation is very similar to that of Clausius and leads to the same result, namely, equation (74), for the limiting value of the coefficient K . M. Berthelot's paper was not received until after the foregoing considerations had been developed, and it was gratifying to find that the equation employed for computation was one that had already been used successfully.

According to the experiments of Chappuis on nitrogen we have, per meter of mercury,

$$\left[\frac{\partial}{\partial p}(pv) \right]_0 = -0.000571 ; \quad \left[\frac{\partial}{\partial p}(pv) \right]_{100} = +0.000347$$

the value of (pv) at 0° and 1 meter of mercury being taken equal to unity. Since it is convenient in using equation (66) to have $\pi=1$, and since all the other quantities involved refer, for nitrogen, to a pressure of 1002 mm, we have merely to increase the above values by 1/500. If we also set $\theta_0 = 273.2$, we have

$$\begin{array}{ll} \theta_0 = 273.2 & K_0 = -0.0005723 \\ \theta_{100} = 373.2 & K_{100} = +0.0002545 \end{array}$$

If these figures be substituted in equation (74), they give

$$\begin{aligned} \frac{a}{R} &= 147.95 \\ b &= 0.001410. \end{aligned}$$

If these values be used in the well-known relations³⁷ deduced from equation (70), to compute the critical pressure from the critical temperature, they give

³⁶ Trav. et Mém. Bur. Int. XIII; 1903.

³⁷ See Preston's Theory of Heat, 2d ed., p. 512.

	Calculated	Observed
p_c (atmospheres)	28.9	33.6

In view of the small range of pressures used, the smallness of the values of K , and the extent of the extrapolation, the agreement of the observed and computed values is remarkably good and bears witness to the admirable precision of Chappuis' work.

We have, then, to make use of the equation

$$K_t = \frac{273.2}{273.2 + t} \left[0.001410 - \frac{147.95}{(273.2 + t)^2} \right] \quad (75)$$

17. THE CORRECTIONS OF THE CONSTANT-VOLUME NITROGEN THERMOMETER.

Having obtained an equation for K , it is now an easy matter by means of equation (66) to find the relation of t_p and t_v . Since the equation for K is given in terms of 1002 mm as the unit of pressure, equation (66) takes the form

$$t_p = t_v \frac{\beta}{a} \frac{1 - K}{1 + K\beta t_v}$$

Chappuis' values of a and β are

$$\beta = 0.00367442$$

$$a = 0.00367315$$

whence we have

$$\frac{\beta}{a} = 1.000346$$

and the equation to be used in computation is

$$t_p = 1.000346 \frac{1 - K}{1 + K\beta t_v} t_v \quad (76)$$

where K is to be found from equation (75).

The results are shown in Table XII. In column 1 are given the values of t_v , the temperature by the centigrade constant-volume scale; in column 2, the values of K computed by equation (75); in column 3, the corresponding values of t_p computed by equation (76); in column 4, the difference $t_v - t_p$; in column 5, the corrections, for a pressure $\pi = 1002$ mm, to be applied to the centigrade constant-

pressure scale; in column 6, the resulting correction to be applied to the centigrade constant-volume scale to reduce it to the centigrade thermodynamic scale. The value $t_\theta - t_r = -0.059$ at -100° is undoubtedly erroneous. This may be due to the failure of equation (75) to represent the compressibility at low temperatures, or it may be that the constant-pressure correction at this temperature as computed is too small. If the mean of Berthelot's and Callendar's corrections for the constant-pressure scale at this point be used, namely, $t_\theta - t_p + 0.453$ instead of $+0.326$, we have $t_\theta - t_r = +0.068$

TABLE XII.

Comparison of Constant-Volume and Constant-Pressure Scales for Nitrogen, $\pi = p_0 = 1002$ mm.

1	2	3	4	5	6
$t_r^\circ C$	K	t_p	$t_r - t_p$	$t_\theta - t_p$	$t_\theta - t_r$
-100	-0.005555	-100.385	+0.385	+0.326	-0.059
+ 50	-0.000005	+ 50.0176	-0.0176	-0.0190	-0.0014
200	+0.000433	199.919	+0.081	+0.105	+0.024
400	+0.000440	399.704	+0.296	+0.435	+0.139
600	+0.000380	599.476	+0.524	+0.830	+0.306
800	+0.000326	799.250	+0.750	+1.265	+0.515
1000	+0.000283	999.024	+0.976	+1.711	+0.735
1200	+0.000249	1198.800	+1.200	+2.163	+0.963

In Table XIII is given a comparison of different values for the constant-volume nitrogen corrections. In column 1 are the centigrade temperatures; in column 2, the corrections as computed by Callendar; in column 3, the corrections according to Berthelot; in column 4, the values from Table XII reduced by $1/500$ so as to make them applicable for an initial pressure $p_0 = 1000$ mm.

The interesting part of the table is that relating to the higher temperatures. The values from 200° to 1200° have been plotted and the resulting curves are shown in Plate II. Curve (A') represents Berthelot's values; curve (B'), those computed by the present writer; curve (C'), Callendar's values. The agreement of the three curves is eminently satisfactory, and their similarity indicates that

TABLE XIII.

Constant-Volume Corrections for Nitrogen, $p_0=1000$ mm.

1	2	3	4
$t^\circ\text{C}$	Callendar, II	D. Berthelot	E. B.
-100	+0.080	+0.125	(-0.059)
+ 50	-0.0059	-0.0086	-0.0014
200	+0.035	+0.046	+0.024
400		+0.194	+0.139
450	+0.189		
600			+0.305
800		+0.56	+0.514
1000	+0.646	+0.77	+0.734
1200			+0.961

a considerable extrapolation by the same methods would be possible before the divergence became of any moment. When we consider that the experimental uncertainty of the melting point of gold at approximately 1065° is generally estimated to be about 2° , we see that the differences between the corrections as computed by the three methods are altogether insignificant.

As in the case of the constant-pressure corrections, it seems probable that we may regard the thermodynamic corrections of the centigrade, constant-volume, nitrogen scale as known with a precision far exceeding the precision as yet attainable in gas thermometry at these high temperatures.

18. CORRECTIONS OF THE CONSTANT-VOLUME HYDROGEN THERMOMETER.

The application to hydrogen of the method just used for nitrogen can not give satisfactory results because, as has been shown in §13, the constant-pressure corrections for hydrogen appear very uncertain. We shall merely give the differences between the constant-volume and constant-pressure scales as computed by the methods of §16 and §17.

Chappuis found from his experiments on hydrogen

$$\left[\frac{\partial}{\partial p} (pv) \right]_0 = +0.000762 ; \left[\frac{\partial}{\partial p} (pv) \right]_{100} = +0.000797$$

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If we set $\theta_0 = 273.06$, the equation for the compressibility of hydrogen at low pressures, as derived from the equation of Clausius, becomes

$$K_t = \frac{273.06}{273.06 + t} \left[0.0008363 - \frac{5.5410}{(273.06 + t)^2} \right] \quad (77)$$

When these coefficients, namely,

$$\frac{a}{R} = 5.5410$$

$$b = 0.0008363$$

are used as before to compute the value of the critical pressure from the equation of Clausius, on the assumption that the critical temperature is $\theta_c = 32^\circ$, we have

	Calculated	Observed
p_c (atmospheres)	13.2	13-19.4

Chappuis' experiments therefore indicate the lower value as the more probable, assuming the validity of the equation of Clausius.

TABLE XIV.

Difference between Constant-Volume and Constant-Pressure Scales for Hydrogen, $t_g - t_p$, $\pi = p_0 = 1000$ mm.

1	2	3	4	5
$t^\circ C$	D. Berthelot	Callendar, I	Callendar, II	E. B.
-240	+1.188			+1.176
-200	+0.190	+0.636	+0.237	+0.200
-150		+0.158	+0.072	+0.063
-100	+0.013	+0.046	+0.024	+0.014
- 50		+0.013	+0.006	
+ 50	-0.0006	-0.0022	-0.0012	-0.0007
+200	+0.002	+0.009	+0.007	+0.003
+300		+0.022	+0.016	+0.008
+400	+0.009			+0.011
+450		+0.042	+0.032	

By means of equations (66) and (77), values of $t_v - t_p$ have been computed for several temperatures. The results are shown in Table XIV. Column 1 gives the temperature; column 2, the values of $t_v - t_p$ according to Berthelot; column 3, the values according to Callendar's first method of computing; column 4, the values according to his second method; column 5, the values computed by the writer.

While the discrepancies between columns 2, 4, and 5 are large, in per cent, the absolute values are so small as to indicate that great experimental precision would be needed to detect any difference of run between the two scales except at the lowest temperatures. The fairly close agreement between Berthelot's values and those in column 5 is a necessary consequence of the approximate identity of the two equations of state upon which the computations were based.

19. CONCLUSION AND SUMMARY.

The time seems to be approaching when, in thermometry of precision, a mere reference of temperatures to "the gas scale" will be insufficient. The "normal scale" serves very well for low and moderate temperatures, but since it can not in practice be used at all for high temperatures we must soon, to secure uniformity of definition, adopt a different standard, and Lord Kelvin's scale is obviously the one indicated. This is especially true, because it is in fact already the only one in practical use at very high temperatures. For the radiation formulas which serve as a basis for all measurements of such temperatures are derived, in so far as they have any theoretical foundation, from the second law of thermodynamics, and the "temperature" which occurs in the equations used is therefore, by definition, the absolute thermodynamic temperature.

In the foregoing pages the attempt has been made, first, to set forth the relations of the actual gas scales to the thermodynamic scale in such simple form as to be readily available to anyone interested in the subject; second, by new computations based on the best available data, and by comparison with the results obtained by others, to show what our actual knowledge of the relation of existing gas scales to the thermodynamic scale is; and third, to point out, by the way, the particulars in which our experimental knowledge is most

deficient and the directions in which new experiments are most needed.

The contents of the paper may be summarized as follows:

(1) After a short statement of the principles of gas thermometry, with definitions of the scales used and of Lord Kelvin's thermodynamic scale (§ 1), the theory of the constant-volume and constant-pressure thermometers, as based on the Joule-Thomson effect, has been developed in the usual manner, emphasis being laid on the different nature of the experimental data most suitable for connecting the two scales with the thermodynamic scale (§§ 2-4).

(2) The best available experimental data on the specific volumes, specific heats, and coefficients of expansion have been quoted (§ 5).

(3) The numerical values of the Joule-Thomson effect have been examined (§ 6), and it has been shown that by the use of the law of corresponding states all the values may be coordinated and represented by a single empirical equation, from which, by an inverse process, values may be obtained which are more probably exact than any that can be deduced from the observations on each gas considered separately (§ 7). This empirical equation has been compared with the empirical equation of Rose-Innes, and with the equation of D. Berthelot, also obtained by the aid of the law of corresponding states (§ 8).

(4) The empirical equation for μ , as well as several simpler approximations, have been utilized in integrating the equations of the constant-pressure thermometer (§ 9). The thermodynamic temperature of the melting point of ice has been computed in several ways, and the resulting values compared with one another and with the values obtained by an altogether independent method, by D. Berthelot (§ 10).

(5) Values have been computed for the thermodynamic corrections of the constant-pressure hydrogen and nitrogen thermometers between the ice and steam points, and compared with the values computed by Rose-Innes, Callendar, and D. Berthelot (§ 11).

(6) Corrections for the same scales have been computed for larger ranges of temperature and compared with the values obtained by Callendar and D. Berthelot. The lack of concordance in the case of hydrogen indicates the need of further experiments on this gas. With nitrogen the agreement of the results of the different methods

of computation is satisfactory (§§ 12, 13). The corrections have been tested by means of the determinations of the boiling point of sulphur by Callendar and Griffiths, and by Chappuis and Harker, with satisfactory results. The centigrade thermodynamic temperature of this point appears to be close to $444^{\circ}9$ (§ 14).

(7) The relation of the constant-pressure and constant-volume scales has been discussed (§§ 15, 16). The differences of the two scales have been computed for nitrogen, and hence the thermodynamic corrections of the constant-volume nitrogen scale deduced from those of the constant-pressure scale already found. Comparison with the results of Callendar and D. Berthelot shows a very close agreement (§ 17). Similar computations and comparisons in the case of hydrogen show merely that the thermodynamic corrections of the constant-volume hydrogen thermometer are in most, if not all, cases negligible, the corrections, while uncertain, being small (§ 18).

WASHINGTON, February 4, 1907.

AN EXACT FORMULA FOR THE MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

By Louis Cohen.

In his paper on Absolute Standards of Inductance, Dr. J. G. Coffin¹ gave, without demonstration, an unpublished formula of Kirchhoff, for the mutual inductance of two symmetrically placed coaxial solenoids, with the remark that this formula is an exact one and very valuable. Although the calculation of inductance by this formula involves a great deal of labor, yet it seemed desirable to calculate by it the mutual inductances of some solenoids and compare the results with the values obtained by other formulæ which are only approximate, and thus get an estimate of the degree of approximation of the other formulæ. The results obtained, however, were disappointing, the values of the mutual inductance by this formula being absurd for every case calculated, thus proving the formula to be seriously in error. Feeling quite certain that Kirchhoff deduced a correct formula, which was probably corrupted in passing through several hands, Professor Rosa asked me to undertake to deduce the correct expression, and so ascertain, if possible, just where Kirchhoff's formula as given by Coffin is wrong.

Himstedt² in his paper on the determination of the ohm gave an exact formula for the mutual inductance of two coaxial coils, although he indicated but very briefly the deduction of this formula. It occurred to me that the formula may be modified so as to give the inductance of solenoids, and this in fact proved to be the case. Before proceeding, however, with the transformation of Himstedt's formula, it will be desirable to give a complete deduction of the formula, since Himstedt does not give the full deduction and the result is of considerable importance.

¹ This Bulletin, 2, p. 125; 1906.

² Himstedt, Pogg. Ann. 28, p. 338; 1886.

The mutual inductance between two circular currents is:

$$M = \iint \frac{\cos \epsilon \, ds \, ds_1}{r} \quad (1)$$

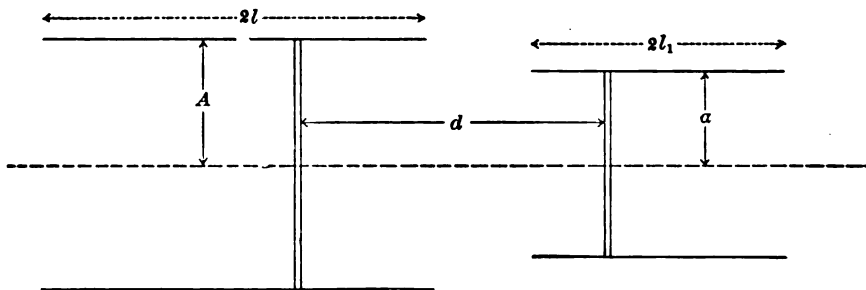


Fig. 1.

If the two circles are parallel and at a distance d from each other, then

$$r^2 = A^2 + a^2 + d^2 - 2Aa \cos(\phi - \phi_1)$$

$$\epsilon = \phi - \phi_1, \quad ds = ad\phi, \quad ds_1 = Ad\phi_1$$

A and a are the radii of the two circles and therefore,

$$M = \int_0^{2\pi} \int_0^{2\pi} \frac{Aa \cos(\phi - \phi_1) \, d\phi \, d\phi_1}{\sqrt{A^2 + a^2 + d^2 - 2Aa \cos(\phi - \phi_1)}}$$

which can be easily put in the following form:

$$M = 4\pi \int_0^\pi \frac{Aa \cos \psi \, d\psi}{\sqrt{A^2 + a^2 + d^2 - 2Aa \cos \psi}}$$

If $2l$ and $2l_1$ are the respective lengths of the two coils, d the distance between the centers of the two coils, x_1 and x_2 the distances of the centers of the coils from the origin, then

$$d = x_1 - x_2$$

and the mutual inductance of the two coils will be given by the following expression:

$$\begin{aligned}
 M &= nn' \int_{d-l}^{d+l} \int_{-l}^l M \, dx \, dx_1 \\
 &= 4\pi nn' \int_{d-l}^{d+l} \int_{-l}^l \int_0^\pi \frac{Aa \cos \psi \, d\psi \, dx_1 \, dx_2}{\sqrt{A^2 + a^2 + (x_1 - x_2)^2 + 2Aa \cos \psi}} \quad (3)
 \end{aligned}$$

n and n' are the number of turns per unit length of the two coils.

By partial integration with respect to ψ equation (3) will reduce to the following:

$$\begin{aligned}
 M &= 4\pi nn' \int_{d-l}^{d+l} \int_{-l}^l \left[\frac{Aa \sin \psi \, dx_1 \, dx_2}{\sqrt{A^2 + a^2 + (x_1 - x_2)^2 - 2Aa \cos \psi}} \right]_0^\pi \\
 &\quad + 4\pi nn' \int_{d-l}^{d+l} \int_{-l}^l \int_0^\pi \frac{(Aa)^2 \sin^2 \psi \, d\psi \, dx_1 \, dx_2}{[A^2 + a^2 + (x_1 - x_2)^2 - 2Aa \cos \psi]^{\frac{3}{2}}} \\
 &= 4\pi nn' \int_{d-l}^{d+l} \int_{-l}^l \int_0^\pi \frac{(Aa)^2 \sin^2 \psi \, d\psi \, dx_1 \, dx_2}{[A^2 + a^2 + (x_1 - x_2)^2 - 2Aa \cos \psi]^{\frac{3}{2}}} \quad (4)
 \end{aligned}$$

Integrating twice with respect to x_1 and x_2 and substituting the limits we get:

$$\begin{aligned}
 M &= 4\pi nn' \int_0^\pi \frac{(Aa)^2 \sin^2 \psi}{A^2 + a^2 - 2Aa \cos \psi} \left\{ \sqrt{A^2 + a^2 + (d+l+l_1)^2 - 2Aa \cos \psi} \right. \\
 &\quad - \sqrt{A^2 + a^2 + (d+l_1-l)^2 - 2Aa \cos \psi} \\
 &\quad - \sqrt{A^2 + a^2 + (d+l-l_1)^2 - 2Aa \cos \psi} \\
 &\quad \left. + \sqrt{A^2 + a^2 + (d-l-l_1)^2 - 2Aa \cos \psi} \right\} \quad (5)
 \end{aligned}$$

When the two coils are concentric, that is $d=0$, equation (5) will reduce to the following:

$$M = 4\pi n n' \int_0^\pi \frac{2(Aa)^2 \sin^2 \psi}{A^2 + a^2 - 2Aa \cos \psi} \left\{ \sqrt{A^2 + a^2 + c^2 - 2Aa \cos \psi} - \sqrt{A^2 + a^2 + c_1^2 - 2Aa \cos \psi} \right\} d\psi \quad (6)$$

where

$$c = l + l_1, \quad c_1 = l - l_1$$

$$M = 4\pi n n' (V - V_1) \quad (7)$$

where

$$V = \int_0^\pi \frac{2(Aa)^2 \sin^2 \psi \sqrt{A^2 + a^2 + c^2 - 2Aa \cos \psi}}{A^2 + a^2 - 2Aa \cos \psi} d\psi$$

and V_1 is obtained from V by putting c_1 in place of c . The evaluation of V can be obtained in the following manner:

Put

$$\phi = 2\psi$$

then

$$\begin{aligned} \frac{V}{8(Aa)^2} &= \int_0^\pi \frac{\sin^2 \phi \cos^2 \phi (A^2 + a^2 + c^2 - 2Aa + 4Aa \sin^2 \phi)}{(A^2 + a^2 - 2Aa + 4Aa \sin^2 \phi) \sqrt{A^2 + a^2 + c^2 - 2Aa + 4Aa \sin^2 \phi}} d\phi \end{aligned}$$

Now make the substitution $\cos \phi = x$ then,

$$\begin{aligned} V &= 8(Aa)^2 \int_0^1 \frac{x^2(1-x^2)\{(A+a)^2 + c^2 - 4Aax^2\}dx}{\{(A+a)^2 - 4Aax^2\}\sqrt{(A+a)^2 + c^2 - 4Aax^2}\sqrt{1-x^2}} \\ &= \frac{8(Aa)^2 w}{(A+a)^2} \int_0^1 \frac{x^2 - (1+k^2)x^4 + k^2 x^6}{(1+nx^2)\sqrt{(1-k^2 x^2)(1-x^2)}} dx \end{aligned}$$

where

$$k^2 = \frac{4Aa}{(A+a)^2 + c^2}, \quad n = \frac{-4Aa}{(A+a)^2}, \quad w^2 = (A+a)^2 + c^2$$

$$\begin{aligned}
 V &= \frac{8(Aa)^2 w}{(A+a)^2} \left[\frac{1}{n} \int_0^1 \frac{1+nx^2-1}{(1+nx^2)\sqrt{(1-k^2x^2)(1-x^2)}} dx \right. \\
 &\quad + \frac{1+k^2}{n^2} \int_0^1 \frac{1-n^2x^2-1}{(1+nx^2)\sqrt{(1-k^2x^2)(1-x^2)}} dx \\
 &\quad \left. + \frac{k^2}{n^3} \int_0^1 \frac{1+n^2x^2-1}{(1+nx^2)\sqrt{(1-k^2x^2)(1-x^2)}} dx \right] \\
 &= \frac{8(Aa)^2 w}{(A+a)^2} \left\{ \frac{1}{n} F - \frac{1}{n} \Pi + \frac{n(1+k^2)+k^2}{n^3} \int_0^1 \frac{1-nx^2}{\sqrt{(1-k^2x^2)(1-x^2)}} dx \right. \\
 &\quad \left. - \frac{1-k^2}{n^2} \Pi + \frac{k^2}{n} \int_0^1 \frac{x^4 dx}{\sqrt{(1-k^2x^2)(1-x^2)}} \right\}
 \end{aligned} \quad (8)$$

Now

$$\int_0^1 \frac{1-nx^2}{\sqrt{(1-k^2x^2)(1-x^2)}} dx = F + \frac{n}{k^2} E - \frac{n}{k^2} F \quad (9)$$

F, E, Π are the elliptic integrals of the first, second, and third kind.

It, therefore, remains only to evaluate the integral

$$\int_0^1 \frac{x^4 dx}{\sqrt{(1-k^2x^2)(1-x^2)}}$$

which can be done in the following manner:

$$\begin{aligned}
 \frac{d}{dx} \left\{ x \sqrt{(1-k^2x^2)(1-x^2)} \right\} &= \left\{ \frac{3k^2x^4}{\sqrt{(1-k^2x^2)(1-x^2)}} \right. \\
 &\quad \left. - \frac{2(1+k^2)x^2}{\sqrt{(1-k^2x^2)(1-x^2)}} + \frac{1}{\sqrt{(1-k^2x^2)(1-x^2)}} \right\} dx \\
 \therefore x \sqrt{(1-k^2x^2)(1-x^2)} \Big|_0^1 &= \int_0^1 \frac{3k^2x^4 dx}{\sqrt{(1-k^2x^2)(1-x^2)}} \\
 &\quad - \int_0^1 \frac{2(1+k^2)x^2 dx}{\sqrt{(1-k^2x^2)(1-x^2)}} + \int_0^1 \frac{dx}{\sqrt{(1-k^2x^2)(1-x^2)}}
 \end{aligned}$$

Hence

$$\int_0^1 \frac{x^4 dx}{\sqrt{(1-k^2 x^2)(1-x^2)}} = \frac{-2(1+k^2)}{3k^4} E + \frac{2+k^2}{3k^4} F \quad (10)$$

Introducing the values from (9) and (10) into equation (8), and collecting the terms containing F , E , and Π separately, we shall finally get

$$V = \frac{1}{2} w \left[F \left\{ (A-a)^2 + c^2 - \frac{3(A^2 - a^2)}{w^2} \right\} + E \left\{ 2(A^2 + a^2) - c^2 \right\} - \frac{3c^2(A-a)^2}{w^2} \Pi \right] \quad (11)$$

The elliptic integral of the third kind Π can be transformed into the complete and incomplete elliptic integrals of the first and second kind, by the following relation:³

$$\frac{k'^2 \sin \theta \cos \theta}{\sqrt{(1-\sin^2 \theta)(1-k^2 \sin^2 \theta)}} \left\{ \Pi - F \right\} = \frac{1}{2} \pi + F \{ F(k'\theta) - E(k'\theta) \} - E F(k'\theta) \quad (12)$$

where

$$k'^2 = 1 - k^2$$

θ is given by the relation

$$n = -1 + k'^2 \sin^2 \theta \quad (a)$$

$F(k', \theta)$ and $E(k', \theta)$ are the incomplete elliptic integrals of modulus k' and amplitude θ .

From equation (a) we get

$$\sin^2 \theta = \frac{n+1}{k'^2} = \frac{(A^2 - a^2)^2 + c^2(A-a)^2}{(A^2 - a^2)^2 + c^2(A+a)^2}$$

The value of Π as obtained from equation (12) is given by the following equation:

$$\Pi = F + \frac{(A+a)\sqrt{(A+a)^2 + c^2}}{c(A-a)} \left[\frac{\pi}{2} + F \left\{ F(k'\theta) - E(k'\theta) \right\} - E F(k'\theta) \right]$$

³ A. Cayley: *Elementary Treatise on Elliptic Functions*, § 183.

if now we introduce the value of Π from the above equations into equation (11) and simplifying, we shall finally get

$$\begin{aligned}
 V = & -(A^2 - a^2)c \left[F \left\{ F(k'\theta) - E(k'\theta) \right\} - E F(k'\theta) \right] \\
 & + \frac{c^2 - (A^2 - 6Aa + a^2)c^2 - 2(A^2 - a^2)^2}{3 \sqrt{(A+a)^2 + c^2}} F + \\
 & \frac{2(A^2 + a^2) - c^2}{3} \sqrt{(A+a)^2 + c^2} E - c(A^2 - a^2) \frac{\pi}{2} \quad (13)
 \end{aligned}$$

To recapitulate we have:

$$M = 4\pi n n' (V - V_1)$$

V is given by equation (13), and V_1 is obtained from V by replacing c by c_1

$$c = l + l_1, \quad c_1 = l - l_1$$

F and E are the complete elliptic integrals of modulus k

$$k^2 = \frac{4Aa}{(A+a)^2 + c^2}$$

$F(k', \theta)$ and $E(k', \theta)$ are the incomplete elliptic integrals of first and second kind of modulus k' and amplitude θ

$$k'^2 = 1 - k^2$$

$$\sin^2 \theta = \frac{(A^2 - a^2)^2 + c^2(A - a)^2}{(A^2 - a^2)^2 + c^2(A + a)^2}$$

Formula (13) is very similar in form to that given in Coffin's paper. The third term is exactly equal to the corresponding term in the Kirchhoff formula, and there is only a small difference in the first term. There is, however, a very great difference between the other two terms in Kirchhoff's formula. When the radii are equal the two formulæ will become identical, because the terms that involve the error then disappear. The test that Dr. Coffin applied to Kirchhoff's formula by deducing from it the self-inductance of a solenoid by making the radii equal is thus seen to be insufficient.

The accuracy of formula (13) is shown by numerical tests given in another paper.⁴

For the case of two solenoids of the following dimensions:

$$l=200, l_1=20. \quad A=15, a=10$$

the value of M given by formula (13) and confirmed by the formulæ of Roiti and of Searle and Airey is

$$M=4\pi nn' \times 6213.4.$$

The value of the mutual inductances for the same case as calculated by Kirchhoff's formula as given in Coffin's paper gives the following:

$$M=-4\pi nn' \times 21399$$

This value is more than three times the true value of M . If we entirely neglect the correction factor due to the end effects, the mutual inductance is:

$$M=4\pi^2 a^2 l_1 nn' = 4\pi nn' \times 6283.19,$$

a little more than 1 per cent greater than the true value of the mutual inductance.

The value of the self-inductance of solenoids can be easily obtained from equation (13), as was done by Coffin, by making $A=a$, $l=l_1$. We thus obtain

$$V = \frac{c^4 + 4A^2 c^2}{3 \sqrt{4A^2 + c^2}} F + \frac{4A^2 - c^2}{3} \cdot \sqrt{4A^2 + c^2} E$$

To obtain V_1 we replace c by c_1 , but in this case

$$c_1 = l - l_1 = 0$$

$$\therefore V_1 = \frac{8A^2}{3} E$$

⁴ This Bulletin, 3.

Since, however, $k^2 = \frac{4Aa}{(A+a)^2 + c^2} = 1$ in this case

$$\therefore E = \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 \phi} \, d\phi = 1$$

$$\therefore V_1 = \frac{8}{3} A^2$$

and

$$L = 4\pi n n' \left\{ \frac{c^4 + 4A^2 c^2}{3\sqrt{4A^2 + c^2}} F + \frac{4A^2 - c^2}{3} \sqrt{4A^2 + c^2} E - \frac{8A^3}{3} \right\} \quad (14)$$

WASHINGTON, *March 14, 1907.*

THE MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

By Edward B. Rosa and Louis Cohen.

Various formulæ for the calculation of the mutual inductance of coaxial solenoids have been given from time to time. Although few of these formulæ are exact, several of the approximate formulæ permit inductances to be calculated with very great accuracy by using a sufficient number of terms of the series by which they are expressed. We have collected a number of these formulæ and propose in this paper to compare and test them, and to give their derivations in some cases where the proofs that have been given are incomplete or wanting altogether.

1. MAXWELL'S FORMULA.¹

CONCENTRIC, COAXIAL SOLENOIDS OF EQUAL LENGTH.

Maxwell's demonstration² of this formula is very incomplete and difficult to understand. We shall give the derivation more fully, and extend the results. It is shown elsewhere³ in this Bulletin that the mutual inductance of the circle S_1 , of radius a Fig. 1, and the solenoid PQ of radius A , and which extends from P to infinity (OP being x), is

$$N_1 = 2\pi^2 a^2 n_1 \left[\left(1 - \frac{x}{(x^2 + A^2)^{\frac{1}{2}}} \right) - \frac{3}{8} \frac{a^2 A^2 x}{(x^2 + A^2)^{\frac{3}{2}}} - \frac{5}{64} \frac{a^4 (3A^4 x - 4A^2 x^3)}{(x^2 + A^2)^{\frac{5}{2}}} - \frac{35}{1024} \frac{a^6 (8A^4 x^3 - 20A^2 x^5 + 5A^6 x)}{(x^2 + A^2)^{\frac{7}{2}}} \right] \quad (1)$$

¹Electricity and Magnetism, II, § 678.

²Quoted by Coffin, this Bulletin, 2, p. 128; 1906.

³Rosa, p. 215, eq. (7).

This is the number of lines of force passing through the circle S_1 due to unit current in the infinite solenoid PQ , the latter having a winding of n_1 turns per cm.

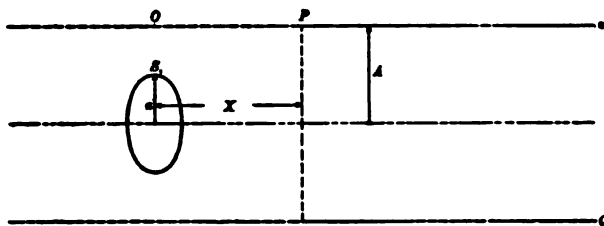


Fig. 1.

To find the number of lines of force due to PQ linked with all the turns of the solenoid RS , Fig. 2, the latter having n_2 turns per cm,

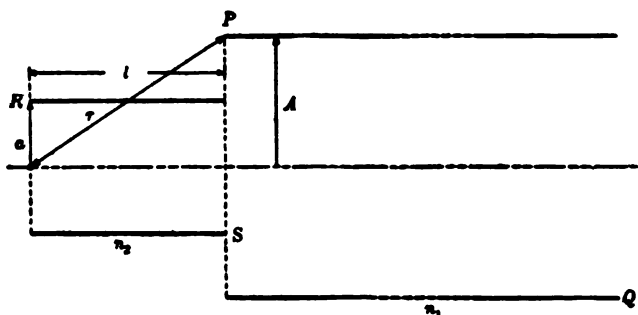


Fig. 2.

we must integrate equation (1) along the solenoid RS , from $x=0$ to $x=l$.

Thus, if

$$N = \int_0^l N_1 n_2 dx$$

$$\begin{aligned}
 N = 2\pi^2 a^2 n_1 n_2 & \left[x - (x^2 + A^2)^{\frac{1}{2}} \right. \\
 & + \frac{1}{8} \frac{a^2 A^2}{(x^2 + A^2)^{\frac{3}{2}}} + \frac{5a^4}{64} \left(\frac{A^4}{(x^2 + A^2)^{\frac{5}{2}}} - \frac{4}{5} \frac{A^2}{(x^2 + A^2)^{\frac{3}{2}}} \right) \\
 & + \frac{35a^6}{1024} \left(\frac{8}{7} \frac{A^2}{(x^2 + A^2)^{\frac{7}{2}}} - \frac{4A^4}{(x^2 + A^2)^{\frac{5}{2}}} + \frac{3A^6}{(x^2 + A^2)^{\frac{3}{2}}} \right) + \dots \left. \right]_0^l \quad (2)
 \end{aligned}$$

Inserting the limits and putting $r = \sqrt{x^2 + A^2}$

$$N = 2\pi^2 a^2 n_1 n_2 \left[(l - r + A) + \frac{a^2 A^2}{8} \left(\frac{1}{r^3} - \frac{1}{A^3} \right) + \frac{5a^4}{64} \left(\frac{A^4}{r^7} - \frac{4}{5} \frac{A^5}{r^5} - \frac{1}{5A^3} \right) \right. \\ \left. + \frac{35a^6}{1024} \left(\frac{8A^8}{7r^7} - \frac{4A^4}{r^5} + \frac{3A^6}{r^{11}} - \frac{1}{7} \frac{1}{A^5} \right) + \dots \right] \quad (3)$$

$$\therefore 2N = 4\pi^2 a^2 n_1 n_2 \left[(l - r + A) - \frac{a^2}{8A} \left(1 - \frac{A^3}{r^3} \right) - \frac{a^4}{32A^3} \left(\frac{1}{2} + 2 \frac{A^5}{r^5} - \frac{5A^7}{2r^7} \right) \right. \\ \left. - \frac{35a^6}{1024A^5} \left(\frac{1}{7} - \frac{8A^7}{7r^7} + \frac{4A^9}{r^5} - \frac{3A^{11}}{r^{11}} \right) + \dots \right] \quad (4)$$

$2N$ is the number of lines of force (due to unit current in the two infinite ends PQ and P'Q' of the larger solenoid, Fig. 3) passing through the short solenoid RS. If, however, the outer solenoid

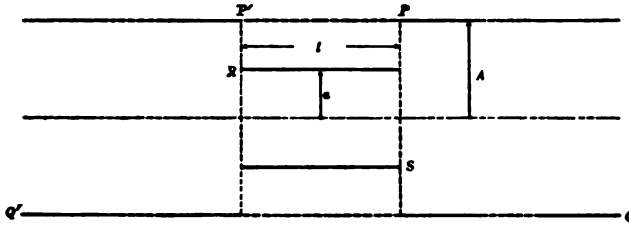


Fig. 3

were continuous and wound uniformly with n_1 turns of wire throughout, it would produce a uniform force within of $4\pi n_1$ when the current is one c. g. s. unit; and the number of lines linked with the $n_2 l$ turns of the inner solenoid would therefore be

$$N_1 = 4\pi n_1 \times \pi a^2 \times n_2 l = 4\pi^2 a^2 n_1 n_2 l. \quad (5)$$

The number due to the short solenoid PP' alone, that is, the mutual inductance of the two coaxial finite solenoids of length l , is $N_1 - 2N$, or

$$M = 4\pi^2 a^2 n_1 n_2 [l - 2Aa] \quad (6)$$

where

$$a = \frac{l - r + A}{2A} - \frac{a^2}{16A^3} \left(1 - \frac{A^3}{r^3} \right) - \frac{a^4}{64A^5} \left(\frac{1}{2} + 2 \frac{A^5}{r^5} - \frac{5A^7}{2r^7} \right) \\ - \frac{35a^6}{2048A^7} \left(\frac{1}{7} - \frac{8A^7}{7r^7} + \frac{4A^9}{r^5} - \frac{3A^{11}}{r^{11}} \right) + \dots \quad (7)$$

Putting

$$M = M_0 - \Delta M$$

M_0 is the mutual inductance of the infinite outer solenoid and the finite inner solenoid, while ΔM is the correction due to the ends. The number N given by equation (3) is $\Delta M \div 2$, the correction for one end.

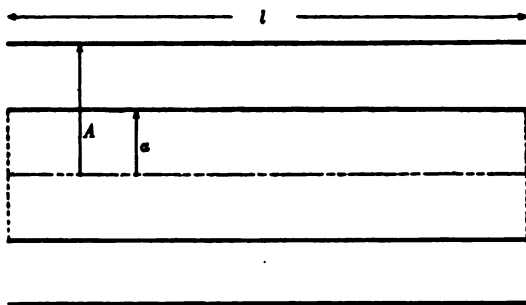


Fig. 4.

Equations (6) and (7) are Maxwell's expressions, except that (7) is here carried out further than the corresponding expression of Maxwell. This expression for M is rapidly convergent when a is considerably smaller than A , Fig. 4. Equation (6) shows that the mutual inductance is proportional to $l - 2Aa$; or the length l must be reduced by Aa on each end. When a is small and l is large, a is $1/2$ approximately. That is, the length l is reduced by A , the radius of the outer solenoid.

When the solenoids are very long in comparison with the radii, formula (7) may be simplified by omitting the terms in A/l , A^3/r^3 , A^5/r^5 , etc. Equation (7) then becomes

$$a = \frac{1}{2} - \frac{a^3}{16A^2} - \frac{a^5}{128A^4} - \frac{5a^7}{2048A^6} - \dots \quad (7a)$$

HEAVISIDE'S VARIATION OF MAXWELL'S FORMULA.

Heaviside gives a formula for the mutual inductance of two coaxial solenoids of equal length which differs in form from the above formula of Maxwell, and does not agree closely with it when applied to a particular case. Heaviside speaks of his formula as an extension of Maxwell's, but it is evidently derived in a somewhat

different manner.⁴ The main formula is the same as Maxwell's (6), the difference coming in the expression for a which, using A and a as the larger and smaller radii and $\rho = a/A$, is as follows:

$$\begin{aligned} 2a &= 1 - \frac{\rho}{8} \left(1 + \frac{\rho}{8} \left(1 + \frac{5\rho}{16} \left(1 + \frac{7\rho}{16} \left(1 + \frac{21\rho}{40} \left(1 + \frac{33\rho}{56} + \dots \right) \right) \right) \right) \right) \\ &= 1 - \frac{\rho}{8} - \frac{\rho^3}{64} - \frac{5\rho^3}{1024} - \frac{35\rho^4}{16384} - \frac{147\rho^5}{131072} - \frac{693\rho^6}{1048576} - \dots \end{aligned}$$

This formula for a is less exact than (7) but better than (7a). A numerical example is given below to test its accuracy.

2. RÖTTI'S FORMULA.

If the inner solenoid is shorter than the outer the limits of integration will be l_1 and l_2 , Fig. 5, when we integrate over the length l of the inner solenoid.

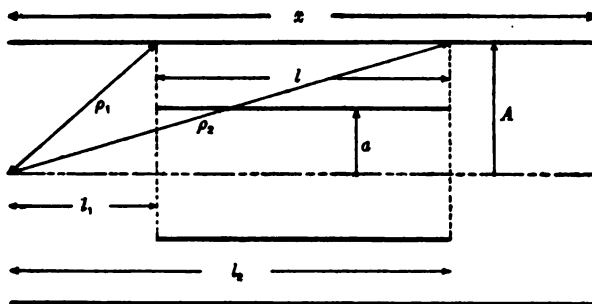


Fig. 5.

$$\begin{aligned} \text{Put } \sqrt{l_1^2 + A^2} &= \rho_1 & \text{where } l_1 &= \frac{x-l}{2} \\ \sqrt{l_2^2 + A^2} &= \rho_2 & l_2 &= \frac{x+l}{2} \\ l_2 - l_1 &= l \end{aligned}$$

⁴There is a misprint in Heaviside, 2, p. 277. The radius of the *inner* solenoid should be c_2 , of the *outer* c_1 , and ρ is c_2/c_1 .

Instead of (3) we shall have

$$N = 2\pi^2 a^2 n_1 n_2 \left[l - (\rho_2 - \rho_1) + \frac{a^2 A^2}{8} \left(\frac{1}{\rho_2^3} - \frac{1}{\rho_1^3} \right) + \frac{5a^4 A^2}{64} \left(\frac{A^2}{\rho_2^7} - \frac{A^2}{\rho_1^7} - \frac{4}{5\rho_2^5} + \frac{4}{5\rho_1^5} \right) + \frac{35a^6}{1024} \left(\frac{8A^2}{7} \left(\frac{1}{\rho_2^7} - \frac{1}{\rho_1^7} \right) - 4A^4 \left(\frac{1}{\rho_2^9} - \frac{1}{\rho_1^9} \right) + 3A^4 \left(\frac{1}{\rho_2^{11}} - \frac{1}{\rho_1^{11}} \right) \right) + \dots \right] \quad (8)$$

For both ends,

$$\Delta M = 2N = 4\pi^2 a^2 n_1 n_2 \left[l - (\rho_2 - \rho_1) + \frac{a^2 A^2}{8} \left(\frac{1}{\rho_2^3} - \frac{1}{\rho_1^3} \right) + \frac{5a^4 A^2}{64} \left(\frac{1}{\rho_2^7} - \frac{1}{\rho_1^7} \right) - \frac{a^4 A^2}{16} \left(\frac{1}{\rho_2^5} - \frac{1}{\rho_1^5} \right) + \frac{5a^6 A^2}{128} \left(\frac{1}{\rho_2^7} - \frac{1}{\rho_1^7} \right) - \frac{35a^6 A^4}{256} \left(\frac{1}{\rho_2^9} - \frac{1}{\rho_1^9} \right) + \frac{105a^6 A^4}{1024} \left(\frac{1}{\rho_2^{11}} - \frac{1}{\rho_1^{11}} \right) + \dots \right] \quad (9)$$

For an infinite outer cylinder and the given inner cylinder

$$M_0 = 4\pi^2 a^2 n_1 n_2 l = M + \Delta M$$

Subtracting ΔM as given by equation (9) from M_0 we have

$$M = 4\pi^2 a^2 n_1 n_2 \left[\rho_2 - \rho_1 + \frac{a^2 A^2}{8} \left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3} \right) - \frac{a^4 A^2}{16} \left(\frac{1}{\rho_1^5} - \frac{1}{\rho_2^5} \right) + \frac{5a^4 A^2}{64} \left(\frac{1}{\rho_1^7} - \frac{1}{\rho_2^7} \right) + \frac{5a^6 A^2}{128} \left(\frac{1}{\rho_1^7} - \frac{1}{\rho_2^7} \right) - \frac{35a^6 A^4}{256} \left(\frac{1}{\rho_1^9} - \frac{1}{\rho_2^9} \right) + \frac{105a^6 A^4}{1024} \left(\frac{1}{\rho_1^{11}} - \frac{1}{\rho_2^{11}} \right) + \dots \right] \quad (10)$$

The first part of this expression is Rødti's formula given by Coffin⁵ (without demonstration). As the proof of this formula has never been published, we have given the derivation above and have extended the formula so that it is now very accurate as well as very convenient. We give below an example to test and illustrate the formula.

⁵ This Bulletin, 2, p. 130; 1906. There is an error in one term of the formula in Coffin's paper. The second coefficient within the brackets should be $-\frac{a^4 A^2}{16}$ instead of $-\frac{a^4 A^4}{16}$.

3. GRAY'S FORMULA.

Gray⁶ gives a general expression for the mutual kinetic energy of two solenoidal coils which may or may not be concentric, and their axes may be at any angle ϕ . The most important case in practice is when the two coils are concentric and coaxial. In that case the zonal harmonic factors in each term reduce to unity, and half the terms become zero. Putting the current in each equal to unity the mutual kinetic energy becomes the mutual inductance M .

Let $2x$ = the length of outer solenoid

$2l$ = " " " inner "

A = radius of outer "

a = " " inner "

n_1 = Number of turns per cm on outer solenoid

n_2 = " " " " " inner "

Gray's expression with these changes becomes

$$M = \pi^2 a^2 A^2 n_1 n_2 [K_1 k_1 + K_2 k_2 + K_3 k_3 + \dots] \quad (11)$$

where K_1, K_2 , etc., are functions of x and A , and k_1, k_2 , etc., are functions of l and a .⁷ When the ratio of the length of the winding of the outer coil to the radius is $\sqrt{3}$ to 1, $K_3 = 0$, and if the same condition holds for the inner coil, $k_3 = 0$. If in addition a is considerably smaller than A , the terms of higher order become negligible and (11) reduces to

$$M = \frac{2\pi^2 a^2 N_1 N_2}{d} \quad (12)$$

where d is half the diagonal of the outer coil, $= \sqrt{x^2 + A^2}$, and the other letters have the meanings given above.

4. SEARLE AND AIREY'S FORMULA.

The following expression for the mutual inductance of two concentric and coaxial coils has been given by Searle and Airey.⁸

⁶ Absolute Measurements, 2, Part I, p. 274, equation 53.

⁷ Rosa, this Bulletin, 3, p. 221.

⁸ The Electrician (London), 56, p. 318; 1905.

$$\begin{aligned}
 M &= g_1 G_1 + g_2 G_2 + g_3 G_3 + g_4 G_4 + \\
 &= \frac{2\pi^2 a^3 N_1 N_2}{d} \left[1 - \frac{A^2}{2d^4} \cdot \frac{4l^2 - 3a^2}{4} \right. \\
 &\quad - \frac{A^2(4x^2 - 3A^2)}{8d^8} \cdot \frac{8l^4 - 20l^2 a^2 + 5a^4}{8} \\
 &\quad - \frac{A^2(8x^4 - 20x^2 A^2 + 5A^4)}{16d^{12}} \\
 &\quad \left. \frac{(64l^8 - 336l^6 a^2 + 280l^4 a^4 - 35a^8)}{64} - \dots \right] \quad (13)
 \end{aligned}$$

The notation of (13) differs slightly from that used by Searle and Airey, being the same as used above, Fig. 6. This is the same

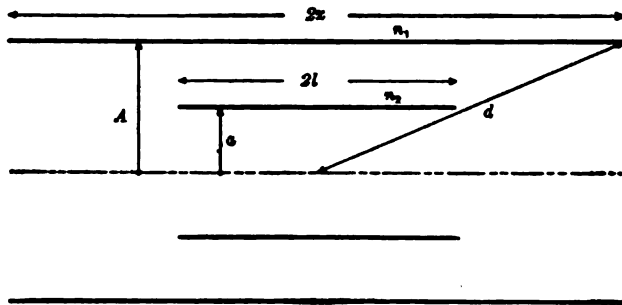


Fig. 6.

result that is obtained by substituting in (11) the values of K_1 , K_2 , K_3 , . . . and k_1 , k_2 , k_3 , . . . as given by Gray for the special case of concentric coils, and putting Z_1 , Z_2 , Z_3 , etc., each equal to unity.⁹ Equation (13) may be put for greater convenience in calculation in the following form:⁹

$$\begin{aligned}
 M &= \frac{2\pi^2 a^3 N_1 N_2}{d} \left[1 + \frac{A^2 a^2}{8d^4} L_2 + \frac{A^4 a^4}{32d^8} X_2 L_4 \right. \\
 &\quad \left. + \frac{A^6 a^6}{32d^{12}} X_4 L_6 + \frac{A^8 a^8}{32d^{16}} X_6 L_8 + \dots \right] \quad (14)
 \end{aligned}$$

⁹ Rosa, this Bulletin, 3, p. 224.

where

$$\left. \begin{aligned} X_2 &= 3 - 4 \frac{x^2}{A^2} \\ X_4 &= \frac{5}{2} - 10 \frac{x^2}{A^2} + 4 \frac{x^4}{A^4} \\ X_6 &= \frac{35}{16} - \frac{35}{2} \frac{x^2}{A^2} + 21 \frac{x^4}{A^4} - 4 \frac{x^6}{A^6} \\ L_2 &= 3 - 4 \frac{l^2}{a^2} \\ L_4 &= \frac{5}{2} - 10 \frac{l^2}{a^2} + 4 \frac{l^4}{a^4} \\ L_6 &= \frac{35}{16} - \frac{35}{2} \frac{l^2}{a^2} + 21 \frac{l^4}{a^4} - 4 \frac{l^6}{a^6} \\ L_8 &= \frac{63}{32} - \frac{105}{4} \frac{l^2}{a^2} + 63 \frac{l^4}{a^4} - 36 \frac{l^6}{a^6} + 4 \frac{l^8}{a^8} \end{aligned} \right\} \quad (15)$$

Examples are given below to test and illustrate these formulæ.

5. HIMSTEDT'S FORMULÆ.

Himstedt has given several formulæ for different cases of coaxial solenoids. The first¹⁰ is for the case of a short secondary on the outside of a long primary, Fig. 7. The formula is very complicated, and tedious to calculate. By putting the shorter coil inside, the formula of Rõiti or of Searle or Airey may be used to much better advantage.

Himstedt's second expression is for the case of two coaxial solenoids, the distance b , Fig. 8, between their mean planes having any value; but the radius of one is supposed to be considerably smaller than the other. This also is a very complicated formula, involving second and fourth derivatives of expressions containing the elliptic integrals F and E . Gray's general equation is much simpler to calculate. This is not, however, an important case in practice, and

Fig. 7.

¹⁰ Wied. Annalen, 26, p. 551; 1885.

we do not therefore give Himstedt's equation. Himstedt's third equation is general and applies to two coaxial solenoids of nearly equal or very different radii, and they may be concentric or not. A limiting case is when the two solenoids coincide, and the expression for M then becomes the self-inductance of a solenoid. This

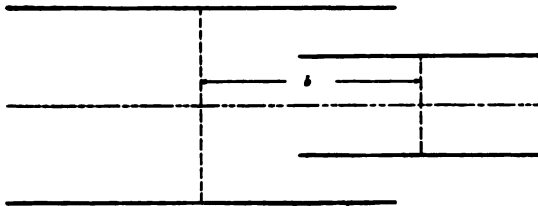


Fig. 8.

expression of Himstedt's consists of four terms, each of which is a somewhat complicated expression involving both complete and incomplete elliptic integrals. A less inconvenient general expression for M in elliptic integrals is given below.

6. COHEN'S FORMULA.

The general formula given by Coffin¹¹ as Kirchhoff's is seriously in error. Cohen has, however, deduced a general expression in elliptic integrals which gives correct results when tested by the formulæ of Ròiti and Searle and Airey; but as it is much more difficult to use in numerical calculations than the formulæ given above, it should only be employed where other formulæ are not sufficiently convergent, or to test other formulæ. Cohen's formula is as follows:

$$\left. \begin{aligned} M &= 4\pi n_1 n_2 (V - V_1) \\ V &= -(A^2 - a^2) c [F\{F(k', \theta) - E(k', \theta)\} - E F(k', \theta)] \\ &+ \frac{c^4 - (A^2 - 6Aa + a^2)c^2 - 2(A^2 - a^2)^2}{3\sqrt{(A+a)^2 + c^2}} F \\ &+ \frac{2(A^2 + a^2) - c^2}{3} \sqrt{(A+a)^2 + c^2} \cdot E - c(A^2 - a^2) \frac{\pi}{2} \end{aligned} \right\} \quad (16)$$

V_1 is obtained from V by replacing c by c_1

$$c = l + l_1 \quad c_1 = l - l_1$$

¹¹ This Bulletin, 2, p. 125, equations (33) and (34); 1906.

F and E are the complete elliptic integrals of the first and second kind to modulus k , where $k^2 = \frac{4Aa}{(A+a)^2 + c^2}$

$F(k', \theta)$ and $E(k', \theta)$ are the incomplete elliptic integrals of modulus k' and amplitude θ

$$k'^2 = 1 - k^2 = 1 - \frac{4Aa}{(A+a)^2 + c^2} = \frac{(A-a)^2 + c^2}{(A+a)^2 + c^2}$$

$$\sin^2 \theta = \frac{(A^2 - a^2)^2 + c^2(A-a)^2}{(A^2 - a^2)^2 + c^2(A+a)^2}$$

7. OTHER FORMULÆ.

The case of the mutual inductance of a solenoid and a coaxial circle within (the limiting case of coaxial solenoids when $l=0$) is discussed elsewhere in this Bulletin.¹² This case is important from its application in the Lorenz experiment and in the Ayrton-Jones absolute electro-dynamometer rather than as a method of obtaining a standard of mutual inductance. Formulæ have been proposed for coils of several layers; but as such coils can not be constructed so as to give very accurate results from their dimensions, the formulæ will not be given here.

8. NUMERICAL TESTS OF THE FORMULÆ.

EXAMPLE 1. MAXWELL'S FORMULÆ, (6) AND (7).

Two solenoids, Fig. 9, of equal length, 200 cm, each wound with a single layer coil.

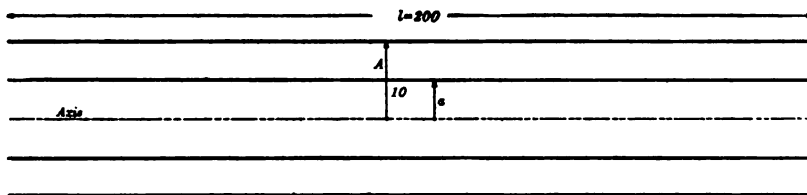


Fig. 9.

$A = 10 =$ radius of outer.

$a = 5 =$ " " inner.

¹² Rosa, this Bulletin, 8, p. 222.

Substituting in (7) for a we have the following:

$$\begin{aligned} a &= 0.487508 - \frac{1}{16} \frac{a^3}{A^3} (0.999875) - \frac{1}{64} \frac{a^4}{A^4} (0.500001) - \frac{35}{2048} \frac{a^5}{A^5} \left(\frac{1}{7}\right) \\ &= 0.487508 - .015610 - .000488 - .000038 \\ &= 0.471372 \end{aligned}$$

$$\therefore M = 4\pi^2 a^3 n^2 (200 - 9.42744)$$

$$\frac{M}{\pi^2 n^2} = 19057.25$$

$$\begin{aligned} \text{If } n &= 10 \text{ turns per cm, } M = \frac{100 \pi^2 \times 19057.25}{10^6} \text{ henry} \\ &= 0.018809 \text{ henry} \end{aligned}$$

By the approximate formula of Maxwell (7a)

$$\begin{aligned} 2a &= 1 - \frac{1}{8.4} - \frac{1}{64.16} - \frac{1}{1024.64} - \dots \\ &= 0.96773 \\ \therefore \frac{M}{\pi^2 n^2} &= 19032.27 \end{aligned}$$

This example by Heaviside's formula is as follows:

$$\begin{aligned} \rho &= \frac{5}{10} = \frac{1}{2} \\ 2a &= 1 - \frac{1}{16} - \frac{1}{64 \times 4} - \frac{5}{1024 \times 8} - \frac{35}{16384 \times 16} - \frac{147}{131072 \times 32} \\ &\quad - \frac{693}{1048576 \times 64} - \dots \\ &= 0.932805 \\ \therefore \frac{M}{\pi^2 n^2} &= 19067.08 \end{aligned}$$

It is thus seen that Maxwell's shorter formula for a gives M too small by 25 in 19000, whereas Heaviside's gives M too large by 10 in 19000.

To show that the result by Maxwell's formulæ (6) and (7) is very accurate for this case we may now calculate M by Cohen's absolute formula:

$$M = 4\pi n^2 (V - V_1)$$

Substituting in (17) for V we have the following terms:

$$\begin{aligned} V &= 7863.79 + 4200532.04 - 4169106.25 - 23561.95 \\ &= 15727.63 \end{aligned}$$

$$V_1 = 1392.18 - 632.16 = 760.02$$

$$\therefore M = 4\pi n^2 (15727.63 - 760.02)$$

$$\frac{M}{\pi^2 n^2} = 19,057.36$$

This agrees with the result by Maxwell's formula to within 1 part in 175000.

This shows that for such a case as this, Maxwell's formula (7) is much more accurate than Heaviside's extension of it. The latter is sufficiently convergent to give a more accurate result than it does, and does not agree with Maxwell's, no matter how long the solenoid be taken. Hence there is some assumption made or some quantity neglected in the derivation of Heaviside's formula that is not strictly correct, aside from the assumption of a long solenoid.

The example of Cohen's formula illustrates the disadvantage of that formula for numerical calculations. Aside from the fact that it is complicated, and involves the use of both complete and incomplete elliptic integrals, the value of M depends on the difference between very large positive and negative terms, so that in order to get a value of M correct to 1 part in 100000 it is necessary in the above example to calculate the large terms to 1 part in 2500000. As a means of testing other formulæ, however, this absolute formula is of great value.

EXAMPLE 2. RÒITI'S FORMULA (10) COMPARED WITH SEARLE AND AIREY'S (13).

We will now calculate the example, Fig. 10, given by Searle and Airey,¹³ by Ròiti's formula.

¹³ Electrician (London), 56, p. 319; 1905.

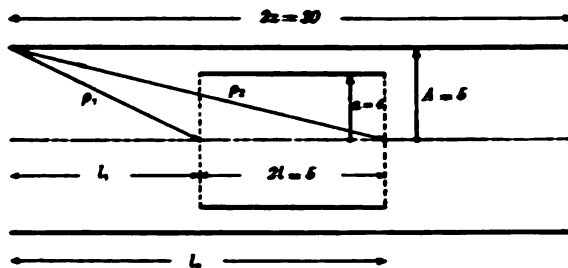


Fig. 10.

$2x = 30$ cm = length of outer solenoid.

$2l = 5$ " = " " inner "

$A = 5$ " = radius " outer "

$a = 4$ " = " " inner "

$$N_1 = 300 \text{ turns } \therefore n_1 = \frac{300}{30} = 10 \text{ per cm}$$

$$N_2 = 200 \text{ " } n_2 = \frac{200}{5} = 40 \text{ per cm}$$

$$l_1 = 12.5 \quad \rho_1 = \sqrt{12.5^2 + 25} = 13.462912$$

$$l_2 = 17.5 \quad \rho_2 = \sqrt{17.5^2 + 25} = 18.200275$$

$$\therefore \rho_2 - \rho_1 = 4.737363$$

$$\frac{A^2 a^2}{8} \left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3} \right) = + .012200$$

$$- \frac{A^4 a^4}{16} \left(\frac{1}{\rho_1^5} - \frac{1}{\rho_2^5} \right) = - .000704$$

$$\left[\frac{5A^4 a^4}{64} + \frac{5A^2 a^4}{128} \right] \left(\frac{1}{\rho_1^7} - \frac{1}{\rho_2^7} \right) = + .000181$$

$$- \frac{35A^2 a^6}{256} \left(\frac{1}{\rho_1^9} - \frac{1}{\rho_2^9} \right) = - .000022$$

$$+ \frac{105A^6 a^6}{1024} \left(\frac{1}{\rho_1^{11}} - \frac{1}{\rho_2^{11}} \right) = + .000002$$

$$\text{Sum} = 4.749020$$

$$4 \pi^2 a^2 n_1 n_2 = 25600 \pi^2$$

$$\therefore M = \frac{25600 \pi^2 \times 4.749020}{10^9} \text{ henry}$$

$$\text{or } M = .001199896 \quad "$$

Searle and Airey give $M = .00119990$ henry.
The difference is inappreciable.

EXAMPLE 3. GRAY'S FORMULA (12) COMPARED WITH RÒITI'S (10).

Let the winding be 20 turns per cm on each coil; $n_1 = n_2 = 20$.

$$A = 25 \text{ cm} \quad N_1 = n_1 A \sqrt{3}$$

$$\therefore N_1 N_2 = 3 n_1 n_2 A a$$

$$a = 10 \text{ cm} \quad N_2 = n_2 a \sqrt{3}$$

$$d = \sqrt{x^2 + A^2} = \frac{A}{2} \sqrt{7}$$

$$\therefore M = \frac{2\pi^2 a^2 N_1 N_2}{d} = 4\pi^2 a^3 n_1 n_2 \left[\frac{3a}{\sqrt{7}} \right]$$

$$M = .0179057 \text{ henry.}$$

We have also calculated the mutual inductance for these coils by Ròiti's formula (10).

The value is, $M = .0179058$ which is practically identical with the value by Gray's formula.

When $A = 25$ cm and $a = 10$ cm, $N_1 = 20a\sqrt{3} = 866.025$ and $N_2 = 20a\sqrt{3} = 346.4$. As there must be an integral number of turns, let us suppose the winding is exactly 20 turns per cm on each coil and the lengths therefore 43.3 cm and 17.3 cm, respectively. Then

$$d = \sqrt{x^2 + A^2} = \sqrt{625 + \left(\frac{43.3}{2}\right)^2} = 33.0715 \text{ cm.}$$

This does not exactly conform to the condition imposed in deriving the simple formula (12) of Gray used above. Hence (12) will not be as exact with these slightly altered dimensions, and we must calculate at least one correction term to get an accurate value of M .

$$\text{By Gray's formula (12), } M = \frac{2\pi^2 100 \times 866 \times 346}{33.0715 \times 10^9} = .0178842$$

The first correction term in (14) increases this value to .0178854 henry.

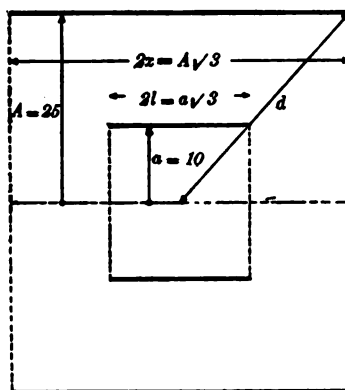


Fig. 11.

We will now calculate the mutual inductance of these coils by Rødti's formula (10)

$$\begin{array}{rclcl}
 A = 25 & 2x = 43.3 & l_1 = 13.0 \text{ cm} & \rho_1 = 28.17800 & \\
 a = 10 & 2l = 17.3 & l_2 = 30.3 \text{ " } & \rho_2 = 39.28218 & \\
 & & & \rho_2 - \rho_1 = 11.10418 & \\
 & & & \text{2nd term} = +.22030 & \\
 & & & \text{3rd} & = -.01781 \\
 & & & \text{4th} & = +.01952 \\
 & & & \text{5th} & = +.00156 \\
 & & & \text{6th} & = -.00453 \\
 & & & \text{7th} & = +.00274 \\
 & & & \text{Sum} & = 11.32596
 \end{array}$$

$$\begin{aligned}
 M &= \frac{4\pi^2 a^2 n_1 n_2 \times 11.32596}{10^9} \text{ henry} \\
 &= .0178853 \text{ henry.}
 \end{aligned}$$

This differs from the result by Gray's formula by only 1 part in 178000.

In taking the dimensions of coils where an accurate value of the mutual inductance is sought it should be borne in mind that the above formulæ have been derived on the supposition that the current is uniformly distributed over the length of the coaxial solenoids; in other words, these formulæ are all current-sheet formulæ. Hence for coils made up of many turns of wire we must meet the conditions imposed by current-sheet formulæ. In calculating self-inductances, this requires an accurate determination of the size of the wire and of the distance between the axes of successive wires, from which we can calculate two correction terms to be combined with the value of L given by the current-sheet formulæ.¹⁴

In the case of mutual inductances, however, there are no correction terms to calculate; but we must take the dimensions of the current sheets that are equivalent to the coils of wire. That is, the radius of each coil will be the mean distance to the center of the wire, and the length of each will be the over-all length, includ-

¹⁴ Rosa, this Bulletin, p. 181; 1906.

ing the insulation, when the coil is wound of insulated wires in contact, or the length from center to center of a winding of $N+1$ turns, where N is the whole number of terms used.¹⁵ It is also very important that the winding on both coils shall be uniform,¹⁶ and that the leads shall be brought out so that there shall be no mutual inductance due to them.

The mutual inductance will of course be different at high frequencies from its value at low frequencies. We assume, however, that for all purposes for which an extremely accurate mutual inductance is desired the frequency of the current would be low, say not more than a few hundred per second. If the value at very high frequency is desired the coil should be wound with stranded wire, each strand of which is separately insulated.

9. RUSSELL'S FORMULÆ.¹⁷

Since the above has been in type we have received the April number of *The Philosophical Magazine* containing Russell's article on *The Magnetic Field and Inductance Coefficients*.

Russell gives a number of formulæ for self and mutual inductance derived by original methods, and includes the case of coaxial cylinders. He derives the axial magnetic force at any point due to a cylindrical current sheet, and integrates this to get the number of lines of force due to one solenoid passing through the various turns of wire on the other, thus obtaining the mutual inductance of the two cylindrical coils considered as current sheets. The formula (41) obtained is as follows:

$$M = \pi b^2 \frac{N_1 N_2}{h_1 h_2} \left[R_1 \left\{ 1 - \frac{1}{2} q_1 k_1^2 - \frac{1}{2} \cdot \frac{1}{4} q_3 k_1^4 - \dots \right\} \right. \\ \left. - R_2 \left\{ 1 - \frac{1}{2} q_1 k_2^2 - \frac{1}{2} \cdot \frac{1}{4} q_3 k_2^4 - \dots \right\} \right] \quad (17)$$

In the notation of this article, where a and A are the radii of the inner and outer cylinders, respectively, $2l_1$ = length of the outer cylinder instead of $2h_1$ and $2l_2$ = the length of the inner instead of

¹⁵ Rosa, this Bulletin, p. 161, 1906; and p. 1, 1907.

¹⁶ Searle and Airey, *Electrician* (London), 56, p. 318; 1905.

¹⁷ Alexander Russell, *Phil. Mag.* Apr. 1907, p. 420.

$2h_2$, and n_1, n_2 are the number of turns of wire per unit of length on the two coils, respectively, we have

$$M = 4\pi^2 a^2 n_1 n_2 \left[R_1 \left\{ 1 - \frac{1}{2} q_1 k_1^2 - \frac{1}{2} \cdot \frac{1}{4} q_3 k_1^4 - \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} q_5 k_1^6 \right. \right. \\ \left. \left. - \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot \frac{5}{8} q_7 k_1^8 - \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \cdot \frac{5}{8} \cdot \frac{7}{10} q_9 k_1^{10} - \dots \right\} \right. \\ \left. - R_2 \left\{ 1 - \frac{1}{2} q_1 k_2^2 - \frac{1}{2} \cdot \frac{1}{4} q_3 k_2^4 - \text{terms with above coeffs.} \right\} \right] \quad (18)$$

where

$$R_1^2 = (A+a)^2 + (l_1 + l_2)^2 \quad k_1^2 = \frac{4}{R_1^2} \frac{Aa}{a}$$

$$R_2^2 = (A+a)^2 + (l_1 - l_2)^2 \quad k_2^2 = \frac{4}{R_2^2} \frac{Aa}{a}$$

$$q_n = \frac{(A+a)^2}{4Aa} q_{n-1} - \frac{1}{n} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \dots \frac{2n-3}{2n-2} \cdot \frac{A}{a}$$

$$q_1 = \frac{(A+a)^2}{4Aa} - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{A}{a}$$

$$q_3 = \frac{(A+a)^2}{4Aa} q_1 - \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{A}{a}$$

etc.

As a test of the formula Russell calculates the value of the mutual inductance of the two coaxial cylinders taken by Searle and Airey, and calculated above also by Ròiti's formula. Russell obtained the value $M = .0011995$ henry, which he supposed correct, evidently thinking that Searle and Airey's formula is less accurate than this own. There is, however, an error in Russell's work (the term $-.00100$ in the second part should be $-.001106$), so that when this error is corrected and each term carried out one more decimal place and two additional terms computed the value obtained for M is $.00119989$ instead of $.0011995$, as given by Russell. This value agrees with the results by the formulæ of Searle and Airey and Ròiti (page 318 above) to within 1 part in 100,000.

Russell's formula is thus seen to give an accurate result for this particular case, but it has serious limitations in other cases. Thus,

for two coils of equal length the second part of the formula is not convergent, and hence it must be replaced by an expression in elliptic integrals. The formula thus becomes (equation 42 in Russell's paper)

$$M = 4 \pi a^2 n_1 n_2 \left[R_1 \left\{ 1 - \frac{1}{2} q_2 k_1^2 - \frac{1}{8} q_3 k_1^4 - \dots \text{as above} \right\} \right. \\ \left. + \frac{8 \pi A a}{3(A+a)} n_1 n_2 [(A^2 + a^2)(F - E) - 2 A a F] \right] \quad (19)$$

This formula gives an accurate result for equal solenoids of considerable length, but Maxwell's formula (6) is just as accurate and much more convenient.

For short coils neither (18) or (19) will apply, but for that case as well as other cases Russell's general formula (38) may be used. There is a serious error, however, in this formula (38) introduced by a wrong value of the integral (5, p. 423) which should be

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 \phi \cos^2 \phi}{A} d\phi = \frac{2 - k^2}{3k^4} E - \frac{2 - 2k^2}{3k^4} F \quad (20)$$

Russell's formula (38) for the mutual inductance of two coaxial solenoids becomes, therefore, when corrected:

$$M = 4 \pi a b \frac{N_1 N_2}{h_1 h_2} \left[R_1 \left(1 - \frac{k_1^2}{c^2} \right) \left(\frac{1 - c^2}{c^2} \right) (F_1 - \Pi_1) \right. \\ + R_1 \left(\frac{1}{3k_1^2} - \frac{1}{c^2} + \frac{1}{3} \right) (F_1 - E_1) + \frac{1}{3} R_1 F_1 \\ - R_2 \left(1 - \frac{k_2^2}{c^2} \right) \left(\frac{1 - c^2}{c^2} \right) (F_2 - \Pi_2) \\ \left. - R_2 \left(\frac{1}{3k_2^2} - \frac{1}{c^2} + \frac{1}{3} \right) (F_2 - E_2) + \frac{1}{3} R_2 F_2 \right] \quad (21)$$

$$R_1^2 = (a+b)^2 + (h_1 + h_2)^2 \quad c^2 = \frac{4ab}{(a+b)^2} \\ R_2^2 = (a+b)^2 + (h_1 - h_2)^2 \quad k_1^2 = \frac{4ab}{(a+b)^2 + (h_1 + h_2)^2} \\ k_2^2 = \frac{4ab}{(a+b)^2 + (h_1 - h_2)^2}$$

F_1 and E_1 are the complete elliptic integrals to modulus k_1 , F_2 and E_2 the same to modulus k_2 , Π_1 and Π_2 are the complete elliptic integrals of the third kind, the values of which may be expressed in terms of complete and incomplete integrals of the first and second kind.¹⁸

N_1 and N_2 are the whole number of turns of wire on the two coils, respectively. $2h_1$ and $2h_2$ are the lengths of the two solenoids, of which a and b are the radii. We have given the equation in Russell's notation in order to avoid confusion in verifying the corrections we have made.

By substituting the values given above for c^2 , k_1^2 , k_2^2 , and Π_1 , Π_2 , this equation may be reduced to Cohen's formula (16) above. It is perfectly general for concentric, coaxial cylinders; but as already stated of (16) is a very tedious formula to employ in numerical calculations.

When the inner cylinder is considerably shorter than the outer, so that R_2 is much larger than $2\sqrt{Aa}$, formula (18) is sufficiently convergent to give very accurate values, although it may sometimes be necessary to calculate a good many terms. Thus, even for so favorable a case as the above problem of Searle and Airey, it is necessary to compute twelve terms to obtain as accurate a result as Searle and Airey's formula gives from three terms, and while Ròiti's is perhaps no more rapidly convergent than Russell's it is in simpler form to calculate. Russell's approximate formulæ (31) and (32) may be useful, but it is doubtful if they are enough simpler than the formulæ of Searle and Airey or Ròiti to compensate for the decreased accuracy.

Russell's paper is, nevertheless, an able and valuable one, and his derivation of the above formulæ, as well as many well-known formulæ by independent methods, is interesting and instructive.

WASHINGTON, March 30, 1907.

¹⁸ Cayley, *Elliptic functions*, p. 139, second edition.

THE PRODUCTION OF HIGH FREQUENCY OSCILLATIONS FROM THE ELECTRIC ARC.

By L. W. Austin.

The first experiments for the production of electrical oscillations from direct currents appear to have been made by Prof. Elihu Thomson,¹ who in 1892 proposed the following method for transforming continuous currents into continuous trains of high frequency oscillations. A source of direct current at 500 volts was connected through high inductive resistances to a spark gap, which was shunted by a condenser and inductance. An air blast or magnet was used to blow out the continuous current arc.

Mr. W. Duddell² in 1900 described experiments showing that when a suitable inductance and capacity are placed in shunt around an ordinary direct current arc a musical tone is given out. According to Duddell the following conditions must be fulfilled in order that the oscillations be satisfactorily produced. The carbons must be solid, not cored; the arc must be of suitable length (1–2 mm); and the current must be of suitable strength (3–5 amp.). In addition, a high resistance must be placed in the main arc circuit and the ohmic resistance of the shunt circuit around the arc should be in general less than 2 ohms. About 5 millihenrys and from 1 to 5 microfarads form a suitable shunt circuit.

The cause of this phenomenon has not been made entirely clear; but according to Duddell's theory, the production of oscillations requires that the V, I curve of the arc should have a falling characteristic; that is, that dV/dI should have a negative value and be greater numerically than r ; V represents the potential difference

¹ U. S. Patent No. 500630; July 4, 1892.

² W. Duddell, Journ. Inst. Elec. Eng., 30, p. 232; 1900.

across the arc, I the direct current, and r the resistance in the shunt circuit around the arc. The simple explanation of the formation of the oscillations according to this theory is as follows. When the shunt is closed on such a circuit as has been described, a part of the current flows into the condenser, thus robbing the arc of a part of its current; but since V increases as the current in the arc decreases, this really increases the potential difference, and the condenser continues to charge. When in turn it discharges through the arc it increases the arc current, and thus decreases the potential difference until entirely discharged. The process then repeats itself.

According to Duddell the frequency of the oscillations in the singing arc depends only on the inductance and capacity in the shunt circuit except as it is modified by the damping, and the frequency is limited by the fact that dV/dI ceases to be negative at a moderately high frequency. Good results may, however, be obtained between 500 and 10,000 per second by a proper adjustment of inductance and capacity.

This theory has been treated mathematically by P. Janet,³ who has arrived at conclusions agreeing with those of Duddell. A second theory of the phenomenon has been advanced by Maisel,⁴ who claims that the sign of dV/dI in the arc is of no importance, but that the occurrence of oscillations depends only on the temperature of the negative electrode. He holds that as the current is withdrawn from the arc by the charging of the condenser the temperature of the electrode falls until the arc is extinguished, being relighted again by the backward surge of the charge from the condenser the next instant, provided the temperature has not fallen too low. He contends that, according to his own observations and those of Corbino,⁵ the current in the shunt circuit is not sinusoidal, and that the frequency can not be determined simply by the formula $1/n = 2\pi\sqrt{CL}$, and also that it has been shown by Salomonson⁶ that a frequency as high as 400,000 may be produced.

³P. Janet, *Comptes Rendus*, 184, p. 821; 1902.

⁴S. Maisel, *Physikal. Zs.*, 5, p. 550; 1904.

⁵M. Corbino, *Atti dell' Assoc. Elett. Ital.*, 7, p. 369, also p. 597; 1903.

⁶Wertheim Salomonson, *Konink. Akad. Wetensch. Amsterdam, Versl.*, p. 381; 1902-3.

In 1902 Prof. R. A. Fessenden⁷ proposed a spark gap or arc fed by continuous current as a source of radiation for wireless telegraphy and afterwards developed the idea, calling attention to the advantages of using high potential and forming the arc under pressure.

H. Th. Simon, and Simon and Reich⁸ have also experimented extensively on this subject, using metal electrodes as well as carbon; and more recently Simon⁹ has published a theory of the phenomenon based on his own theory of the arc. In this he agrees with Duddell and Janet as to the necessity of a falling characteristic for the arc curve, and concludes that high frequency oscillations can probably best be obtained by using high voltage and low current and electrodes, which are good conductors of heat like the metals.

Since the present work was begun, V. Poulsen¹⁰ has announced the successful production of oscillations ranging as high as one million per second and of considerable power, which have been used successfully in wireless telegraphy. In the Poulsen arc the anode is copper and the cathode carbon, while the arc is formed in an atmosphere of hydrogen which appears considerably to increase the intensity of the oscillations.

ARRANGEMENT OF APPARATUS.

During the first portion of the work the source of current was for the most part a 240-volt constant potential circuit, though at times 120 volts were substituted for this. The current was cut down by a variable resistance and the primary of a small transformer was introduced into the main circuit as a choking coil. A diagram of the connections is shown in Fig. 1. Here A is a d. c. ammeter, B a d. c. voltmeter across the arc, H a hot wire ammeter reading to 10 amp. with a resistance of about 0.3 ohm. The capacity C consisted of one or two small mica condensers of 0.04 mf capacity each. The inductance L, generally used in the oscillatory circuit, consisted of three turns of No. 16 wire wound on a battery jar. The

⁷U. S. Patents No. 706742, Aug. 12, 1902; No. 730753, June 9, 1903; No. 793649, July 4, 1905.

⁸Simon and Reich, *Physikal. Zs.*, 8, p. 278; 1902; 4, p. 364; 1903.

⁹H. Th. Simon, *Physikal. Zs.*, 7, p. 433; 1906.

¹⁰V. Poulsen, *Electrotech. Zs.*, 27, p. 1040; 1906.

inductance of this coil and the connecting wires amounted to 0.009 millihenry approximately. The frequency of the oscillations was

always measured by the resonance method in a secondary circuit so loosely coupled to the primary that the reaction between the circuits was negligible, as shown by special experiments. This secondary circuit consisted of a variable air condenser C' , a hot wire ammeter H' reading to 0.23 amp. and having a resistance of 7 ohms, a coupling inductance L' of 25 turns of No. 16 wire wound on a jar, and sometimes a variable roller inductance in addition.

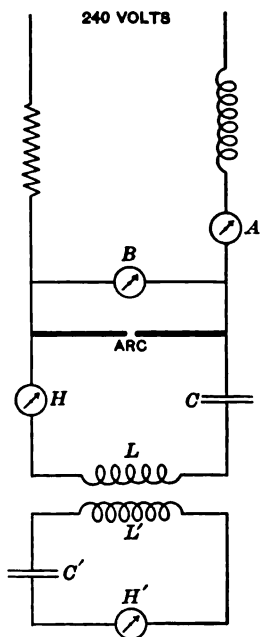


Fig. 1.

Below are given the data of a typical experiment:

PRIMARY CIRCUIT.

Electrodes	graphite, ends flat
Diameter	12 mm
Resistance in series with arc	about 50 ohms
Arc current	4 amp.
P. D. across arc	26 volts
P. D. open circuit	242 volts
Arc length	about 0.3 mm
Inductance in shunt circuit	0.009 millihenry
Capacity	0.4 ohm
Alternating current in shunt	4.0 amp

SECONDARY CIRCUIT.

Inductance
Capacity
Resistance
Current

0.168 millihenry
variable
7 ohms
0.1-0.2 amp

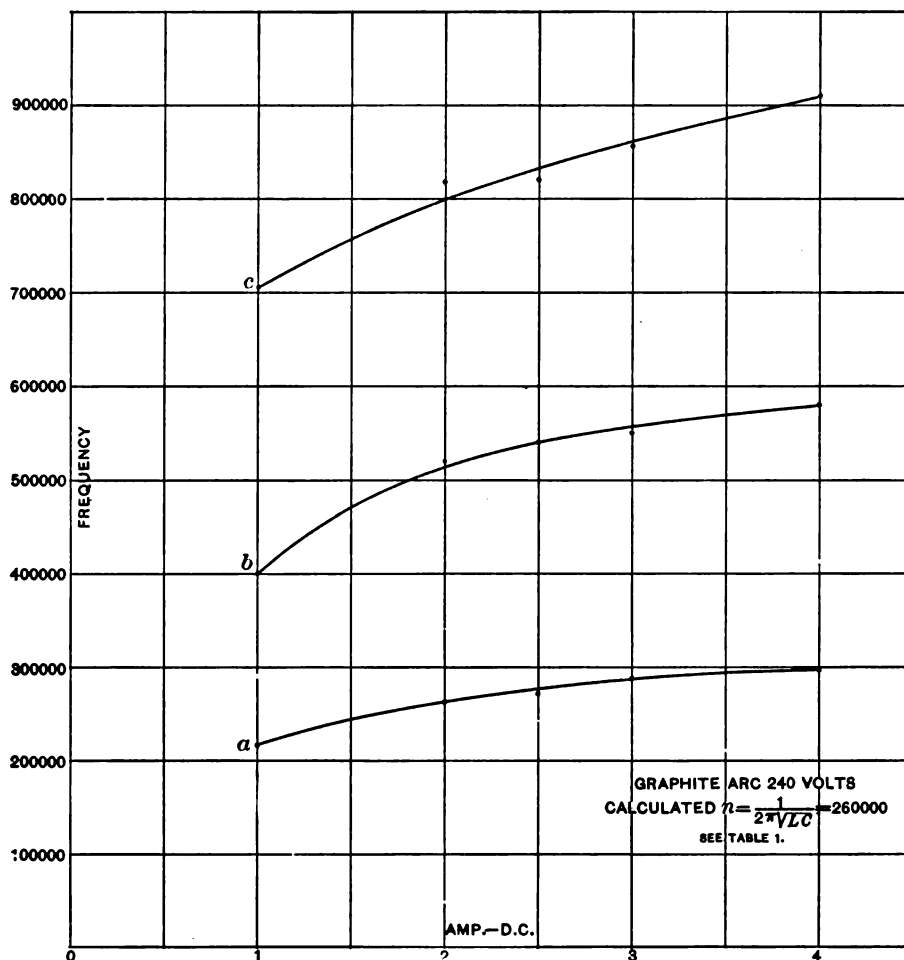


Fig. 2.

It was found that under the above conditions the high frequency oscillations were readily obtained and were very constant, sometimes running for half an hour without attention before the oscillations disappeared. When the frequency was measured by the secondary

resonance circuit it was discovered that there were well-marked maxima corresponding to frequencies of 295,000, 580,000, and 910,000 per second. The lowest frequency appears to carry the largest part of the energy, while the highest was the weakest. The frequency calculated from the inductance and capacity of the primary circuit was 260,000. It seems possible that there may be other frequencies present of even shorter period and less intensity, but I was unable to detect them.

Change of Frequency.—According to Simon's results with low frequency arcs the frequency should increase with increasing cur-

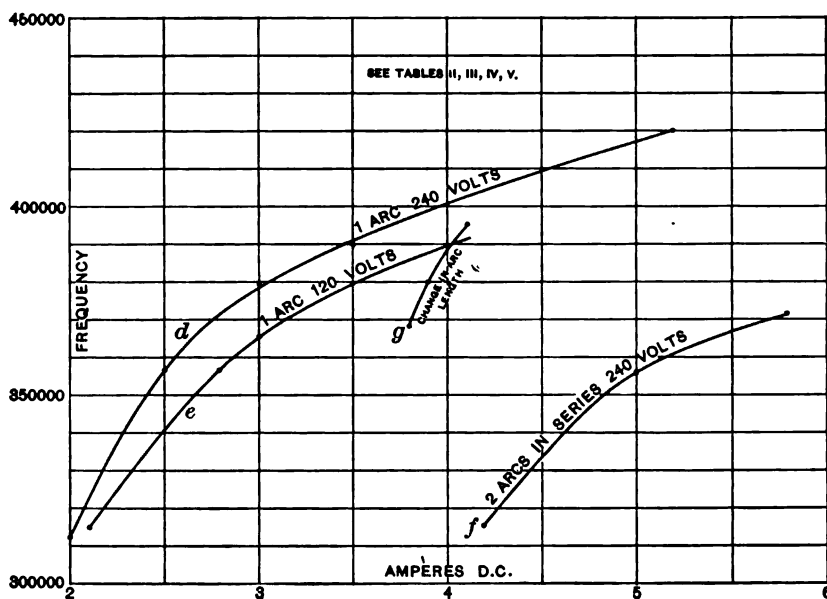


Fig. 3.

rent and with decreasing arc length. These points were investigated, and the results are shown in Tables I and II and in the curves of Figs. 2 and 3. Table I and the curves *a*, *b*, *c*, Fig. 2, show the effect on the three maxima due to changing the resistance in the main circuit. It is to be noted that the lower frequency changes less than the other two, and that the three frequencies roughly approximate to the ratios 1:2:3. It was not considered possible to measure the change in arc length with any accuracy, as the total range of length over which the oscillations could be obtained was

only a few tenths of a millimeter. Table II and curve *g*, Fig. 3, therefore show only the relation between the change in frequency of the second maximum and the current change produced by changing the arc length. Curve *g* shows that the frequency increases with decreasing arc length much more rapidly than would be the case if the change were entirely due to the resulting change in current, showing that the shortening of the arc produces by itself an increase in frequency.

Experiments were also made on the effect of using the 120-volt circuit instead of the 240 volt, and also on the effect of putting two arcs in series. The capacity in the primary circuit in these experiments and in the one shown in Table II was 0.08 mf. The results are shown in Tables III, IV, V, and in the curves of Fig. 3. It is seen that the curves for one arc at 120 volts and 240 volts lie near together. The observed difference may be due to a slight difference in the adjustment of the arc length. The introduction of a second arc in series with the first produced a marked lowering in the frequency while the alternating current rose from 5 to 7 amp.

Resistance in the Shunt Circuit.—Under the conditions given on page 328, adding an ohm to the resistance of the shunt circuit destroyed the oscillations entirely, while one-fourth ohm reduced them about one-half. No change in frequency as far as could be observed attended the changes in resistance and alternating current in the shunt oscillatory circuit.

In connection with the graphite arc some experiments were made in wireless telephony. A wire was connected to the secondary oscillatory circuit and run up outside the window of the laboratory while the other side of the secondary was earthed. The current in the antenna was about 0.1 amp. A microphone was inserted in the antenna, and when this was spoken into the words could be understood in a telephone connected to an electrolytic wireless receiver attached to a similar antenna wire on a neighboring building about 50 meters distant. The articulation was loud and fairly distinct; but the hissing noises from the arc were disturbing. This arrangement was of course merely experimental. By making better use of the energy and using proper aerials and microphone a much greater distance could have been covered, probably several miles.

THE ARC IN HYDROGEN.

According to Poulsen much more powerful oscillations can be produced with the arc in hydrogen than can be obtained in air. This point was tested in a simple but effective way suggested by Poulsen by forming the arc in the interior of a gas flame. It was found at once that the high frequency oscillations formed much more readily with the carbon and carbon-copper electrodes than in air. The best results were obtained using copper anode with carbon cathode or with both electrodes of graphite.

While the pressure of the hydrogen enabled the oscillations to be produced more easily and to be maintained with greater arc length (up to about 0.8 mm in the case of graphite at 240 volts), the most remarkable effect was the increase in the energy which could be drawn from the oscillating circuit. As has been before mentioned, no resistance of more than one or two ohms could be inserted directly in the oscillatory circuit without destroying the oscillations. Large resistances can, however, be placed in the secondary coupled circuit and in this way power can be derived from the primary circuit. With the graphite arc in air at 240 volts only about six or eight watts can be absorbed in this way. With the arc in hydrogen supplied in sufficient quantities this power may amount to more than 100 watts, in one experiment bringing a 110-volt 32-cp lamp to full brightness.

Below are given the data of an experiment with the arc in hydrogen which may be compared with the data on page 327 when the arc was in air

PRIMARY CIRCUIT.

Direct current without oscillations	5 amp
Direct current with oscillations	2.5 amp
Potential difference without oscillations	40 volts
Potential difference with oscillations	90-100 volts
Potential difference, open circuit	240 volts
Alternating current in primary	9 amp
Capacity	0.08 mf

SECONDARY CIRCUIT.

Inductance	0.168 millihenry
Capacity	0.00011 mf
Resistance	110 v. 32-cp lamp
Alternating current	1 amp

The most noticeable thing in the above experiment, after the marked increase in power developed, is the marked decrease in direct current and increase in fall of potential across the arc. This depends on the amount of load in the secondary, and with the secondary removed the differences would have been less. The same changes can be noted in the graphite arc in air, but then the current drops only about 0.1 amp. and voltage increases but a few volts.

The presence of the hydrogen, at least when poured in a stream directly into the arc, as in the present case, has a very detrimental effect on the sharpness of tuning. The maxima frequently become so unsteady that it is almost impossible to determine the frequency. This may be due to the blowing about of the arc by the gas stream. Very peculiar pressure conditions certainly exist in the arc as is shown by the remarkable way in which the gas flame in which the arc is formed is blown outward when the oscillations are taking place.

The cause of the beneficial effect of hydrogen in the arc is not entirely clear. It seems certain that it is partly due to the high heat conductivity of the gas which helps to cool the electrodes between the surges of the current. According to Poulsen, in addition to the cooling effect, there is a marked influence on the electrical conductivity of the arc which he considers of great importance; but this is a question which will require further investigation before conclusions can be reached.

THE ARC IN STEAM.

Experiments were also made in which the arc was formed in steam. The graphite electrodes were used in some cases and in others the copper anode and carbon cathode. It was found possible to duplicate all of the effects observed with the arc in hydrogen. Oscillations were obtained as readily and with approximately the same power. Oscillations of great power were also observed when the arc was formed under water, where the steam was produced by the heat of the discharge. In this case frequently cooling became too rapid, so that the arc blew itself out.

THE HIGH POTENTIAL ARC IN COMPRESSED AIR.

In the following experiments I have made use of the ideas of Fessenden¹¹ and Simon¹² in putting the arc under pressure, using a

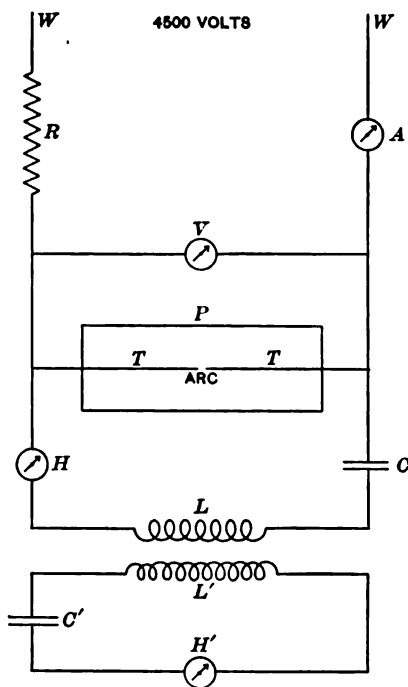


Fig. 4.

compressed air condenser having a capacity of 0.0044 mf, L the three turn jar inductance of 0.009 millihenry and R the 30,000 ohm series resistance.

The leads W W run to the terminals of the set of small dynamos. The air chamber P is of massive iron with removable ends and with small windows of heavy glass at the front and back for observing the arc. The electrodes proper E were thin caps of metal, shaped as shown in Fig. 5, and screwed down by nuts and rubber washers to the ends of the electrode tubes T. These were hollow and arranged for water circulation. The electrode tubes were introduced into the pressure chamber through tightly fitting

high potential circuit, small current, and metallic electrodes. The source of current was a set of ten 500-volt direct current dynamos in series driven by a 2-hp motor. A fixed resistance of about 30,000 ohms was in series with the machines, and on short circuit the direct current was 0.15 amp. The voltage on open circuit, as measured on a Braun electrostatic voltmeter, was 4,500 volts.

In Fig. 4 the general arrangement of the circuits is shown. Here T is the compressed air, arc, A a d. c. ammeter, V a Braun electrostatic voltmeter, C a Fessenden com-

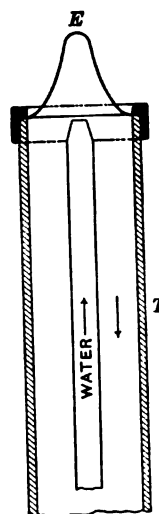


Fig. 5.

¹¹ Fessenden, l. c.¹² Simon, l. c.

brass sleeves, the space between the two being filled with mutton tallow to make an air-tight joint. The sleeves were insulated from the pressure chamber by hard rubber rings. Air pressures up to about seven atmospheres were obtained by means of a double cylinder automobile pump.

The observations below give the complete data of a typical experiment:

Air pressure in arc	6.8 atm
Electrodes	silver, 0.8 mm thickness
Direct current without oscillations	0.15 amp
Direct current with oscillations	0.11 amp
Potential difference of arc without oscillations	90 volts
Potential difference of arc with oscillations	2,000 volts
Potential difference, open circuit	4,500 volts
Arc length	about 0.4 mm
Inductance in shunt circuit	about 0.009 millihenry
Capacity	0.0044 mf
Alternating current	11.5 amp
Frequency	850,000

The most noticeable thing in these observations is the remarkable rise in arc potential difference when the oscillations occur. It is evidently a more marked example of the phenomenon described above with the arc in hydrogen. It is also seen that the oscillations cause the direct current to drop in the same way that was observed in the low potential arc. It has always been found to be one of the characteristics of the singing arc, and also of the low potential, high frequency arc, that no resistance of more than an ohm or two can be introduced into the shunt oscillatory circuit without causing the oscillations to cease. In the high potential arc this is not the case. Resistances amounting to several hundred ohms may be introduced without any detrimental effect.

The question of the presence of other frequencies than the fundamental was investigated by means of the secondary resonance circuit, but none was found. Unlike the low potential arc this arc showed no change of frequency when the direct current was reduced by more than half by increasing resistance. There was also no change in frequency when the arc length was varied.

Efficiency of the Oscillations.—A 110-volt 100-cp lamp was placed in the shunt oscillatory circuit, and at a frequency of 650,000 an alternating current of 2.1 amp. was observed. By means of direct-current observations this was determined to repre-

sent a consumption of about 200 watts. As the available power, after subtracting that absorbed in the series resistance, amounted to but a little over 300 watts, the efficiency was more than 60 per cent.

Experiments were also made at higher frequencies by the same method, 180 watts being obtained at 1,500,000 and 140 watts at 3,500,000. At about 4,500,000 the oscillations became irregular and the observations were not pushed further.

Effect of Air Pressure.—Figure 6 shows the remarkable increase in the oscillations produced by the increased air pressure. Starting with the apparatus and data indicated in the typical experiment, page 328, already cited, the air pressure was gradually decreased

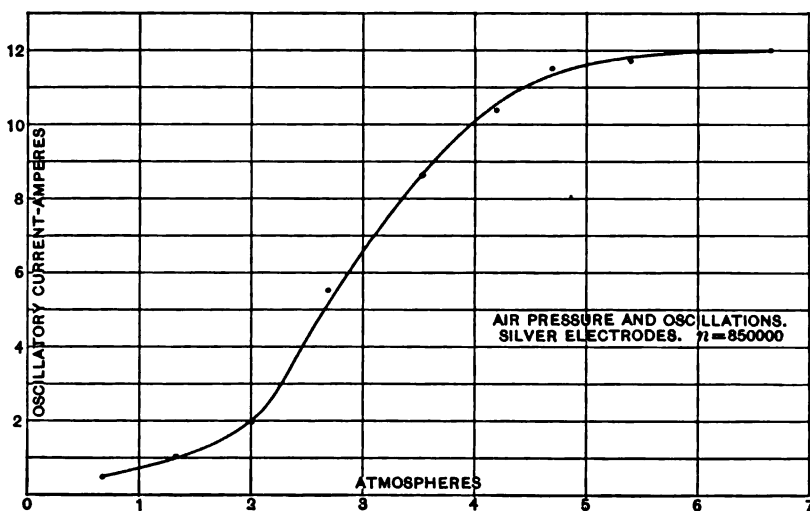


Fig. 6.

with the result shown in the figure. Above 5 atm. little change takes place, but below this the oscillations fall off rapidly and become much less steady. Below 2 atm. this unsteadiness becomes so great that no accurate readings can be taken.

During the work several different metals were tried as electrodes. Of these copper gave results nearly as good as silver, while zinc, aluminum, and platin-iridium (10 per cent Ir.) were all much poorer. This seems to indicate that metals with the higher heat conductivities are best suited for the purpose.

Continuity of Wave Trains.—It has been generally assumed that oscillations formed from the direct current arc are undamped, i. e.,

that the wave trains are continuous. While this is undoubtedly the case at low frequencies (singing arc), there is a point above which the arc breaks up into successive sparks, probably depending on the magnitude of the direct current. The high potential arc in compressed air is certainly not continuous. Using the circuit, the data for which are given on page 14, the damping coefficient γ , measured by Drude's method, amounted to about 0.1 per period. We have in this case simply a spark discharge with the sparks following each other very rapidly. Attempts were made to determine whether the waves were undamped in the case of the low potential graphite arc at frequencies of 300,000 and 100,000. The observations indicated that damping existed; but the constant slight shifting back and forth of the frequency due to changes in the arc might very well have stimulated this effect. Therefore, the indications can not be considered conclusive.

SUMMARY OF RESULTS.

1. The experiments show in agreement with Salomonson ⁶ *that the ordinary singing arc with carbon, or better, graphite electrodes is capable of producing oscillations of several hundred thousand per second.*

WITH THE GRAPHITE ARC.

2. The oscillations as investigated in a secondary resonance circuit show three frequencies of decreasing intensity corresponding approximately to the first three harmonics. This shows that the wave is not only not sinusoidal, but not even symmetrical.

3. The frequency increases with increasing direct current. Figs. 2 and 3.

4. The frequency increases with diminishing arc length. Fig. 3.

5. Putting arcs in series reduces the frequency. Fig. 3 (equivalent to increasing arc length).

6. The frequency does not, as far as has been observed, vary with the intensity of the oscillations.

7. No considerable resistance can be placed in the shunt oscillatory circuit without destroying the oscillations. Resistance may, however, be introduced into the secondary coupled circuit, thus giving means of obtaining power.

8. The power is much increased when the arc is formed in hydrogen or steam.

WITH HIGH VOLTAGE ARC IN COMPRESSED AIR.

9. When an arc is formed in air at a pressure of 6 atm. between silver or copper electrodes, the direct current being less than 0.2 amp. and the voltage of the circuit 4,500 volts, the arc takes on all the characteristics of a very rapid spark discharge.

a. The frequency is independent of the direct current and of arc length and depends only on the inductance and capacity of the oscillatory circuit.

b. No other frequency than the fundamental has been observed. The wave is probably sinusoidal.

11. Unlike the low potential arc, resistances of several hundred ohms may be inserted in the shunt oscillatory circuit, and thus power amounting to more than 60 per cent of that calculated to be available can be derived from the oscillations.

12. The high-voltage arc is nearly free from the hissing noises always heard at low voltages. It is therefore particularly well suited for use in wireless telephone experiments.

TABLE I (Fig. 2).

Current changed by changing resistance.

240 volt circuit. Potential difference across arc about 34 volts.

Currents		Frequency
D. C.	A. C.	n
1.0 amp.	2.2 amp.	217,000
		400,000
		705,000
2.0	3.2	263,000
		520,000
		820,000
2.5	3.5	274,000
		540,000
		820,000
3.0	3.5	289,000
		550,000
		860,000
4.0	4.0	295,000
		580,000
		910,000

TABLE II (Fig. 3, Curve G).

Direct current changed by changing arc lengths.

120-volt circuit.

D. C.	Potential difference across arc	A. C.	Frequency
			n
3.8 amp.	40 volts	1.5 amp.	368,000
3.9	34	2.3	380,000
4.1	26	4.8	394,000

TABLE III (Fig. 3, Curve E).

1 arc, 120-volt circuit.

Potential difference across arc about 30 volts.

Currents		Frequency
D. C.	A. C.	n
2.1 amp.	1.0 amp.	315,000
2.8	1.5	356,000
3.0	1.5	365,000
3.5	2.0	379,000

TABLE IV (Fig. 3, Curve D).

1 arc, 240-volt circuit.

Potential difference across arc about 34 volts.

Currents		Frequency
D. C.	A. C.	n
2.0 amp.	2.2 amp.	312,000
2.5	2.5	356,000
3.0	3.2	378,000
3.5	4.1	390,000
5.2	4.9	420,000

TABLE V (Fig. 3, Curve F).

2 arcs in series, 240-volt circuit.

Potential difference across arc about 54 volts.

Currents		Frequency
D. C.	A. C.	n
4.2 amp.	5.8 amp.	316,000
5.0	6.7	355,000
5.8	6.2	371,000

WASHINGTON, April 1, 1907.

AN EXPLANATION OF THE SHORT LIFE OF FROSTED LAMPS.¹

By Edward P. Hyde.

It has long been recognized that the useful life of a frosted incandescent lamp, taken to 80 per cent of its initial candlepower, is only a little more than one-half the life of a corresponding plain-bulb lamp. The only explanation which has been advanced, so far as the writer knows, is that the temperature of the frosted lamp is higher, due to the increased absorption by the bulb, and that therefore the lamp reaches any given point in its life, e. g. the 80 per cent point, in a shorter time than that required for the corresponding plain bulb lamp.

Without entering, at present, into a discussion of the possible temperature effects, it is sufficient to notice that if the shortened useful life can be attributed to this cause, it is very probable that, due to the same cause, the total life of a frosted lamp up to the time when the filament burns out would also be very much less than that for the plain lamp. On the contrary, the writer has recently been informed by Mr. S. E. Doane, chief engineer of the National Electric Lamp Association, that numerous tests made at the engineering laboratories of the Association indicate that if the life is taken to the time when the filament burns out, the average life of frosted lamps is the same as that for plain lamps.

It would therefore seem that some other explanation than that of the effect of temperature must be sought. A possible explanation of the phenomenon recently occurred to the writer, and was subjected to a test which showed conclusively that at least a large part of the effect, if not the entire effect, could be accounted for by the

¹ Printed in the *Electrical Review*, Apr. 6, 1907, p. 556.

proposed explanation. A complete discussion of the investigation, with the detailed numerical results obtained, will be published shortly in an extended paper on spherical reduction factors and on the effect produced by plain and frosted bulbs of various shapes on the distribution of light around filaments of different forms. It was thought desirable, however, on account of the general interest in the question of the short useful life of frosted lamps, to publish this preliminary note outlining the explanation of the effect.

If we consider the case of a new plain bulb lamp, a certain small percentage of the total flux of light emitted by the incandescent filament is absorbed by the glass envelope surrounding the filament. When the lamp is frosted, a relatively large part of the light which in the plain bulb would pass through the outer surface of the bulb is diffusely reflected back through the glass. In this way a relatively large part of the total flux of light passes through the glass more than once, and so the frosted bulbs show an absorption about 5 per cent greater than that for the plain bulbs.

Now the actual absorption coefficient of glass is quite small, so that it is readily seen that if in any way the absorption coefficient is increased appreciably the apparent absorption due to frosting would be increased greatly. This is what happens when with increasing life the strongly absorbing carbon film is deposited on the inside of the bulb. The effect is the same as if the absorption coefficient of the glass had been increased greatly. In the case of an old plain lamp all the light passes through the carbon film once, a small percentage passes through twice, a still smaller percentage three times, and so on. When the lamp is frosted, the percentage that passes through more than once is very much larger, and so the carbon film has an opportunity to absorb a much larger percentage of the total flux. According to this theory, although at any time the filament in the frosted bulb may be emitting the same total flux of light as that emitted by the filament in the plain bulb, the absorption of light in the carbon film is much greater in the one case than in the other, and so the apparent intensity of the frosted lamp at any time during life is less than that of the plain lamp, the difference in intensity increasing with the number of hours the lamps have burned.

In order to test the hypothesis in a rough way a frosted spherical diffusing globe was mounted on the photometer bench. At first a

new 4 candlepower lamp, whose intensity was known, was placed inside the globe, and the apparent absorption of the globe was measured. Then an old 4 candlepower lamp, on the inside of whose bulb a very perceptible deposit of carbon had formed, was substituted for the new lamp. Measurements on this lamp showed a much larger apparent absorption of the frosted globe. Then the new lamp was coated with a layer of lacquer which absorbed some of the light, and a third determination of the absorption of the globe was made. As before the apparent absorption was greater than with the bare lamp.

Although in this way the presence of the effect was plainly noticeable, no quantitative results could be obtained because the conditions were quite different from those existing when the bulb of the lamp is frosted. On account of the small size of the neck of the diffusing globe compared with the diameter of the globe, the largest lamp that could be used filled only a relatively small part of the inside of the globe, and so of the light diffusely reflected back by the frosted globe only a part passed through the lamp bulb. Therefore the magnitude of the effect measured in this way was necessarily small compared with that which would occur with ordinary frosted lamps.

In order to make a quantitative determination of the effect, 10 comparatively new lamps and 12 old lamps that had dropped to 80 per cent in candlepower were carefully measured for mean horizontal and mean spherical candlepower. They were then sent to a lamp factory and frosted by the acid process, care being exercised to see that the frosting was as nearly uniform for the different lamps as it was possible to obtain. The lamps were then returned to the Bureau of Standards and measured. The new lamps were found to have decreased in mean horizontal intensity by about 4 per cent on the average, the individual lamps agreeing among themselves to within less than 2 per cent. On the other hand, the older lamps were found to have decreased in mean horizontal intensity by 18 per cent, or 14 per cent more than the new lamps. In other words, the apparent absorption of the frosting was approximately $4\frac{1}{2}$ times as great for the old lamps at the end of their useful life as for the new lamps. This means that if we were to assume no difference in the physical or mechanical properties of plain and frosted

lamps, a lot of plain lamps which would decrease 20 per cent in candlepower in a definite number of hours, would in the same number of hours decrease approximately 32 per cent if they were first frosted. The useful life of the frosted lamps to 80 per cent of initial candlepower would be about 60 or 70 per cent of the useful life of the plain lamps, a value in approximate agreement with that commonly accepted. This shows that whatever effects may be produced in the lamps by frosting them, the mere absorption by the deposited carbon film as explained above is sufficient to account for a decrease in useful life of about 30 or 40 per cent.

A single set of measurements of mean spherical intensity indicated that the new lamps had decreased about the same in mean spherical candlepower as in mean horizontal, whereas the old lamps had decreased several per cent more in mean spherical intensity. This is probably due to the uneven distribution of the carbon on the inside of the bulb.

The following experiment has been planned to determine whether there are any other elements entering than that of the increased absorption of the carbon film due to internal reflections. A large number of carefully selected lamps will be accurately measured, of which 20 lamps will be burned to the "peak" point in the curve, 20 more to a point corresponding to about 60 per cent of the normal useful life of the lamps, and 40 more to the end of their useful life, measured to the 80 per cent point. Twenty lamps will not be burned at all. All of the lamps, except 20 of those at the end of their useful life, will now be frosted at the same time and by the same process, and then all will be burned until the filaments break. In this way different points in the life curve can be reached by different paths. Thus, for example, it will be interesting to note whether the 20 lamps frosted at the end of their useful life will have the same average value as the 20 lamps frosted initially and then burned for a number of hours corresponding to the average useful life of the plain lamps. In this way it is hoped to determine whether any other elements enter to partly account for the relatively short useful life of frosted lamps.

WASHINGTON, D. C., March 23, 1907.

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44. Purity and Intensity of Monochromatic Light Sources. *P. G. Nutting*
45. Radiometric Investigations of Infra-Red Absorption and Reflection Spectra. *W. W. Coblentz*
46. A Vacuum Radiometer. *W. W. Coblentz*
47. On the Geometrical Mean Distances of Rectangular Areas and the Calculation of Self-Inductances. *E. B. Ross*
48. The Compensated Two-Circuit Electrodynamometer. *E. B. Ross*
49. Complete Form of Peckner's Law. *P. G. Nutting*
50. A Comparison of the Unit of Luminous Intensity of the United States with those of Germany, England, and France. *E. P. Hyde*
51. Geometrical Theory of Radiating Surfaces with Discussion of Light Tubes. *E. P. Hyde*
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53. On the Colorimetric Determination of Iron with Special Reference to Chemical Reagents. *H. N. Stokes and J. R. Cuth*
54. On Sulphocyanic Acid. *H. N. Stokes and J. R. Cuth*
55. Radiation from and Melting Points of Palladium and Platinum. *C. W. Walfer and G. K. Burgess*
56. The Mutual Inductance of a Circle and a Coaxial Single-Layer Coil—The Lorentz Apparatus and the Ayton-Jones Absolute Electrodynamometer. *F. B. Ross*
57. On the Establishment of the Thermodynamic Scale of Temperature by Means of the Constant-Pressure Thermometer. *Edgar Buckingham*
58. An Exact Formula for the Mutual Inductance of Coaxial Solenoids. *Louis Cohen*
59. The Mutual Inductance of Coaxial Solenoids. *E. B. Ross and L. Cohen*
60. The Production of High Frequency Oscillations from the Electric Arc. *L. W. Jewett*
61. An Explanation of the Short Life of Protected Lamps. *E. P. Hyde*

BUREAU CIRCULARS

- No. 1. Verification of Standards and Measuring Instruments.
- No. 2. Verification of Metal Tapes.
- No. 3. Verification of Standards of Mass.
- No. 4. Verification of Standards of Capacity.
- No. 5. Testing of Clinical Thermometers.
- No. 6. Verification of Electrical Standards and Measuring Instruments.
- No. 7. Pyrometer Testing and Heat Measurements.
- No. 8. Testing of Thermometers.
- No. 9. Testing of Glass Volumetric Apparatus.
- No. 10. Legal Weights (40 pounds) per Bushel of Various Commodities.
- No. 11. Samples of Standardized Iron and Steel.
- No. 12. Verification of Polarographic Apparatus.

MISCELLANEOUS

International Metric System (chart)
Table of the Equivalents of Customary and Metric Weights and Measures
The International Metric System of Weights and Measures. (Pamphlet.)
First Conference on the Weights and Measures of the United States.
Second Annual Conference on the Weights and Measures of the United States.

LIST OF TECHNICAL PAPERS ISSUED BY THE BUREAU OF STANDARDS, DEPARTMENT OF COMMERCE AND LABOR

1. Comparison of the United States Prototype Meter *L. A. Fischer*
2. A Study of the Silver Voltmeter *K. E. Guthe*
3. The So-Called International Electrical Units *Frank J. Wolf*
4. The Spectra of Mixed Gases *P. G. Nutting*
5. On Secondary Spectra and the conditions under which they may be produced *P. G. Nutting*
6. Some new Rectifying Effects in Conducting Gases *P. G. Nutting*
7. On Filters resembling Quartz in their Elastic Properties *K. E. Guthe*
8. On the Temperature of the Arc *C. W. Waidner and G. K. Burgess*
9. The Absolute Measurement of Inductance *E. B. Ross and F. W. Greiner*
10. The Absolute Measurement of Capacity *E. B. Ross and F. W. Greiner*
11. Optical Pyrometry *C. W. Waidner and G. K. Burgess*
12. On the Theory of the Matthews and the Russell-Lemard Photometers for the Measurement of Mean Spherical and Mean Hemispherical Intensities *E. P. Hyde*
13. The Testing of Clinical Thermometers *C. W. Waidner and L. A. Fischer*
14. Measurement of Inductance by Anderson's Method, Using Alternating Currents and a Vibration Galvanometer *E. B. Ross and F. W. Greiner*
15. Use of Serration in Standards of Inductance *E. B. Ross and F. W. Greiner*
16. The Silver Coulometer *K. E. Guthe*
17. History of Standard Weights and Measures of the United States *L. A. Fischer*
18. Watmeter Methods of Measuring Power Expended upon Condensers and Circuits of Low Power Factor *E. B. Ross*
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23. The Positive Charges Carried by the Canal Rays *L. W. Austin*
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25. A Pye-Thomson Volt Generator Set *P. G. Nutting*
26. Talbot's Law as applied to the Rotating Secured Disc *E. P. Hyde*
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31. Calculation of the Self-Inductance of Single-Layer Coils *Edward B. Ross*
32. Heat Treatment of High Temperature Mercantal Thermometers *Robert C. Dickinson*
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34. On "Amor Lines" in Light Sources in Polariscopic Measurements *Frederick Hites*
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39. A Pocket Spectrophotometer *P. G. Nutting*
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43. On the Determination of the mean Horizontal Intensity of Incandescent Lamps by the Rotating Lamp Method *E. P. Hyde and P. G. Nutting*

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DEPARTMENT OF COMMERCE AND LABOR

Vol. 3

BULLETIN

No. 3

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

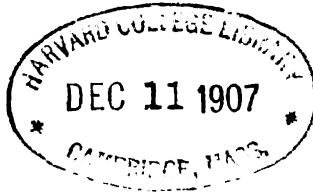
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1907



FROM ELO
J. C. GORDON

MELTING POINTS OF THE IRON GROUP ELEMENTS BY A NEW RADIATION METHOD.

By Geo. K. Burgess.

Recently¹ Dr. Waidner and the author suggested two radiation methods for the determination of high melting points of substances in minute quantities. The first depends on the total radiation from a surface as measured in terms of some instrument making use of the Stefan-Boltzmann law. The second method is based on the measurement of the intensity of a particular monochromatic radiation from platinum or other substance; that is, it makes use of an instrument based on Wien's equation.²

Some preliminary measurements made at that time by the second method on the metals of the iron group showed it to possess considerable accuracy and to be particularly applicable to the easily oxidized elements if the melts were obtained in an atmosphere of pure hydrogen. It was also demonstrated that a compound of one of the refractory elements, which could be reduced to the metal by hydrogen, might serve as material for determining the melting point of the pure metal.

The present paper is an account of the application of this second method to the determination of the melting points in hydrogen of the members of the iron group, using, for the measurement of temperature a Holborn-Kurlbaum optical pyrometer, and the data on the monochromatic radiation from platinum published in this Bulletin.³

¹ Waidner and Burgess: On the Determination of Melting Points by Radiation Method, *Phys. Rev.*, **22**, p. 359; 1906.

² A discussion of the laws of radiation will be found in this Bulletin, **1**, p. 189; 1905.

³ Waidner and Burgess, this Bulletin, **3**, p. 1; 1907.

Experimental Method.—The arrangement of the apparatus is shown in Fig. 1, and is as follows: Within the blackened brass cylinder C is a strip of pure platinum P, about 6 cm long, 4 mm wide, and 0.02 mm thick. This platinum strip is heated electrically to any desired temperature, and by means of a rheostat, which has a fine adjustment, very delicate control of the temperature of P is obtained. The cylinder C is connected to an electrolytic hydrogen generator in series with an alkaline pyrogallate solution and a drv-

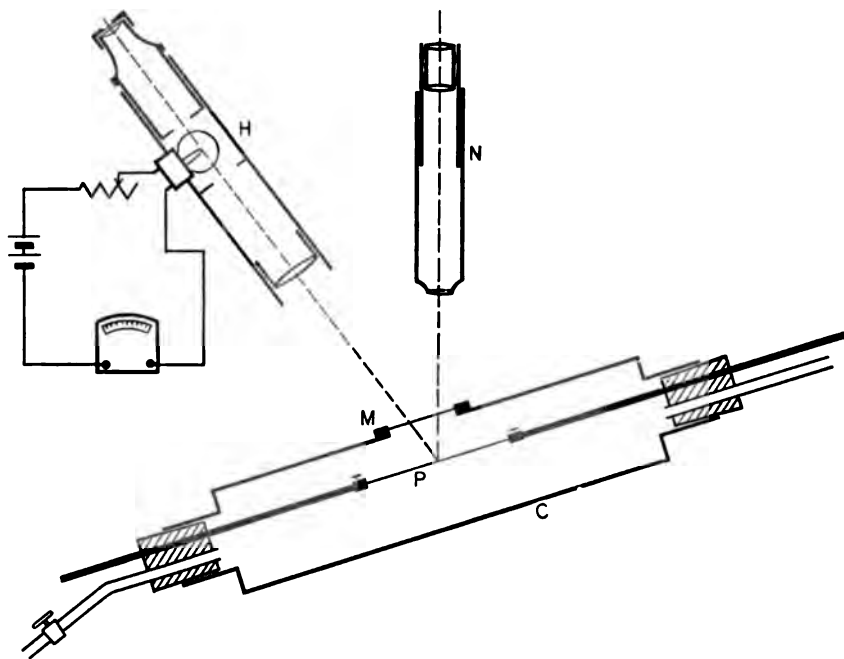


Fig. 1.

ing tube of calcium chloride. At M is a removable window of thin mica, giving access to the cylinder and permitting observation of P both with the microscope N and the optical pyrometer H.

To take an observation of a melting point, a minute quantity of a metal or its oxide, usually about 0.001 mg in the form of powder, is placed at the center of the platinum strip P, the window M screwed down, the cylinder closed and filled with hydrogen, and then the electric circuit is made through P. One observer watches the metallic speck or dust through the microscope and gradually

increases the current until the melting point is reached, while, simultaneously, another observer reads this temperature with the pyrometer. Only one melt is taken on a given platinum strip.

Temperature Scale.—The Holborn-Kurlbaum pyrometer was calibrated by sighting upon an improved form of the electrically heated experimental black-body due to Lummer and Kurlbaum, the temperature of which was given by the electromotive forces of two platinum, platinum-rhodium thermoelements as measured on a potentiometer. The temperature scale of the thermoelements was obtained by assuming the following melting points: $\text{Zn}=419^{\circ}\text{C}$, $\text{Sb}=630^{\circ}\text{C}$, $\text{Cu}=1084^{\circ}\text{C}$, all these metals being of "Kahlbaum" purity. As an upper fixed point, the melting point of platinum was taken as 1753°C , determined from measurements with the optical pyrometer. In the optical pyrometer two lamps were used which gave practically identical values for every temperature observed. As the establishment of the high temperature scale of the Bureau of Standards is treated at length elsewhere⁴ in this Bulletin, further discussion of this scale need not be given here.

The indications of the optical pyrometer, when sighted upon platinum through a mica window, are subject to two corrections, one for the reflection and absorption of the mica, the other for the selective emission of the platinum for the light used—in this case, red light $\lambda=0.66\mu$. The former correction, which is about 15° , is easily determined and checked between measurements of the melting points by taking observations of the platinum brightness at a definite temperature both with and without the mica interposed, with a second mica strip always in place to prevent air currents.

The correction for the selective emission of platinum is discussed in the paper above cited. The magnitude of this correction varies from 110 to 160°C and is known to at least 10° within the temperature range here used.

Sources of the Materials.—Chemically pure elements are difficult to get, and it is desirable to use products from different sources and of known analyses. These experiments were greatly facilitated by the willingness of scientists possessing very pure samples to put them at our disposal.

⁴Waidner and Burgess, this Bulletin, **3**, p. 1; 1907.

M. Copaux of Paris, who has recently⁵ completed a most comprehensive study of the properties of cobalt and nickel, and produced these metals containing less than 1/2000 part impurities, furnished samples of these elements. Dr. H. Goldschmidt of Essen Ruhr, Germany, sent pieces of his pure aluminothermic metals, namely, chromium and manganese, all of an average purity of 98 to 99 per cent; and Prof. C. F. Burgess of the University of Wisconsin supplied samples of his pure electrolytic iron. The other samples were from Kahlbaum and of "Kahlbaum" grade except as otherwise noted.

IRON.

Iron from two sources was used, "Kahlbaum" iron in the form of powder, and electrolytic iron in lumps, from Prof. C. F. Burgess, from which particles weighing slightly more than 0.001 mg were chipped. This electrolytic iron had the following composition, as determined from an analysis of a similar sample by Mr. A. A. Blair, of Wisconsin:

Sulphur,	none.	Manganese, none.
Silicon,	0.013 per cent.	Carbon, 0.012.
Phosphorus,	0.004.	Hydrogen, 0.072.

Iron as it approaches the melting point becomes plastic, so that the actual temperature of melting is not sharply defined, there being a considerable transition region.

The very considerable variation in the separate melts appears to be due mainly to the influence of size and shape of the particles on the viscosity and surface tension; also lack of homogeneity in chemical constitution of such minute samples may play some rôle, as well as variations in the temperature of adjacent areas of the platinum strip. That the electrolytic iron melts at a slightly higher temperature seems probable. Any impurity likely to be formed in iron will lower its melting or freezing point; but the total impurities in the electrolytic iron here used will not cause a lowering as great as 5° C.

⁵ H. Copaux, *Recherches experimentales sur le Cobalt et sur le Nickel*, *Ann. Chem. et Phys.*, 6, p. 508; 1905.

The corrected values found for all the separate melts are given in Table I.

TABLE I. Melting Point of Iron.

C. F. Burgess Electrolytic.	"Kahlbaum" Powder.
1493°	1509°
1519	1494
1518	1512
1512	1483
1512	1507
1514	1491
1489	—
1508	Mean . . . 1499
1508	
1494	
—	

Mean . . . 1507

Among previous determinations of the melting point of iron have been the following:

1600° Roberts Austen, 1899.

1575° Le Chatelier, 1901.

1550° Osmond.

1555° (1490°) Tammann, 1904.

The value given by Tammann,⁶ 1555°, is that, assuming the nickel point to be 1484°, as determined by Holborn and Wien.⁷ Tammann's value as obtained by extrapolation of his own thermocouple calibration from the gold point (= 1064°) would be about 1490°.

COBALT.

Three kinds of cobalt were used: chips from M. Copaux's sample, "Kahlbaum" cobalt, and the metal obtained from Kahlbaum's nickel-free cobalt-carbonate transformed to cobalt-oxide by heating in air, and reduced to metallic cobalt by hydrogen, in place on the platinum strip on which the melt was taken.

⁶ Guertler and Tammann, *Zs. Anorg. Chem.*, **45**, p. 205; 1905.

⁷ Holborn and Wien, *Wied. Ann.*, **56**, p. 360; 1895.

In great contrast to iron, cobalt possesses a very sharp melting point, and the resulting precision is correspondingly increased, showing that the greater deviations in the measurements on iron are not due to the fault of the method nor to its lack of sensitiveness.

TABLE II. Melting Point of Cobalt.

Co from CoCo_3 (Kahlbaum Ni-frei).	"Kahlbaum" Powder.	Copaux pieces from crucible melt.
1456°	1470°	1457°
1472	1466	1468
1467	1468	1463
1468	1466	1466
1453	1459	1462
1470	1461	
Mean . . . 1464	Mean . . . 1465	Mean . . . 1463

Copaux himself gets 1530° for the cobalt point by interpolation between the gold and platinum points, assuming the latter to be 1780°. Recent measurements⁸ of the platinum point show 1780° to be high by about 70° on the thermocouple scale, or by about 30° on the optical scale, as usually defined in terms of Wien's law. Holborn and Valentiner,⁹ however, have recently obtained 1789° for the melting point of platinum, using Wien's equation, the constants of which were newly determined by comparison with the indications of a gas thermometer up to 1600°. Guertler and Tammann get 1505° for cobalt on the Holborn and Wien scale, or reduced, as explained in the case of iron, their value for cobalt would be about 1455° on the thermoelectric scale.

Nickel is usually present in cobalt to at least 2 per cent, causing a proportional lowering of the freezing point¹⁰ of less than 1° C. Iron, even if present in considerable quantities, has a negligible effect on the cobalt melting point. Cobalt is less oxidized in the air at high temperatures than are the other elements of the iron group, and

⁸ Harker, *Chem. News*, **91**, p. 62; 1905. Holborn and Henning, *Sitzber. Berlin Akad.*, **12**, p. 311; 1905. Nernst, *Berichte Deut. Phys. Ges.*, **4**, p. 48; 1906. Waidner and Burgess, l. c.

⁹ Holborn and Valentiner, *Ann. d. Physik*, **22**, p. 1; 1907.

¹⁰ Guertler and Tammann, *Zs. Anorg. Chem.*, **42**, p. 353; 1904.

cobalt oxide is only slightly soluble in the metal.¹¹ These properties, taken together with its very sharp melting point, the insignificant effect of its usual impurities on this temperature, and its relative cheapness as compared with the platinum metals, make cobalt, even of a commercial purity, a desirable metal to use for a fixed point in pyrometric measurements.

NICKEL.

The melting point of nickel was determined with the pure metallic nickel of M. Copaux and with Kahlbaum's cobalt-free nickel oxide reduced by hydrogen on the platinum strip. The melting point is about as well defined as that of cobalt.

TABLE III. Melting Point of Nickel.

Pressed Nickel. Copaux.	Ni from the oxide. Kahlbaum.
1434°	1433°
1435	1434
1434	1439
1436	1434
1436	
Mean . . . 1435	Mean . . . 1435

The preliminary measurements of Dr. Waidner and the author showed that "Kahlbaum" nickel also gave the same value as the above.

The following thermoelectric determinations of the nickel point may be noted:

- 1420° Le Chatelier, 1887.
- 1484° Holborn and Wien, 1895.
- 1427° Harker, 1905.
- (1425–1430°) Tammann, 1905.
- 1470° Copaux, 1905.

The measurements of Copaux and of Holborn and Wien are subject to the same corrections as discussed under cobalt. Le Chatelier's determination is based on a too low value of the gold point. Tam-

¹¹ Copaux, l. c.

mann's value is that deduced from his own observations as in the case of cobalt and iron. According to the measurements of Waidner and Burgess (l. c.) the optical is 8° higher than the usual thermoelectric scale at 1427° , so that the nickel freezing point, as determined by Harker using thermocouples, is in exact agreement with the above optical determination.

The presence of cobalt in nickel as impurity will raise the melting point,¹² while iron as an impurity in nickel will lower its melting point.

CHROMIUM.

Crystallized chromium from Kahlbaum was used and also Dr. Goldschmidt's aluminothermic chromium of 98–99 per cent purity and carbon free. As Table IV shows, it is about as difficult with chromium as with iron to get a sharp melt. Possibly the greater impurity of the crystallized chromium increases this effect for that sample.

TABLE IV. Melting Point of Chromium.

Chromium crys. Kahlbaum.	Cr. 98–99 per cent Goldschmidt.
1501°	1497°
1472	1483
1480	1486
1505	1484
1475	1494
Mean . . . 1487	Mean . . . 1489

A thermoelectric determination of the chromium melting point has been made by Lewis,¹³ using 99 per cent chromium obtained by the Goldschmidt process. The melt was made in a lime crucible in an oxy-coal-gas flame; two melts gave 1510° and 1520° .

MANGANESE.

Two samples of manganese were used: "Kahlbaum" manganese, and a sample from Dr. Goldschmidt, 98 per cent manganese, carbon free and technically iron free. The manganese point, although

¹² Tammann, l. c.

¹³ E. A. Lewis, Chem. News, 86, p. 13; 1902.

the lowest in the series, is the least satisfactory to determine on account of the very great temperature interval (over 25° C) in which the melt takes place.

TABLE V. Melting Point of Manganese.

Manganese "Kahlbaum."	Mn 98 per cent Goldschmidt.
1226°	1207°
1203	1210
1197	
1198	

Mean . . . 1207

First signs of melting were evident as low as 1195° C, and the melt of relatively large pieces was not complete at 1220° C.

Heraeus¹⁴ found 1245°, by observing with a telescope the melt in hydrogen within an electric furnace, whose temperature was given with a thermoelement and pyrometer-galvanometer. Levin and Tammann,¹⁵ also using a thermoelement, get between 1190° and 1200° for the manganese melting point.

CONCLUSION.

The values found by the radiation method described above, for the melting points, in an atmosphere of hydrogen of the elements of the iron group in contact with platinum, are as follows :

TABLE VI. Approximate Melting Points of the Iron Group.

Metal.	Melting point.	Purity.
Iron	1505°C	99. 95
Chromium	1489	98-99
Cobalt	1464	99. 95
Nickel	1435	99. 95
Manganese	1207	98-99

Of these, the cobalt and nickel melting points appear to be correct to within 5°, while the uncertainty of the iron, chromium, and manganese points is probably less than 10° C. It is not probable

¹⁴ Heraeus, Zs. Elec. Chem., 8, p. 185; 1902.

¹⁵ Levin and Tammann, Zs. Anorg. Chem., 47, p. 136; 1905.

that any of these metals combines with platinum, or alloys with it until the metal is melted at its natural melting point. The fresh melts are all white, changing to the color characteristic of each metal after standing in the air.

It is not claimed that the above numerical values are final, and they are given mainly as an illustration of the possibilities of a method which may be the only one available in certain cases. The above values are, however, in excellent agreement with certain of the latest determinations by the thermoelectric method, notably Harker's determination of the nickel point, and the determinations of Tammann and his associates for iron, nickel, cobalt, and manganese when these latter results are reduced by extrapolation from the gold point and account is taken of the difference between the thermoelectric and optical scales.¹⁶ It would be highly desirable to determine the melting point of cobalt with the greatest accuracy for use as a fixed point in pyrometry.

By this radiation method less than 0.001 mg is required for a single observation, making it readily applicable for the relatively exact determination of the melting points of the rarer refractory elements, and it is the only sensitive method yet suggested when the pure element can be obtained only in very small quantities. It is our intention to continue such determinations whenever it is possible to obtain the refractory elements of sufficient purity.

There may be considerable variations in size of the particles to be melted without appreciable change in the apparent melting point. Furthermore, the method gives the same melting point for the pure metal, and for the metal reduced from the pure oxide by hydrogen in the apparatus used, and requires no transfer or handling of the microscopic quantity of the metal so formed.

For the determination of the melting points of those elements having a higher fusing point than platinum, iridium strips may be used and in some cases perhaps carbon; but it is hoped that we may shortly be able to use tungsten strips. Tungsten, as we have found,¹⁷ appears to have the highest melting point—above 3000° C—of any element that can be melted without considerable previous evapora-

¹⁶ Waidner and Burgess, this Bulletin, 3, p. 1; 1907.

¹⁷ Waidner and Burgess: Temperature and selective radiation of incandescent lamps, *Electrical World*, Nov. 10, 1906, p. 915. This Bulletin, 2, p. 319; 1906.

tion, and appears to possess the further advantage of being unattacked by hydrogen. The above apparatus could evidently be arranged, if necessary, to work with a vacuum instead of an atmosphere of hydrogen or other gas.

It might also be possible, as in the case of steels, to apply the method to a determination of carbon or other content from the melting point measurements.

To Mr. J. R. Fehr credit is due for efficient aid in carrying out the above experiments.

WASHINGTON, *December 14, 1906.*

ON THE DETERMINATION OF THE MEAN HORIZONTAL INTENSITY OF INCANDESCENT LAMPS.

By Edward P. Hyde and F. E. Cady.

INTRODUCTION.

In a paper "On the Determination of the Mean Horizontal Intensity of Incandescent Lamps by the Rotating Lamp Method," published several months ago in abstract in the *Electrical World*,¹ and subsequently in full in this Bulletin,² the authors, after reviewing briefly the various methods employed in the determination of mean horizontal intensity, described an investigation, undertaken at the Bureau of Standards, of the possible errors of the rotating lamp method. This method is the one in almost universal use in the United States. Recently another paper³ on the same subject has been published by Uppenborn in Munich. Inasmuch as the results given in the latter paper are at variance with those obtained at the Bureau of Standards, and have, moreover, been quoted in the United States as discrediting the determination of mean horizontal intensity by the convenient method of rotating the lamp about its axis of figure, it seemed desirable to present some further data obtained at the Bureau, and also to call attention to several points of difference in the two methods employed to study the accuracy of the rotating lamp method.

According to the results obtained previously at the Bureau in the study of this method, two possible sources of error were found when the lamp was rotated, one due to a distortion of the filament produced by centrifugal forces, and one due to the flickering nature of

¹ *Electrical World*, Nov. 17, 1906; p. 956.

² This Bulletin, 2, p. 415.

³ *Electrotechnische Zeitschrift*, Feb. 14 and 21, 1907; pp. 139, 168.

the illumination of the photometer screen, produced primarily by the nonuniformity of the horizontal distribution curve of the lamp. Of these two effects the first was found to be quite small at moderate speeds of rotation for all types of filaments examined, the maximum error at 500 or 600 revolutions per minute being only about 1 per cent.

The error due to flicker, however, was found to be quite large for lamps of certain types in which the horizontal distribution curves deviate greatly from a circle. Moreover, the error was found to be largely a function of the individual, some observers reading a flickering light too high and others too low. Except, however, for a few types of lamps (of the number examined) in which the flicker is very annoying, the combined error due to both bending and flicker is probably not much over 1 per cent, particularly if the speed of rotation is increased to 300 or 400 r. p. m. By the use of a single auxiliary mirror as described in the original paper, the flicker pertaining to all types of filaments is greatly reduced, so that even those lamps with the most irregular horizontal distribution curves can be photometered with accuracy at a speed of only 200 or 300 r. p. m., at which the effect of bending is negligible for all types of lamps studied.

In the investigation at the Bureau a substitution method was employed, which did not necessitate a determination of the actual candlepower of the lamps studied. Any change in candlepower, due either to the bending of the filament or to the flicker could be detected and measured separately. For the nine different types of lamps investigated, as described and illustrated in the detailed paper referred to above, the greatest change in mean horizontal candlepower due to bending at a speed of 550 r. p. m. was found to be only about 1 per cent. For most of the types of filaments the effect of bending was to decrease the mean horizontal intensity, but for one type the effect was to increase it by about 1 per cent.

In Uppenborn's investigation the lamps were measured for actual mean horizontal candlepower, by taking the mean of 36 readings made every 10° in the horizontal plane. The values obtained in this way were taken as the true values, and a comparison of the candlepower values obtained by other methods with these standard values gave the errors incident to the several methods studied.

In this way 8 lamps of different sizes, types, and voltages were measured for mean horizontal candlepower, first by the point-to-point method, and subsequently by the rotating lamp method. A comparison of the results indicated that the values obtained by the rotating lamp method were uniformly too low, the amount of the error ranging from 3.3 to 9.9 per cent. Uppenborn attributes the errors to the distortion of the filament on rotation; but since the experiments at the Bureau had shown in no case an error much over 1 per cent due to this cause, it seemed desirable to make some further experiments by a different method, and using lamps of other types than those previously investigated. The differences in the results obtained at the two laboratories could not have been due to differences in speed used, for in the experiments at the Bureau various speeds were employed up to 550 or 600 r. p. m., whereas in Uppenborn's work, although no definite statement is made of the speed at which the results quoted above were obtained, it is very probable that no greater speed than 450 or 500 r. p. m. was employed, since this is the limiting speed stated in another part of the paper.

METHOD AND RESULTS OF RECENT EXPERIMENTS.

Before conducting experiments upon types of filaments differing from those investigated previously a few measurements were made on some of the same types of filaments, in order to check the results obtained by the former method using the rotating mirrors, with those obtained by the new method. In this new method the lamp was placed in a universal lamp-holder which could be set at different angles, but which also could be rotated. In general, readings were made first with the lamp spinning, then at every 10° in the horizontal plane with the lamp at rest, and again with the lamp spinning. The speed was then gradually increased, in some experiments until the filament touched the bulb. This way of comparing the point-to-point method with the rotating lamp method was chosen as being somewhat similar to that employed by Uppenborn, with the advantage that being a substitution method it does not involve the actual calibration of the comparison lamp such as is done in Uppenborn's experiments.

The first lamp studied was a 110-volt 16-cp oval anchored filament lamp (Fig. 1, No. 1). In expressing the results of this, as of subsequent experiments, since only relative values are desired, the convenient method will be used of adopting an arbitrary unit such that the intensity under some one condition will be taken as unity, and then all the other values will be expressed in terms of this unit. In this way the percentage changes will be evident immediately.

The lamp was first rotated at about 420 r. p. m. and readings were taken. Then measurements were made every 10° in the horizontal plane with the lamp stationary, and finally the lamp was rotated again at 420 r. p. m. If we call the first rotating value 1.000, then the value of mean horizontal intensity obtained by the point-to-point method is 1.002, and the value found on subsequent rotation 0.999. In other words, the results obtained by the two methods, the speed of rotation being 420 r. p. m., agree to within 0.2 or 0.3 per cent, which is within the range of experimental error.

Since in this type of lamp the filament is well supported, one would not expect as large errors as for some other types, such as the double filament lamp (Fig. 1, No. 2). In this type of lamp, which was next studied, the filament consists of two long, slender loops, entirely unsupported except at the leading-in wires. Denoting by 1.000 the value of mean horizontal candlepower obtained at the beginning at a speed of 350 r. p. m., the value found by the point-to-point method every 10° is 1.005, and the value obtained at the end at a speed of 350 r. p. m. is 0.999. This decrease of 0.5 per cent at a speed of 350 r. p. m. is in very good agreement with that obtained in the previous investigation using the rotating mirror method. The mean of a number of determinations using this method indicated a decrease of 0.9 per cent at a speed of 500 or 600 r. p. m.

Another lamp of the double filament type gave the following results: Mean horizontal intensity at 180 r. p. m., 1.000 at the beginning of the set of measurements, 1.003 in the middle of the set, and 0.998 at the end; by the point-to-point method 1.002, or an average decrease of 0.2 per cent, which also agrees with the values previously obtained for this type of lamp.

Having shown now that the error at a speed of 180 r. p. m. for this type of filament is negligibly small, the two lamps were mounted successively in the rotator and the speed was increased until the

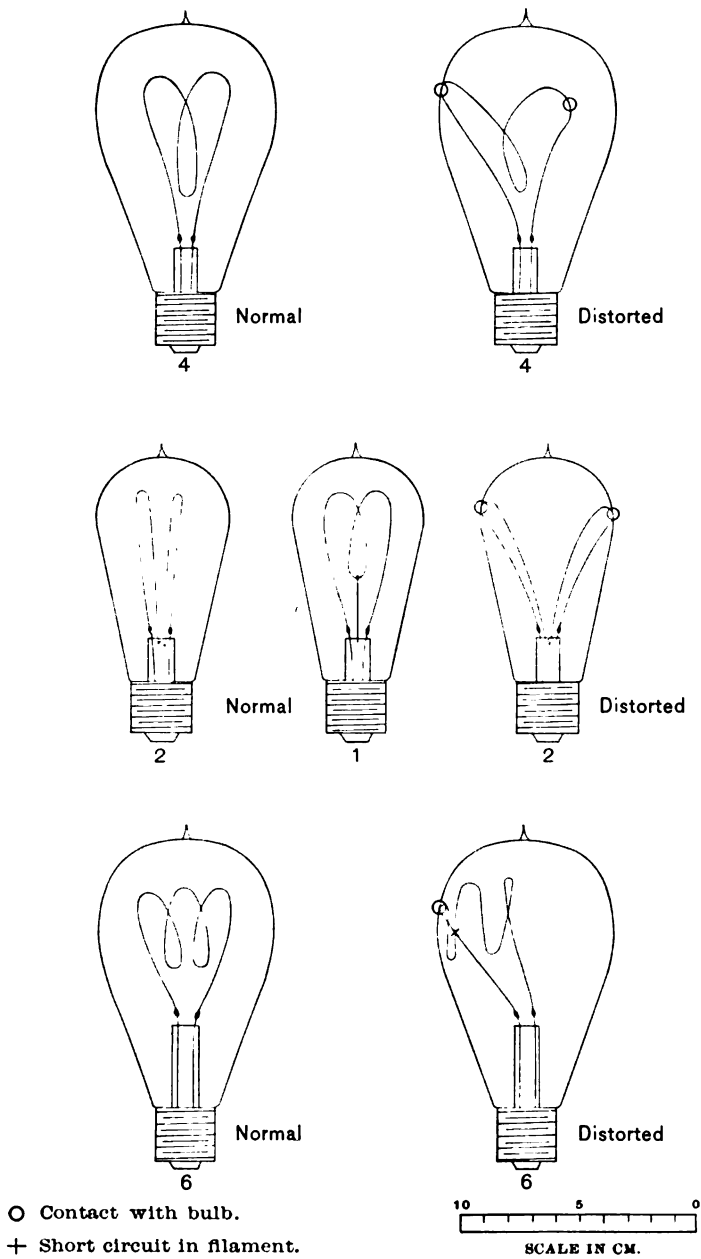


Fig. 1.

TABLE I.

Values of Mean Horizontal Intensity of Double Filament Lamps at Different Speeds of Rotation.

LAMP NO. 2.

Speed	Relative intensity	Current	Remarks
180 r. p. m.	1.000	0.5232	
400 r. p. m.	0.990	0.5232 ₅	
750 r. p. m.	0.951	0.5237 ₅	Both loops touching bulb.
180 r. p. m.	0.997	0.5231	Loops shaken loose. Filament has a "set."
400 r. p. m.	0.988	0.5231	
470 r. p. m.	0.986	0.5231	
200 r. p. m.	0.999	0.5231	
			(Speed gradually increased with following results:)
	0.985		Just before either loop touched bulb.
	0.978	0.5232	One loop touching bulb.
	0.966	0.5232 ₅	Both loops touching bulb.
			(Lamp stopped, loops shaken loose from bulb, and cycle repeated.)
	0.986	0.5231 ₅	Just before either loop touched bulb.
	0.981	0.5231 ₅	One loop touching bulb.
	0.949	0.5233 ₅	Both loops touching bulb.
	0.995	0.5231	Both loops shaken loose.

LAMP NO. 3.

200 r. p. m.	1.000	0.5217	
510 r. p. m.	0.985	0.5216 ₅	
725 r. p. m.	0.964	0.5217 ₅	One loop touching bulb.
180 r. p. m.	0.995	0.5216 ₅	Lamp stopped and loop shaken loose.
330 r. p. m.	0.989	0.5216 ₅	
500 r. p. m.	0.989	0.5216	
750 r. p. m.	0.966	0.5216	One loop touching bulb.
820 r. p. m.	0.959		Second loop spreading. Possibly touched bulb.
180 r. p. m.	0.997	0.5216 ₅	Lamp stopped and loop shaken loose.
400 r. p. m.	0.992		
180 r. p. m.	0.997		
500 r. p. m.	0.991		
775 r. p. m.	0.971		One loop touching bulb.

filaments touched the bulb, in order to see how large a decrease in mean horizontal intensity could be obtained with this type of lamp, and what the speed must be in order to accomplish this maximum reduction. The results are given in Table I. In the first column are given the different speeds used, in the second column the values of mean horizontal intensity in terms of an arbitrary unit, in the third column the currents under the different conditions, and in the fourth column brief explanatory notes. Usually when the filament was forced against the bulb it sealed itself to the glass (Fig. 1, No. 2, distorted) and had to be shaken loose after stopping the lamp.

An inspection of the table shows that the maximum decrease in mean horizontal intensity obtained under any condition was about 5 per cent, when at a speed of 750 or 800 r. p. m., the two loops of the filament were each separately touching the bulb, one at one side and one at the other. In no case did the decrease amount to more than 1.5 per cent so long as the filament did not touch the bulb. In order to test this point directly, the two series of readings recorded near the end of the table for lamp No. 2 were made. One observer made the electrical measurements, a second observer read the photometer, and a third observer increased the speed gradually and noticed when the filament touched the bulb. It is seen that as each loop touched, a decrease in candlepower resulted, but it was impossible to say *a priori* how much the intensity would decrease when the filament touched.

These various results are not surprising when we consider the effect of a contact of the filament with the bulb. On touching the bulb the filament is cooled, and so a decrease in candlepower is to be expected. Moreover the amount of the cooling, and consequently the decrease in candlepower, depends upon whether the filament touches in only one point, or whether it lies against the glass for an appreciable part of its length, so that the decrease in candlepower is not always the same when a filament touches the bulb.

Independent evidence of the cooling of the filament and the consequent decrease in candlepower is afforded by a study of the current values. Although a change in voltage of from 100 to 110 volts produced an increase in resistance of only 2 parts in 1000, yet the cooling due to the contact of a small part of the filament with the bulb was sufficient to increase the current appreciably, in one case

the change amounting to 1 part in 1000. But although the resistance changes are indicative of the cooling, quantitative relations could not be deduced because of the very complicated nature of the phenomenon.

It would seem that the double filament lamp would probably show a greater decrease in mean horizontal candlepower due to bending than any other type of lamp made in the United States. The two loops of which the filament is composed are long and slender, and being unsupported are susceptible to centrifugal forces. Moreover the candlepower in the direction of the tip is very small and consequently the decrease in the effective length of the sides of the loop on rotation is not compensated to any extent by the tip. A lamp having a relatively large tip candlepower would not be expected to show as much decrease in mean horizontal candlepower due to bending as a lamp with a small tip candlepower. But although the double filament lamp seemed to the writers to represent one of the worst cases for high-speed rotation, since the large errors found by Uppenborn were obtained with filaments having a turn in them somewhat similar to the common oval anchored filament, it was thought desirable to make some tests with lamps of this type.

The maximum error which Uppenborn obtained, -9.3 per cent, was found for a 32-cp 110-volt $1\frac{1}{2}$ -turn oval filament with no anchor. Since the writers do not know of any type of lamp made in the United States corresponding exactly with that described by Uppenborn, they obtained through the courtesy of the Franklin Electric Manufacturing Company two special lamps (Fig. 1, No. 4), which were identical in every respect with the regular 32-cp 110-volt, oval anchored lamp except that the anchor wire was omitted, in order that the filament might be left as unstable as possible. The dimensions of the filament agreed approximately with those given by Uppenborn, the total height of the filament used at the Bureau being about 75 mm instead of 65 mm, and the vertical axis of the oval being 50 mm instead of 40 mm.

These two lamps were studied in the same way as the double filament lamps described above, except that the readings by the point-to-point method were made every 15° instead of every 10° as in the preceding determinations. Lamp No. 4 was first measured

at different speeds up to 600 r. p. m., the greatest change in mean horizontal candlepower, as compared with the value determined by the point-to-point method, being only 1 per cent. In these determinations the filament did not touch the bulb, although it approached close to it. Subsequently the lamp was kept at a speed of 600 r. p. m. for some minutes, and though at first the filament was separate from the bulb it gradually spread and ultimately came into contact with the glass. Just before the filament reached the bulb the mean horizontal intensity was about 1 or 1.5 per cent low; but after coming into contact the intensity decreased to a value 3 or 4 per cent lower than that determined by the point-to-point method. The current also underwent a very appreciable change when the filament touched the bulb.

The speed was now increased to 800 r. p. m., at which both loops were in contact with the glass (Fig. 1, No. 4, distorted). The mean horizontal intensity was found to be about 7 per cent low, and the current to have undergone another change in the same direction as before. While rotating at the high speed of 800 r. p. m. a star crack developed and the filament burned out so that it is possible that a part of the observed 7 per cent change in candlepower may have been due to the constantly deteriorating vacuum.

The other special 110-volt 32-cp lamp (No. 5) showed approximately the same characteristics as lamp No. 4, except that the filament touched the bulb first in one point and then in a second point, the mean horizontal intensity decreasing to about 97 per cent of its original value as determined by the point-to-point method. The current also showed a marked change due to the cooling of the filament.

In addition to the two special 32-cp 110-volt lamps, the Franklin Electric Manufacturing Company furnished also two special 32-cp 220-volt lamps without anchors. The filaments were of the $2\frac{1}{2}$ -turn type such as are usually provided with two anchors (Fig. 1, No. 6). Without the anchors the filaments are quite unstable and entirely unsuited for high-speed rotation. Consequently they were forced against the bulb at the comparatively low speed of 400 or 500 r. p. m. Before touching the bulb the decrease in candlepower was not more than 1 per cent for either lamp, and, in fact, the error due to the flicker, which is somewhat annoying for this type of

lamp at the low speed of 200 r. p. m., was of the same order of magnitude as that due to bending. It is interesting to note in this connection that the same kind of error due to flicker was observed in these experiments as that described in the former paper. Thus while one observer obtained for the flickering light a value higher than that obtained by the point-to-point method, the other observer obtained a value lower than the true value. Moreover the two observers deviated from the true values in the same way as they did previously in the original investigation. This point will be mentioned again later. It is sufficient to note here that at the higher speeds this error due to flicker disappeared and the different observers obtained approximately the same errors due to bending.

At speeds of 600 or 700 r. p. m., when the filaments were well over against the glass, the decrease in candlepower reached a value as high as 4 per cent for one of the lamps; but whenever the change in candlepower was large there was always a very appreciable change in resistance, indicating that the change in candlepower was due probably, to a large extent, to a cooling of parts of the filament. In Fig. 1, lamp No. 6 is shown both when new and after having touched the bulb and burned out. At the high speed of 675 r. p. m. one loop was short-circuited by the next and the filament burned out at the point of contact.

From a consideration of all the data given in the preceding paragraphs it is seen that the maximum error observed, due to a bending of the filament, was not in any case greater than 1.5 per cent, provided the filament did not touch the bulb, even though the filament was bent so as almost to touch the bulb. The speed of rotation necessary to force the filament against the bulb was not less than 600 r. p. m. except for the special 32-cp 220-volt lamps without anchors, and for these it was about 400–450 r. p. m. For most common types of lamps the speed would be much higher than 600 r. p. m., and it is probable that for some types a rupture of the filament would occur before the filament would touch the bulb.

When the filament touches the bulb changes in candlepower of as much as 6 or 7 per cent may take place, but as these large changes are always accompanied by relatively large changes in resistance, it is probable that they are due to a cooling of the filament owing to the contact with the glass.

EFFECT OF BENDING.

As explained in detail in the original paper referred to above, the difference between the value of mean horizontal intensity obtained by the point-to-point method and that obtained by the rotating lamp method may be ascribed to two possible causes—the effect of bending of the filament on rotation, and the effect of the flicker on the ability of the eye to estimate correctly the true mean values. In the method used at the Bureau in the original investigation these two elements could be separated, but in the method used for the present experiments, as also in the method of Uppenborn, both elements were present and the resultant effect was obtained. Since in the experiments at the Bureau the flicker, with the types of lamps studied and at the high speeds used, was always practically eliminated, the conclusions in regard to bending at the high speeds, obtained by comparison with the point-to-point values, should be correct. At the lower speeds, judging from results obtained previously, the error would probably be negligible for all types studied except the 32-cp 220-volt lamps to which attention was called in passing.

Uppenborn recognizes the two effects of bending and flicker and attempts to study them separately, but the methods which he used will not yield correct quantitative results, and in the case of the effect of flicker will fail to show, even in a qualitative way, the most important element.

Thus in studying the change in mean horizontal intensity of different types of filaments at different speeds, use is made of the observation that, corresponding to a decrease in mean horizontal intensity on rotation, the end-on candlepower, or intensity in the direction of the tip of the lamp, is increased. Although this is true in a general way, there is no definite constant relation between the changes in mean horizontal and end-on candlepower. This can be seen readily by considering a simple hypothetical case.

If we suppose a single straight filament mounted vertically, the mean horizontal intensity would be the intensity normal to the filament in any azimuth, being the same in all azimuths. The end-on candlepower would be zero. Now, if on rotation the filament was bent slightly the change in mean horizontal intensity would be

quite small, whereas the percentage increase in end-on candlepower would be infinite. Even if we compared the actual change in candles the end-on change would be very much larger than the change in mean horizontal intensity. If now the filament initially had been inclined slightly so that the initial end-on candlepower was different from zero, then for a definite decrease in mean horizontal candlepower on rotation the percentage increase in the end-on candlepower would be a finite quantity. It is therefore evident that at least theoretically the percentage change in end-on candlepower is no indication of the percentage decrease in mean horizontal intensity.

In order to show what differences could exist in practice, the double filament lamp No. 2 was measured for end-on candlepower at rest and when rotating at 350 r. p. m. Whereas at this speed the mean horizontal intensity showed a decrease of 0.6 per cent, the end-on candlepower showed an increase of 17.5 per cent. The actual change in mean horizontal candlepower was from 16.0 to 15.9, and the change in end-on candlepower from 7.7 to 9.05. On the other hand, a lamp of the downward light type showed no appreciable change in end-on candlepower at a speed of 425 r. p. m., although it was found in the previous investigation that for this type of filament the effect of rotation was to increase the mean horizontal intensity by about the same percentage amount as the observed decrease for lamps of the double filament type.

It is evident therefore that the percentage change in end-on candlepower can not be taken as a criterion for determining the effect of rotation on the mean horizontal intensity.

EFFECT OF FLICKER.

One effect of a fluctuating illumination producing the sensation of flicker is a diminished sensibility in making a photometric setting. This is the element which Uppenborn investigated. There is another and more important element, however, which he failed to observe. Not only is the sensibility decreased but the mean reading may itself be in error by several per cent. As shown in the previous paper on this subject some observers ascribe too high a candlepower value to a flickering light, others too low a value, and sometimes every individual setting of one observer may be entirely

without the range of the individual settings of another observer. A large number of readings will reduce the probable error due merely to lack of sensibility, but will not correct for the erroneous judgment of the point of balance of a flickering light.

For many of the common types of lamps in use in the United States the error due to flicker at a speed of 300 or 400 r. p. m. is probably not very great with most experienced observers, and since the error due to bending is quite small at these speeds the rotating lamp method is convenient and reliable in commercial testing. For the photometry of lamps having an annoying flicker, or for the more accurate measurement of lamps with a moderate flicker the simple expedient of employing an auxiliary mirror as described in detail in the former paper referred to above will be found very satisfactory.

WASHINGTON, *April 30, 1907.*

SIMULTANEOUS MEASUREMENT OF THE CAPACITY AND POWER FACTOR OF CONDENSERS.

By Frederick W. Grover.

1. INTRODUCTION.

In a perfect condenser—that is, a condenser without absorption or leakage—the phase of the current is 90° ahead of that of the impressed electromotive force. Although many condensers closely approximate to this ideal case, it is only with well-insulated air condensers that the angle of advance may be regarded as sensibly 90° . In condensers having for a dielectric paper filled with paraffine or beeswax, and even with mica condensers, there is an appreciable energy component of the current in phase with the electromotive force.

In what follows the small angle θ , by which the phase angle ϕ falls short of 90° , will be called for convenience the *phase difference* of the condenser.

The power factor of the condenser is equal to $\cos \phi = \sin \theta$.

A condenser having absorption is equivalent, therefore, in its effect on the phase of the current, to a capacity C in series with a resistance ρ , Fig. 1, of such a value that

$$\tan \theta = p C \rho \quad (1)$$

(where $p = 2\pi$ times the frequency) or to a capacity C in parallel with a resistance W , Fig. 2, such that

$$\tan \theta = \frac{1}{p C W} \quad (2)$$

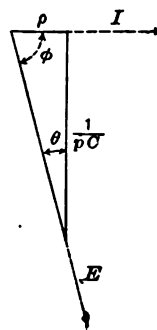


Fig. 1.

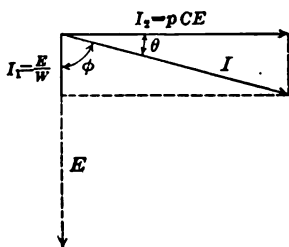


Fig. 2.

Since θ is usually a very small angle, it is evident that for ordinary frequencies ρ is a small resistance, while W is a very large one. We shall sometimes use one of these conventions and sometimes the other, according as one or the other gives greater simplicity to the formulæ.

The power factor of a condenser may be determined—

1. By measuring the energy loss and computing θ .
2. By measuring the angle θ directly.

Since the energy loss in a condenser is very small, it is difficult to obtain accurate results by the first method. Such determinations have, however, been successfully made.

By the use of two methods in which the condenser to be tested was made part of a resonance circuit, Rosa and Smith¹ were able to measure the loss in the condenser with a wattmeter. The mean power factor of six condensers having a dielectric of beeswax and rosin was found to be about 0.02 at 30° . This corresponds to a value of θ of about 1° .

The same observers² measured directly, by a continuous-flow calorimeter, the amount of heat generated in these same condensers, and obtained results in good agreement with those by the wattmeter methods.

Steinmetz³, using one of the wattmeter methods employed by Rosa and Smith, found values for the power factors of some General Electric Company paraffined paper condensers in the neighborhood of $\frac{1}{2}$ per cent, the result obtained being dependent on the frequency. For these condensers, therefore, θ was of the order of magnitude of $0^\circ 15'$.

It is, however, less difficult to measure the phase difference than the energy expended in the dielectric. By means of one of the Rowland electrodynamicometer methods, Potts⁴ measured the increase in effective resistance which was produced by introducing a condenser in series with a bridge arm which already contained an inductance and the hanging coil of the electrodynamicometer. To calculate the angle θ it is necessary to know, in addition to this, not

¹Phys. Rev., Jan., 1899.

²Phys. Rev., Feb., 1899.

³Lond. Electrician, July 5, 1901.

⁴American. Jour. Sci. (4) 10, p. 91; 1900. Phys. Zs. 2, p. 301; 1901.

only the steady current resistance of the coils and their leads, but also the change of this resistance, with the frequency. Further, the unavoidable temperature changes of the copper wire of the coils requires that their resistances be frequently determined.

Rosa⁵ has developed a number of dynamometer methods for the measurement of θ , and has shown that they give results in good agreement with one another, and with those he found by the calorimetric method.

In 1891, Max Wien gave⁶ several alternating current bridge methods for the measurement of inductances and capacities. Among these is an arrangement for comparing the capacities of two condensers and simultaneously obtaining the absorption of the condenser

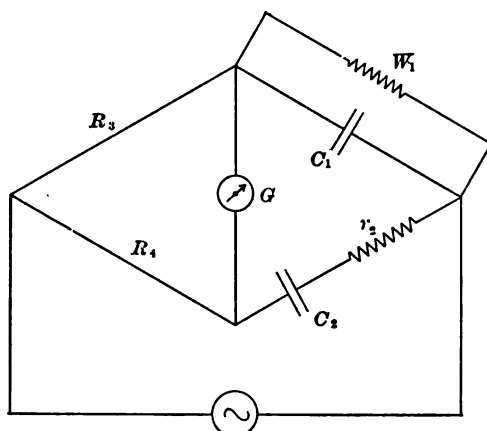


Fig. 3.

C_1 , Fig. 3, the absorption of C_2 being assumed equal to zero. He assumed the absorption of C_1 to be represented by a resistance W_1 in parallel with the condenser. The bridge is balanced by alternately adjusting r_2 and one of the ratio arms. When the current in the tuned galvanometer G is zero

$$\frac{C_1}{C_2} = \frac{R_4}{R_3} - \frac{r_2}{W_1} \quad (A)$$

$$p^2 C_1 C_2 W_1 r_2 = 1 \quad (B)$$

(Wien's nomenclature is here changed to agree with that of this article.)

He also shows how this method may be used to determine *two capacities simultaneously*. To do this, a resistance r_1 is inserted in parallel with C_1 , such that, to a first approximation, the resistance

⁵This Bulletin, 1, p. 383; 1905.

⁶Wied. Annalen, 44, p. 681; 1891.

W_1 , due to the absorption of the condenser, may be neglected. To correct for the absorption of C_1 , a second reading is taken with the resistance r_1 removed. In an example given by Wien the error due to the absorption of C_1 was neglected.

The first-mentioned arrangement of this method has since been used by Hanauer⁷, who measured the change of the capacity and the resistance W_1 , with change of frequency, for a number of solid and liquid dielectrics. The few results obtained for mica gave values of θ as large as several degrees.

This method in a modified form has been used by the author since 1904.

Since the work in this paper was begun Monasch has published⁸ the results of an elaborate investigation (using Wien's method) of the energy losses in cables. Pressures of about 1,500 volts were ordinarily used, although occasionally values as high as 9,000 volts were employed.

In this very interesting paper Monasch describes his variable standard air condenser, constructed to withstand the high voltages used, and discusses the effect of errors due to the capacity of the resistances and to the electrostatic capacity of the bridge with respect to its surroundings. The former of these difficulties he met by the use of resistances coils wound in the manner suggested by Chaperon, and he gives measurements to show that the effect of the latter was in his apparatus very small. The cables tested had a capacity of a few thousandths of a microfarad.

The present paper deals particularly with the determination of the power factor of condensers to be used for precision measurements of inductance and capacity. Especial attention has been paid to the elimination of constant errors, and the different methods have been compared with one another to ascertain how well this has been accomplished.

Examples will be given to show that the power factor gives a good idea of the quality of the condenser, the order of magnitude of its residual charges, and the change of capacity with the frequency and method of charge, all of which depend upon the absorption. It

⁷Wied. Annalen, **65**, p. 789; 1898.

⁸Inaugural Dissertation, Dantzig, 1906, and Annalen der Phys. **22**, p. 905, 1907.

is the best single test which can be made, and has the great advantage that the power factor can be determined simultaneously with the capacity.

2. THE COMPARISON OF CAPACITIES.

The arrangement shown in Fig. 4 has long been used with steady currents, with intermittent currents (using a rotating commutator in the battery and galvanometer circuits), or with alternating currents. In the latter case a telephone has frequently been employed as a detector of current in the galvanometer arm AB. As it is generally impossible, even with a sine wave, to make the current in this arm zero, it has been customary to determine that ratio of R_3 and R_4 which will bring the current to a minimum. The capacity ratio is then determined from the equation

$$\frac{C_1}{C_2} = \frac{R_4}{R_3} \quad (3)$$

When the absorption of the two condensers is nearly the same, the sound in the telephone can be made nearly to disappear, and fairly good settings may be obtained; but, with such differences in the power factor of the two condensers as often occur, the minimum is by no means small, and its position is far from being sharp. This is illustrated in the following example. Suppose

$$\begin{aligned} C_1 &= 1 \text{ mf} & C_2 &= 1.05 \text{ mf} \\ \theta_1 &= 30' & \theta_2 &= 1' \\ R_4 &= 1,000 \text{ ohms.} & p &= 2\pi \text{ times the frequency} = 600 \\ E &= 100 \text{ volts} = \text{emf. impressed on the bridge} \\ g &= 200 \text{ ohms} = \text{resistance of galvanometer circuit.} \end{aligned}$$

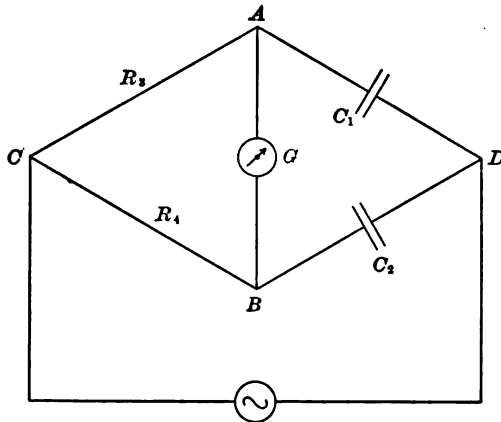


Fig. 4.

This is the case of the comparison of a rather poor paper condenser with a good mica condenser.

From these data we find by equation 1,

$$\rho_1 = 14.545 \text{ ohms}$$

$$\rho_2 = 0.462 \text{ ohms}$$

and it is not difficult, though rather tedious, to calculate the galvanometer current for different values of R_3 . If the condensers were perfect the bridge should be balanced with $R_3 = 1,050$ ohms (equation 3). The following results were obtained:

R_3	Current in galvanometer arm	R_3	Current in galvanometer arm
<i>Ohms</i>	<i>Micro-amperes</i>	<i>Ohms</i>	<i>Micro-amperes</i>
1,045.0	20.184	1,050.1	14.324
1,049.0	14.506	1,050.25	14.333
1,049.75	14.336	1,051.0	14.502
1,049.9	14.325	1,055.0	18.870
1,050.0	14.323		

This table shows that the value of R_3 , corresponding to minimum current, is substantially the same as it would be if the condensers were perfect. The minimum is, however, so "flat" that with an ordinary telephone where 1 micro-ampère can be detected it is doubtful if settings could be made closer than to the nearest ohm; and, if the wave form of the supply deviates very much from a sine curve, the settings would be still worse. If, instead of a telephone, we use a vibration galvanometer it will still be difficult to make close measurements. The vibration galvanometer used in these experiments, which responds to the frequency assumed in this example, will detect 0.2 of a micro-ampère. On the assumption that this corresponds to a deflection of 0.1 mm, the deflection produced by the minimum current in the example would be about 7.1 mm; and, as it would probably be impossible to detect a change in this of less than 0.1 mm, a difference of 1 ohm in R_3 on either side of the balance point would just begin to have a noticeable effect on the galvanometer current. It is, therefore, doubtful if the balance point could be estimated much closer than to the nearest half ohm—that is, to 5 parts in 10,000.

The reason why it is impossible to balance the bridge shown in Fig. 4 is evident from the following considerations. In order that the current in the arm AB, Fig. 4, shall, at every moment, be zero, it is necessary that the potentials of points A and B shall always be the same. This requires that the potential differences AC and BC must not only be equal, but must have the same phase. The same is true of the potential differences AD and BD. In other words, the currents in the two sides of the bridge must have the same phase.

Fig. 5 shows the vector diagrams of the two sides of a bridge containing condensers without absorption or leakage, the galvanometer being removed, and the voltage impressed on the bridge being CD.

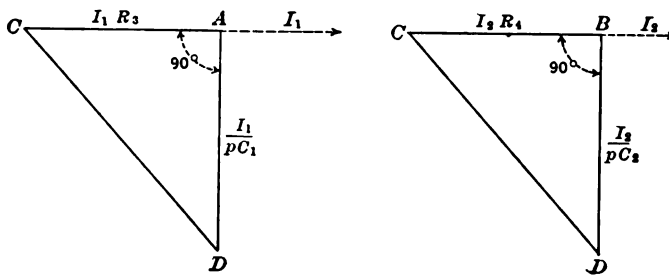


Fig. 5.

In order that A and B shall always be at the same potential at the same moment, triangles ACD and BCD must be equal. CD being common to both, this condition is satisfied if

$$I_1 R_3 = I_2 R_4$$

$$\frac{I_1}{p C_1} = \frac{I_2}{p C_2}$$

$$\therefore p C_2 I_1 = p C_1 I_2 \text{ and } \frac{C_2}{C_1} = \frac{I_2}{I_1} = \frac{R_3}{R_4}$$

as in equation 3.

If, however, we are comparing condensers which have an appreciable absorption, the angles CAD and CBD, Fig. 5, will no longer be right angles, as in the preceding case, and will not be equal unless the condensers have equal power factors. Such a case is illustrated in Fig. 6. The triangles EAD and FBD are the phase diagrams of the condensers C_1 and C_2 , respectively, their

absorptions being represented by the fictitious resistances ρ_1 and ρ_2 . By equation (1), their power factors are

$$\cos EAD = \sin \theta_1$$

$$\cos FBD = \sin \theta_2$$

Since these angles depend on the absorption of the condensers, it will be impossible, simply by adjustment of the ratio $R_3 : R_4$, to make the triangles CAD and CBD equal. That is, it is impossible to

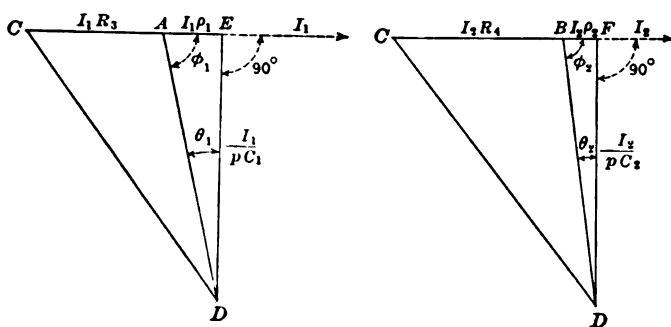


Fig. 6.

make the potentials of A and B the same. It is only possible, then, to find a value of the ratio $R_3 : R_4$ which will make the galvanometer current a minimum. Therefore, in order to bring the points A and B to the same potential it is necessary to shift the phases of the condenser currents. This can be done by any one of three methods.

3. SERIES RESISTANCE METHOD (WIEN).

An adjustable resistance r is inserted in the arm with the condenser having the smaller power factor, and the bridge is balanced by alternately adjusting one of the resistances R_3 , R_4 , and the resistance r .

This is easily and quickly accomplished, if, with each adjustment of one of the two resistances, the observer systematically finds a value of the resistance which will make the galvanometer current a minimum. The successive minima thus obtained very rapidly approximate to zero.

This is illustrated in Fig. 7, in which, as in Fig. 6, the phase diagrams of C_1 and C_2 are represented by the triangles AED and BFD, respectively.

The galvanometer current i_g is proportional to the distance AB.

The locus aa of the point A, when R_s is varied, is one of a family of circles, passing through C and D, and having for a parameter the angle EAD. We will call these *equal phase curves*. Similarly, with R_s as a parameter and r as a variable, we may draw a second family of circles, which are, when angle EAD is nearly 90° , very closely orthogonal to the first and may be called *equal resistance curves*.

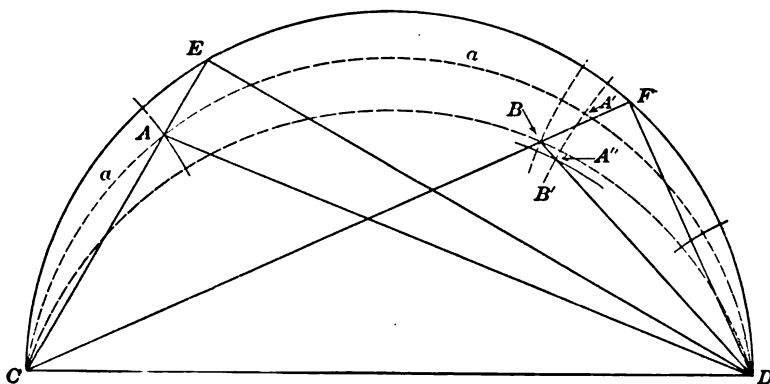


Fig. 7.

Starting from the point A, suppose the resistance R_s is varied until the galvanometer current is a minimum. A will move on its equal phase curve aa to a point A' which satisfies the condition that a line drawn through it, tangent to aa , is perpendicular to the line BA' . A' will lie near, but not upon, the equal resistance curve which passes through B. When, now, r is varied till i_g is again a minimum, a point A'' is found, where the equal resistance curve $A'B'$ is tangent to a line perpendicular to BA'' . Again varying R_s , a minimum value, BA''' , of the current i_g will be obtained, which is still smaller than the preceding minima BA' and BA'' , and thus the condition of zero current is rapidly approximated.

If, however, there be a component of the galvanometer current, due, for instance, to electrostatic charges on some part of the bridge, which does not depend on the capacities and power factors of the

arms, it will be impossible to bring the current i_g to zero by any adjustment of R_3 and r . Not only will this reduce the sensitiveness (see p. 376), but if the value of this disturbing current is affected by changes in R_3 or r , a minimum current may be obtained with values of these which differ appreciably from those corresponding to the capacity ratio and the power factors.

This emphasizes the advantage of compensating for the phase difference of the condensers, for when the galvanometer current is

zero it is certain that the balance depends only on the constants of the four arms of the bridge, and that no external disturbance is appreciable.

Wien derives the conditions for balance in this case on the assumption that the absorption may be represented by a resistance in *parallel* with the condenser. I have, however, adopted the convention of a *series* resistance, because it leads

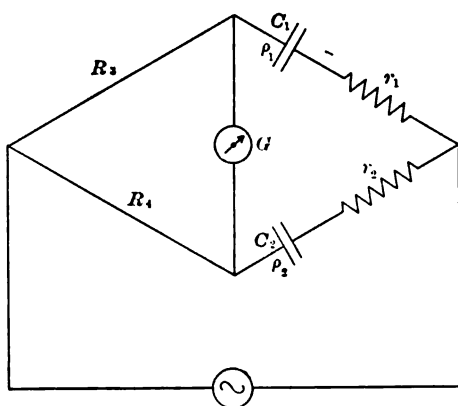


Fig. 8.

to simpler and more symmetrical formulæ.

To avoid ascertaining which of the two condensers has the smaller power factor, it is convenient to insert resistance in series with each condenser, as shown in Fig. 8.

Derivation of the formulæ.—Let the absorptions of the condensers be represented by fictitious resistances ρ_1 and ρ_2 . Then the impedances of the different arms are

$$a_1 = (\rho_1 + r_1) - \frac{i}{pC_1}$$

$$a_2 = (\rho_2 + r_2) - \frac{i}{pC_2}$$

$$a_3 = R_3$$

$$a_4 = R_4 \text{ where } i = \sqrt{-1}$$

It has been shown by Lord Rayleigh⁹ and others that the condition for balance in such a bridge is

$$a_2 a_3 - a_1 a_4 = 0 \quad (4)$$

We have, therefore,

$$R_3 \left[(\rho_2 + r_2) - \frac{i}{pC_2} \right] - R_4 \left[(\rho_1 + r_1) - \frac{i}{pC_1} \right] = 0$$

Separating the real and imaginary parts,

$$\begin{aligned} R_3(\rho_2 + r_2) - R_4(\rho_1 + r_1) &= 0 \\ -\frac{R_3}{pC_2} + \frac{R_4}{pC_1} &= 0 \\ \therefore \frac{\rho_2 + r_2}{\rho_1 + r_1} &= \frac{R_3}{R_4} \end{aligned} \quad (5)$$

$$\frac{C_2}{C_1} = \frac{R_3}{R_4} \quad (6)$$

From these it follows that

$$\begin{aligned} \frac{\rho_2 + r_2}{\rho_1 + r_1} &= \frac{C_1}{C_2} \\ \therefore pC_2(\rho_2 + r_2) &= pC_1(\rho_1 + r_1) \\ \text{or} \quad pC_2\rho_2 - pC_1\rho_1 &= pC_1r_1 - pC_2r_2 \end{aligned}$$

which by (1) becomes

$$\tan \theta_2 - \tan \theta_1 = pC_1r_1 - pC_2r_2$$

For such small angles, however, we may write

$$\tan \theta_2 - \tan \theta_1 = \tan (\theta_2 - \theta_1)$$

and the above formula becomes

$$\tan (\theta_2 - \theta_1) = pC_1r_1 - pC_2r_2 \quad (7)$$

⁹ Proc. Roy. Soc. 49, p. 203; 1885.

To test the approximation, let us take the unfavorable case

$$\theta_1 = 0^\circ 3' \quad \theta_2 = 1^\circ 0'$$

Then $\tan \theta_1 \tan \theta_2 = 0.000015$

This quantity is negligible in the measurement of power factor.

It may easily be shown that (7) is equivalent to Wien's expression (B) for the power factor. For that case $\theta_2 = 0$, and therefore $\rho_2 = 0$, so that (7) becomes

$$\rho C_2 r_2 = \tan \theta_1$$

Wien's expression is

$$\rho^2 C_1 C_2 W_1 r_2 = 1$$

which may be written

$$\rho C_2 r_2 = \frac{1}{\rho C_1 W_1}$$

and by (2) we have

$$\rho C_2 r_2 = \tan \theta_1$$

Sources of Error.

- (a) Inductance or capacity of R_3 and R_4 .
- (b) Error in the ratio $R_3 : R_4$.
- (c) Inductance or capacity of r_1 and r_2 .
- (d) Electrostatic induction between the bridge and its surroundings.

We will consider these separately, since their elimination is accomplished in different ways.

(a) **Inductance or capacity of R_3 and R_4 .**—For this case

$$a_3 = R_3 + i\rho l_3$$

$$a_4 = R_4 + i\rho l_4$$

Then for balance of the bridge

$$(R_3 + i\rho l_3) \left[(\rho_2 + r_2) - \frac{i}{\rho C_2} \right] - (R_4 + i\rho l_4) \left[(\rho_1 + r_1) - \frac{i}{\rho C_1} \right] = 0.$$

The real part of this gives

$$R_3(\rho_2 + r_2) + \frac{\rho l_3}{\rho C_2} - R_4(\rho_1 + r_1) - \frac{\rho l_4}{\rho C_1} = 0$$

which from (6) and (1) may be written approximately

$$\tan (\theta_2 - \theta_1) = (\dot{p}C_1r_1 - \dot{p}C_2r_2) + \left(\frac{\dot{p}l_4}{R_4} - \frac{\dot{p}l_3}{R_3} \right) \quad (8)$$

$$= (\dot{p}C_1r_1 - \dot{p}C_2r_2) + (\tan \phi_4 - \tan \phi_3) \quad (8a)$$

The imaginary part gives

$$-\frac{R_3}{\dot{p}C_2} + \dot{p}l_3(\rho_2 + r_2) + \frac{R_4}{\dot{p}C_1} - \dot{p}l_4(\rho_1 + r_1) = 0$$

or

$$\frac{C_2}{C_1} = \frac{R_3}{R_4} + \frac{\dot{p}^2 C_2}{R_4} \left\{ l_4(\rho_1 + r_1) - l_3(\rho_2 + r_2) \right\}$$

which may be written approximately

$$\frac{C_2}{C_1} = \frac{R_3}{R_4} \left[1 + \frac{\dot{p}l_4}{R_4} \cdot \dot{p}C_1(\rho_1 + r_1) - \frac{\dot{p}l_3}{R_3} \cdot \dot{p}C_2(\rho_2 + r_2) \right] \quad (9)$$

$$= \frac{R_3}{R_4} \left[1 - \tan \phi_4 \tan (90^\circ - \phi_1) + \tan \phi_3 \tan (90^\circ - \phi_2) \right] \quad (9a)$$

where ϕ_1 , ϕ_2 , etc., are the phase differences of the currents and impressed electromotive forces of the arms 1, 2, etc., an angle of lag being taken as positive.

As an example, suppose

$$C_2 = C_1 = 1 \text{ mf}$$

$$r_1 = 1 \text{ ohm}$$

$$R_3 = R_4 = 1,000 \text{ ohms}$$

$$\dot{p} = 628$$

$$\rho_1 = 49 \text{ ohms}$$

$$l_3 = -0.00004 \text{ henry}$$

$$\rho_2 = 5 \text{ ohms}$$

$$l_4 = -0.0015 \text{ henry}$$

A *negative* inductance l signifies that the capacity effect in the coil is predominant, and that its effect on the phase of the current is equal and opposite to that of an inductance l .

This case is realized where R_4 consists of a single coil of bifilar winding, and R_3 is made up of a number of coils, the subdivision reducing its effective electrostatic capacity.

From equation (5) $r_2 =$ approximately 45 ohms, and by (9) the correction to the capacity ratio is nearly 3 parts in 100,000.

By (8), $\tan(\theta_2 - \theta_1) = 0.0276 - .00092$, that is, the correction to the difference of phase of the two condensers is more than 3 per cent, a quantity which would be important if the power factors of the two condensers were nearly equal. However, by taking for R_3 and R_4 coils wound similarly, and by making another setting with R_3 and R_4 interchanged, this error may be much reduced.

(b) **Error in the ratio $R_3:R_4$.**—Suppose that R_3 and R_4 are each divided into two parts such that

$$R_3 = W_3 + w_3,$$

$$R_4 = W_4 + w_4,$$

where W_3 and W_4 have the same nominal value, and are nearly equal to R_3 and R_4 , and where w_3 and w_4 are small variable resistances of perhaps 100 ohms. Then if the latter be fairly well adjusted, W_3 and W_4 do not need to be accurately known, provided that the bridge be balanced a second time with W_3 and W_4 interchanged, but with w_3 and w_4 in their original positions.

For, if $W_3 = W + \delta_3$ and $W_4 = W + \delta_4$, where δ_3 and δ_4 are the differences between W_3 and W_4 and their nominal value W , then

$$\frac{C_1}{C_2} = \frac{W_4 + w_4'}{W_3 + w_3'} = \frac{W + \delta_4 + w_4'}{W + \delta_3 + w_3'}$$

or

$$\frac{C_1}{C_2} = \frac{W + \delta_4 + w_4'}{W + \delta_3 + w_3'} = \frac{W + \delta_3 + w_4''}{W + \delta_4 + w_3''}$$

By addition we have

$$\begin{aligned} \frac{C_1}{C_2} &= \frac{2W + (w_4' + w_4'') + (\delta_3 + \delta_4)}{2W + (w_3' + w_3'') + (\delta_3 + \delta_4)} \\ &= \frac{W + \frac{1}{2}(w_4' + w_4'') + \frac{1}{2}(\delta_3 + \delta_4)}{W + \frac{1}{2}(w_3' + w_3'') + \frac{1}{2}(\delta_3 + \delta_4)} \end{aligned} \quad (10)$$

or, very nearly

$$\frac{C_1}{C_2} = \frac{W + \frac{1}{2}(w_4' + w_4'')}{W + \frac{1}{2}(w_3' + w_3'')} \quad (10a)$$

To calculate the ratio of the capacities from (10a) it is necessary to know only the means of the settings of w_3 and w_4 and the nominal

value W of the main coils. Usually only one of the resistances w_3 and w_4 would be varied and the formula would be somewhat simplified.

As a test of error from using (10a) let us take as a rather extreme case

$$\begin{aligned} \delta_3 &= 1 \text{ ohm} & W + \frac{w_4' + w_4''}{2} &= 1100 \text{ ohms} \\ \delta_4 &= 2 \text{ ohms} & W + \frac{w_3' + w_3''}{2} &= 1050 \text{ ohms.} \end{aligned}$$

From formula (10) $\frac{C_1}{C_2} = 1.04765$.

From formula (10a) $\frac{C_1}{C_2} = 1.04755$, an error of only 1 in 10,000. With well-adjusted coils and with errors δ_3 and δ_4 of opposite sign the error would be much smaller.

(c) **Inductance or capacity of r_1 and r_2 .**—Formula (7) shows that for a given difference of phase ($\theta_2 - \theta_1$), the difference of the resistances r_1 and r_2 will be greater the smaller the capacities measured. For instance, for

$$\begin{aligned} C_1 &= C_2 = 0.001 \text{ mf} \\ \phi &= 628 \\ \theta_2 - \theta_1 &= 0^\circ 5' \end{aligned}$$

we will have $(r_1 - r_2) = 2,316$ ohms. Examples will be given below to show that much greater values of θ may occur in small mica condensers. Measurements of the change of phase produced by coils of 10,000 ohms and greater show they have quite an appreciable capacity. For instance, the equivalent capacity of a 10,000-ohm coil—that is, the capacity which must be shunted across the terminals of a resistance of 10,000 ohms in order to produce a phase displacement equal to that observed in the coil—was found to be 0.0019 mf. For a 100,000-ohm coil the value was 0.00058 mf. It is therefore interesting to investigate the effect of introducing such capacities into the arm with the condenser.

The impedance of a resistance r in parallel with a capacity k is

$$\frac{-\frac{i}{pk}r}{r - \frac{i}{pk}} = -\frac{ir}{pkr - i} = \frac{r - ipkr^2}{1 + p^2k^2r^2}$$

Therefore, putting

$$a_1 = \left(\rho_1 - \frac{i}{pC_1} \right) + \frac{r_1 - ipk_1 r_1^2}{1 + p^2 k_1^2 r_1^2}$$

$$a_2 = \left(\rho_2 - \frac{i}{pC_2} \right) + \frac{r_2 - ipk_2 r_2^2}{1 + p^2 k_2^2 r_2^2}$$

$$a_3 = R_3$$

$$a_4 = R_4$$

$$R_3 \left[\left(\rho_2 + \frac{r_2}{1 + p^2 k_2^2 r_2^2} \right) - i \left(\frac{1}{pC_2} + \frac{pk_2 r_2^2}{1 + p^2 k_2^2 r_2^2} \right) \right] \\ - R_4 \left[\left(\rho_1 + \frac{r_1}{1 + p^2 k_1^2 r_1^2} \right) - i \left(\frac{1}{pC_1} + \frac{pk_1 r_1^2}{1 + p^2 k_1^2 r_1^2} \right) \right] = 0.$$

Separating the real and imaginary parts, we have

$$R_3 \left(\rho_2 + \frac{r_2}{1 + p^2 k_2^2 r_2^2} \right) - R_4 \left(\rho_1 + \frac{r_1}{1 + p^2 k_1^2 r_1^2} \right) = 0 \quad (11)$$

$$-R_3 \left(\frac{1}{pC_2} + \frac{pk_2 r_2^2}{1 + p^2 k_2^2 r_2^2} \right) + R_4 \left(\frac{1}{pC_1} + \frac{pk_1 r_1^2}{1 + p^2 k_1^2 r_1^2} \right) = 0 \quad (12)$$

Equation (12) may be written

$$\frac{C_2}{C_1} = \frac{R_3}{R_4} \left[1 + \frac{p^2 C_2 k_2 r_2^2}{1 + p^2 k_2^2 r_2^2} - \frac{p^2 C_1 k_1 r_1^2}{1 + p^2 k_1^2 r_1^2} \right]$$

or, neglecting the quantities $p^2 k_1^2 r_1^2$ and $p^2 k_2^2 r_2^2$ with respect to unity,

$$\frac{C_2}{C_1} = \frac{R_3}{R_4} \left[1 + (p^2 C_2 k_2 r_2^2 - p^2 C_1 k_1 r_1^2) \right] \quad (13)$$

and from (11), using the approximation $\frac{R_3}{R_4} = \frac{C_2}{C_1}$ we may derive

$$\tan (\theta_2 - \theta_1) = \frac{pC_1 r_1}{1 + p^2 k_1^2 r_1^2} - \frac{pC_2 r_2}{1 + p^2 k_2^2 r_2^2}$$

or, approximately,

$$\tan (\theta_2 - \theta_1) = pC_1 r_1 [1 - (pk_1 r_1)^2] - pC_2 r_2 [1 - (pk_2 r_2)^2] \quad (14)$$

For large capacities, where r_1 , r_2 , and, therefore, k_1 , k_2 , are small, the correction terms in (13) and (14) will be negligible. In any case, if the condensers be of nearly equal capacity and power factor, the corrections will be unimportant, since r_1 and r_2 and, therefore, k_1 and k_2 will be nearly equal.

When, however, condensers with capacities as small as 0.001 mf, and with quite different power factors, are to be compared, these corrections may become appreciable.

In one extreme case, when a poor mica condenser of 0.003 mf was being compared with a good mica condenser of the same capacity, the uncorrected capacity was found (by calculation from the approximate values of the capacities of r_1 and r_2) to be in error by 7 per cent. The correction to the measured value of the power factor, due to the same cause, was 15 per cent. Usually, however, the errors are much smaller.

(d) **Electrostatic capacity of the bridge.**—The errors, due to the electrostatic capacity of the different parts of the bridge with respect to the earth, are in some cases troublesome, but may be largely eliminated. Owing to this cause the balance point of the bridge depends on the potential of the points A and B, Fig. 4, with respect to the earth, and changes when resistance is placed unsymmetrically in the galvanometer circuit.

Although these effects are especially important only when capacities smaller than 0.1 mf are being compared, the following method for the elimination of such errors is so simple that it is recommended in all cases.

The substitution method.—To compare two nearly equal condensers, an auxiliary condenser C_2 is employed. The condenser to be tested, C_1' , and the standard, C_1'' , are then successively balanced against C_2 .

Neglecting residual inductances or capacities in the resistances, we will have from (6)

$$\frac{C_1'}{C_2} = \frac{R_4}{R_3'} \quad \text{and} \quad \frac{C_1''}{C_2} = \frac{R_4}{R_3''}$$

Therefore

$$\frac{C_1'}{C_1''} = \frac{R_3''}{R_3'} \quad (15)$$

where R_3' and R_3'' are the two settings of R_3 found in the two measurements.

For the power factor we have from (7)

$$\begin{aligned}\tan \theta_1' - \tan \theta_2 &= pC_1'r_1' \\ \tan \theta_1'' - \tan \theta_2 &= pC_1''r_1'' \\ \therefore \tan \theta_1' - \tan \theta_1'' &= pC_1''r_1'' - pC_1'r_1'\end{aligned}$$

or

$$\tan (\theta_1' - \theta_1'') = pC_1''r_1'' - pC_1'r_1' \quad (16)$$

$$= pC_1'' \left(r_1'' - \frac{R_3''}{R_3'} \cdot r_1' \right) \quad (16a)$$

r_1' and r_1'' being the two observed values of r_1 .

Either (16) or (16a) is convenient to use, the logarithms of the capacities C_1' , C_1'' , and of their ratio $\frac{R_3''}{R_3'}$ having been already found in the calculation of the capacities. From its symmetry, formula (16) is perhaps to be preferred, when a number of condensers have been measured, the quantity pC_1r_1 being calculated for each of the condensers, and subtracted from the corresponding quantity for the standard.

Each of the ratio arms consists of two parts—a resistance W whose nominal value is the same for both arms, and a much smaller variable resistance w , one of which, w_3 , is used to make the final setting. With condensers whose capacities do not differ by more than a few per cent, the resistance needs only to be approximately known, since the difference of the capacities is proportional to the difference in the observed values of w_3 . Additional measurements may be taken with the two coils W reversed (independently of w_3 and w_4) as previously described, but this is now of advantage only to obtain check measurements for reducing the effect of the accidental errors in balancing the bridge. The same end is reached by taking an additional pair of readings with the condenser arms interchanged.

Of the series resistances r_1 , r_2 , the latter is kept constant and the phases of the condenser currents are brought into unison by adjusting r_1 . The difference in the power factors of the two condensers is, therefore, measured by the difference of the settings of r_1 , and, consequently, the leads in the condenser arms do not need to be known.

Of course, if any condenser is used with permanent leads of its own, their resistance should be determined and a correction applied to the apparent power factor if necessary.

The effects of the electrostatic capacity of the bridge itself will enter into the measurements on the two condensers in the same manner, since they are both measured in the same arm of the bridge; and, if their capacities are not very different, the value of the error in the two cases will be sensibly the same and will not affect either the measurement of the capacity or the power factor. If, however, the condensers are of small and quite unequal capacity, the electrostatic effects may be different enough as a result of the necessary change in R_s , when one condenser is substituted for the other, to cause an appreciable error. A modification of the method which overcomes this difficulty will be described later.

In deriving the formulæ for the substitution method, the residual inductances and capacities of the ratio arms have been neglected. This is entirely allowable, since the resistances included in R_s' and R_s'' are identical except for the small resistance ($R_s' - R_s''$) which has negligible inductance. The correction terms, therefore, in (8) and (9) do not appreciably change when one condenser is substituted for the other, and since their absolute value has been made small, by using similarly wound coils in R_s and R_o , this error may be considered as entirely eliminated.

In order to avoid introducing capacity into the condenser arms outside of the condensers themselves, Professor Rosa has suggested the following method, which may be regarded as more generally applicable than the Series Resistance Method. It presents, however, some practical inconveniences when condensers of very large power factor are to be measured.

4. SERIES INDUCTANCE METHOD.

As its name suggests, this method makes use of a variable inductance to compensate for the difference of phase of the currents in the condensers. It has the advantage that the compensating device is not placed in the arm with the condenser, and can not, therefore, modify the capacity of that branch. It is, for this reason, of especial value in the comparison of very small capacities.

Fig. 9 shows the arrangement of the bridge when two inductances are employed, one of fixed value and the other variable. Only one inductance is necessary, provided the condensers are not of too nearly equal power factor. In such a case, the single inductance, which must of course be variable, is placed in the ratio arm which is adjacent to the condenser with the larger power factor. By using *two* coils, however, the necessity for knowing, in advance, which condenser has the greater absorption is removed. Further, when the condensers are of nearly equal power factor, and, consequently, the

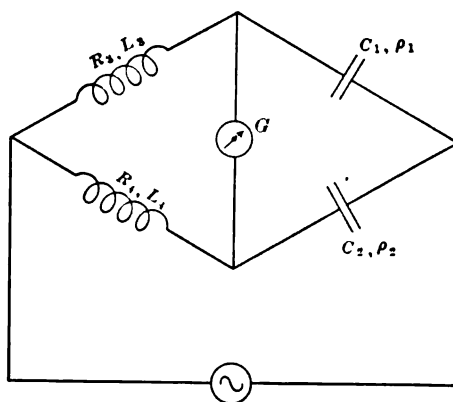


Fig. 9.

value of the inductance of the single coil would be small, the insertion of a coil in the other arm obviates the inconvenience of working with small inductances, and the inductance of the leads does not need to be considered.

Derivation of the formulæ.—To derive the conditions for a balance of the bridge, we proceed as before. From Fig. 9

$$\begin{aligned} a_1 &= \rho_1 - \frac{i}{pC_1} & a_3 &= R_3 + ipL_3 \\ a_2 &= \rho_2 - \frac{i}{pC_2} & a_4 &= R_4 + ipL_4 \end{aligned}$$

and substituting in the equation $a_1a_4 - a_2a_3 = 0$, we have

$$(R_3 + ipL_3)\left(\rho_2 - \frac{i}{pC_2}\right) - (R_4 + ipL_4)\left(\rho_1 - \frac{i}{pC_1}\right) = 0$$

Separating the real and imaginary parts, we obtain

$$R_3\rho_3 - R_1\rho_1 + \frac{L_3}{C_2} - \frac{L_1}{C_1} = 0 \quad (17)$$

$$-\frac{R_3}{pC_2} + \frac{R_1}{pC_1} + pL_3\rho_3 - pL_1\rho_1 = 0 \quad (18)$$

From (17) we have

$$\tan(\theta_1 - \theta_3) = \frac{pL_3}{R_3} - \frac{pL_1}{R_1} \quad (19)$$

and from (18)

$$\begin{aligned} \frac{C_1}{C_2} &= \frac{R_1}{R_3} + \frac{p^2 C_1}{R_3} (L_3\rho_3 - L_1\rho_1) \\ &= \frac{R_1}{R_3} + \frac{pL_3}{R_3} pC_2\rho_2 \frac{C_1}{C_2} - \frac{pL_1}{R_1} pC_1\rho_1 \frac{R_1}{R_3} \\ &= \frac{R_1}{R_3} \left[1 + \frac{pL_3}{R_3} \tan \theta_2 - \frac{pL_1}{R_1} \tan \theta_1 \right] \end{aligned} \quad (20)$$

$$= \frac{R_1}{R_3} \left[1 + \tan \phi_3 \tan \theta_2 - \tan \phi_1 \tan \theta_1 \right] \quad (20a)$$

Although the usual simple formula (3) for the ratio of the capacities does not hold exactly, the correction terms in (20) are exceedingly small in practice. As an example let us take

$$\begin{aligned} R_3 &= R_1 = 1,000 \text{ ohms} \\ L_1 &= 0.01 \text{ henry} \\ p &= 628 \\ \theta_1 &= 0^\circ 5' \text{ and } \therefore \tan \theta_1 = 0.001454 \\ \theta_3 &= 0^\circ 1' \quad \tan \theta_3 = 0.000291 \end{aligned}$$

We then find

$$\frac{pL_1}{R_1} = 0.00628, \text{ and from (19)}$$

$$\frac{pL_3}{R_3} = \tan 4' + 0.00628 = 0.001792$$

The correction term in (20) is, therefore,

$$0.001792 \times 0.000291 - 0.00628 \times 0.001454 = -0.0000085$$

which is negligible.

What has been said concerning the elimination of errors in the Series Resistance Method is applicable here. As in the previous method, also, the comparison of the condenser with the standard by means of a substitution method is to be recommended.

The substitution method.—Using the previous nomenclature, we may derive from (19)

$$\begin{aligned}\tan (\theta_1' - \theta_2) &= \frac{\rho L_3'}{R_3'} - \frac{\rho L_4}{R_4} \\ \tan (\theta_1'' - \theta_2) &= \frac{\rho L_3''}{R_3''} - \frac{\rho L_4}{R_4} \\ \text{and } \therefore \tan (\theta_1' - \theta_1'') &= \frac{\rho L_3'}{R_3'} - \frac{\rho L_3''}{R_3''}\end{aligned}\quad (21)$$

and from (20)

$$\begin{aligned}\frac{C_1'}{C_2} &= \frac{R_4}{R_3'} \left[1 + \tan \theta_2 \cdot \frac{\rho L_3'}{R_3'} - \tan \theta_1' \cdot \frac{\rho L_4}{R_4} \right] \\ \frac{C_1''}{C_2} &= \frac{R_4}{R_3''} \left[1 + \tan \theta_2 \cdot \frac{\rho L_3''}{R_3''} - \tan \theta_1'' \cdot \frac{\rho L_4}{R_4} \right]\end{aligned}$$

or, approximately,

$$\frac{C_1'}{C_1''} = \frac{R_3''}{R_3'} \left[1 + \tan \theta_2 \left(\frac{\rho L_3'}{R_3'} - \frac{\rho L_3''}{R_3''} \right) - \frac{\rho L_4}{R_4} \tan (\theta_1' - \theta_1'') \right]$$

which by (21) becomes

$$\frac{C_1'}{C_1''} = \frac{R_3''}{R_3'} \left[1 + \tan (\theta_1' - \theta_1'') \left\{ \tan \theta_2 - \frac{\rho L_4}{R_4} \right\} \right] \quad (22)$$

The correction factor in this equation is, as before, very small, so that usually we calculate the capacity ratio by the simple formula (3). It is to be noticed that the correction depends almost entirely on the power factors of the two condensers, and their difference. As an example let us take the constants of the preceding problem. The correction term in (22) becomes

$$\tan 4' (\tan 1' - 0.00628) = -0.000007$$

which is usually negligible.

The substitution method just described possesses the following advantages:

(a) Ordinary errors of adjustment of the resistances R_3 and R_4 are of negligible effect.

(b) The resistances of the inductances need not be accurately measured. This is a great advantage, since inductances are generally wound with copper, and it is difficult to determine accurately enough the temperature of the coils. This advantage is not shared by the method where the condensers are compared directly.

(c) Errors due to the electrostatic capacity of the bridge (see p. 389) are practically eliminated.

(d) The absolute values of the inductances need not be known very accurately. It is merely necessary that the relative values of the inductance in L_3 be determined. If L_3 is a well-calibrated variable inductance this condition can be fulfilled. When, however, the power factor of the condenser to be tested differs very much from that of the standard, some inconvenience may be experienced in finding the proper value of the inductance. This difficulty may be lessened, when a limited number of fixed inductances are at hand, by varying both ratio arms by equal amounts to aid in obtaining a balance. If, however, the difference in θ_1' and θ_1'' is much over a degree of arc, the frequency is below 100 cycles per second, and the condensers are not smaller than about 0.01 mf, the Series Resistance Method will be found more convenient.

The formula (21) for the power factor is easy to use. Having calculated the capacities, it is only necessary to determine the quantity $\frac{pL_3}{R_3}$ for each condenser and to take the differences. With this method it is easier to judge of the order of magnitude of the power factor, while the observations are being taken, than with the Series Resistance Method, because the setting of the inductance does not depend on the *magnitude* of the capacity.

5. PARALLEL RESISTANCE METHOD.

In this method the difference of phase in the condenser currents is compensated by a variable resistance R in parallel with the condenser which has the smaller power factor. From equation (2) it follows that, at a frequency of 100 cycles, to represent the absorption of

a 1-microfarad condenser, for which $\theta = 0^\circ 5'$, the fictitious resistance W must be taken as greater than 1 megohm. The smaller the capacity and the power factor the greater will this resistance be. In order, therefore, to avoid using extremely large resistances, it is better to place 100,000 ohms in parallel with the condenser to be tested, in which case the adjustable resistance in parallel with the standard will be of convenient value. This arrangement of the bridge is shown in Fig. 10, the absorption resistances W_1 and W_2 of the condensers being shown dotted.

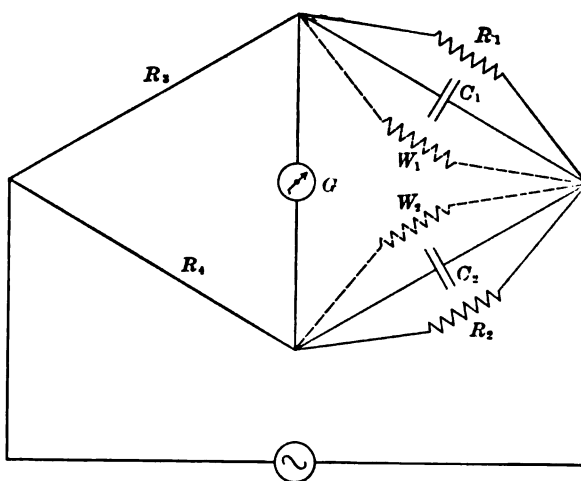


Fig. 10.

This is similar to a bridge method given by Wien for the simultaneous measurement of two capacities. For this purpose, however, R_1 and R_2 must be taken so small that the effect of W_1 and W_2 on the resistances of those arms is small or negligible.

Derivation of the formulæ.—In Fig. 10 let N_1 and N_2 be the joint resistances of the condenser arms.

$$\frac{1}{N_1} = \frac{1}{W_1} + \frac{1}{R_1}$$

$$\frac{1}{N_2} = \frac{1}{W_2} + \frac{1}{R_2}$$
(23)

We have, then,

$$a_1 = \frac{1}{\frac{1}{N_1} + \frac{1}{-i - pC_1}} = \frac{1}{\frac{1}{N_1} + ipC_1}$$

$$a_2 = \frac{1}{\frac{1}{N_2} + ipC_2}$$

$$a_3 = R_3$$

$$a_4 = R_4$$

For balance of the bridge

$$\frac{R_3}{\frac{1}{N_2} + ipC_2} - \frac{R_4}{\frac{1}{N_1} + ipC_1} = 0$$

or

$$\frac{R_3}{N_1} + ipC_1R_3 - \frac{R_4}{N_2} - ipC_2R_4 = 0$$

Separating the real and imaginary parts

$$\frac{R_3}{N_1} = \frac{R_4}{N_2} \quad (24)$$

$$\frac{C_1}{C_2} = \frac{R_4}{R_3} \quad (25)$$

From these it follows that

$$\frac{N_1}{N_2} = \frac{C_2}{C_1} \quad \text{or} \quad \frac{\frac{1}{N_1}}{\frac{1}{N_2}} = \frac{\frac{1}{C_2}}{\frac{1}{C_1}}$$

Substituting for N_1 and N_2 their values in (23) this becomes

$$\frac{\frac{1}{W_1} + \frac{1}{R_1}}{\frac{1}{W_2} + \frac{1}{R_2}} = \frac{\frac{1}{C_2}}{\frac{1}{C_1}}$$

$$\frac{\frac{1}{pC_1W_1} - \frac{1}{pC_2W_2}}{\frac{1}{pC_1R_1} - \frac{1}{pC_2R_2}} = \frac{\frac{1}{C_2}}{\frac{1}{C_1}}$$

or

$$\tan \theta_1 - \tan \theta_2 = \frac{\frac{1}{pC_2R_2}}{\frac{1}{pC_1R_1}}$$

and very approximately

$$\tan(\theta_1 - \theta_2) = \frac{1}{\rho C_2 R_2} - \frac{1}{\rho C_1 R_1} \quad (26)$$

In order that we may use the simple formula (25) it is necessary that the capacities k_1 , k_2 of the resistances R_1 , R_2 shall be negligible compared with C_1 and C_2 . If this condition is not fulfilled, these capacities, which are in parallel with the condenser arms, must be included in (25), which becomes

$$\frac{C_1 + k_1}{C_2 + k_2} = \frac{R_2}{R_1}$$

or very approximately

$$\frac{C_1}{C_2} = \frac{R_2}{R_1} \left[1 + \left(\frac{k_2}{C_2} - \frac{k_1}{C_1} \right) \right] \quad (27)$$

If the power factors are nearly equal, R_1 and R_2 will be also nearly equal, and by using similarly wound coils in these resistances, the differences between k_1 and k_2 will be small, so that the correction term in (27) may become negligible.

The substitution method.—This is the same in principle as in the Series Resistance and Series Inductance Methods. Adopting the usual nomenclature, the formulæ for this case follow at once from (26) and (27). They are

$$\frac{C_1'}{C_1''} = \frac{R_2''}{R_2'} \left[1 + \left(\frac{k_1''}{C_1''} - \frac{k_1'}{C_1'} \right) \right] \quad (28)$$

$$\tan(\theta_1' - \theta_1'') = \frac{1}{\rho C_1'' R_1''} - \frac{1}{\rho C_1' R_1'} \quad (29)$$

The substitution method possesses the following previously discussed advantages:

- (a) The ratio arms need not be accurately known.
- (b) The effect of residual inductances or capacities in R_2 and R_1 is eliminated.
- (c) Electrostatic induction effects are eliminated.

And, further, (d), by using this arrangement, the correction term of (28) is much reduced, and in many cases becomes negligible, since k_1' and k_1'' differ only by the value of the capacity of the resistance ($R_1' - R_1''$).

6. ADAPTATION OF ANDERSON'S METHOD.

In each of the three methods just described a simple bridge is used, the condensers being included in two of the arms, and the ratio coils in the other two, and their differences lie in the means by which the difference of phase of the condenser currents is brought to zero. The scheme about to be described is considerably different in principle, and is, therefore, aside from its other advantages, especially valuable as a check on the other three methods.

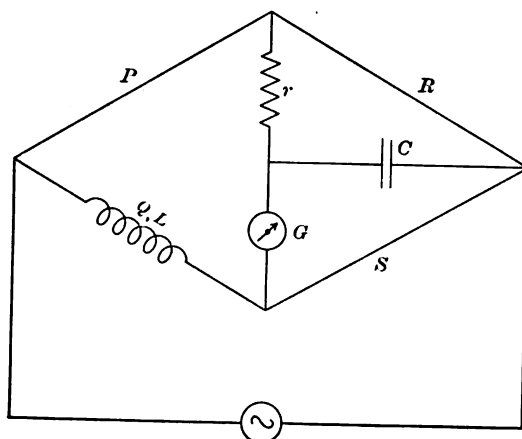


Fig. 11.

The conditions for balance in the ideal Anderson's bridge, Fig. 11, are¹⁰

$$L = CS \left[\frac{r(P+R)}{R} + P \right] \quad (30)$$

$$\frac{Q}{S} = \frac{P}{R} \quad (31)$$

The expression for L is very little modified by the power factor of the condenser,¹¹ but under certain circumstances the deviation of the resistance Q from the value given by (31) may become considerable.

¹⁰ Anderson, *Phil. Mag.*, **31**, p. 329; 1891.

¹¹ Rosa and Grover, *this Bulletin*, **1**, p. 312, equations 18 and 20; 1905.

The modified formulæ are

$$L = CS \left[\frac{r(P+R)}{R} + P \right] (1 - \tan^2 \theta) \quad (32)$$

$$Q = \frac{PS}{R} + p^2 L \rho C = \frac{PS}{R} + pL \tan \theta \quad (33)$$

or, when $P=R$,

$$L = CS (2r + P) (1 - \tan^2 \theta) \quad (34)$$

$$Q = S + pL \tan \theta \quad (35)$$

where θ is the phase difference of the condenser used in the bridge.

If the bridge be balanced successively with two condensers having a phase difference $(\theta_1 - \theta_2)$, the same inductance being used in both cases, we obtain at once the ratio of the capacities and the difference of their power factors.

For, from (34)

$$L = C_1 S (2r_1 + P) (1 - \tan^2 \theta_1)$$

$$L = C_2 S (2r_2 + P) (1 - \tan^2 \theta_2)$$

$$\therefore \frac{C_2}{C_1} = \frac{2r_1 + P}{2r_2 + P} \cdot \frac{(1 - \tan^2 \theta_1)}{(1 - \tan^2 \theta_2)}$$

or, very approximately,

$$\frac{C_2}{C_1} = \frac{2r_1 + P}{2r_2 + P} \left[1 + (\tan^2 \theta_2 - \tan^2 \theta_1) \right] \quad (36)$$

and in many cases it is sufficient to write

$$\frac{C_2}{C_1} = \frac{2r_1 + P}{2r_2 + P} \quad (36a)$$

From (35)

$$Q_1 = S + pL \tan \theta_1$$

$$Q_2 = S + pL \tan \theta_2$$

$$\therefore Q_2 - Q_1 = \Delta Q = pL (\tan \theta_2 - \tan \theta_1)$$

or nearly enough

$$\Delta Q = pL \tan (\theta_2 - \theta_1) \quad (37)$$

from which is obtained

$$\tan (\theta_2 - \theta_1) = \frac{\Delta Q}{pL} = \frac{\Delta Q'}{pL} \quad (38)$$

where $\Delta Q'$ is the observed change in the resistance Q' included in the Q arm with the coil.

It is very seldom that the correction factor in (36) need be applied, since for the rare case that $(\theta_2 - \theta_1) = 1^\circ$ its value is only 3 parts in 10,000, and this decreases nearly with the square of $(\theta_2 - \theta_1)$.

This is inherently a substitution method, and possesses the advantages that, for condensers of the same nominal value—

(a) Only the relative values of r_1 and r_2 need be *closely* known in the determination of the ratio of the capacities. It is sufficient, with fairly well adjusted resistances, to use the nominal values of P and r .

(b) Since only the small change in Q is desired, this is given accurately enough by the difference of the nominal values of the two observed values of Q' , the resistance exclusive of the resistance of the coil.

(c) Errors due to the change of resistance of the inductance coil with the frequency are eliminated, together with the effect of residual inductances or capacities in the bridge arms.

(d) Errors due to the electrostatic capacity of the bridge are practically eliminated.

Formula (38) shows that in order to obtain $(\theta_2 - \theta_1)$ accurately, the change ΔQ in the resistance of the Q arm, when one condenser is substituted for the other, must be large enough to be determined with precision.

For a given capacity at a given frequency, this requires that L be chosen large enough to satisfy this condition. The higher the frequency the smaller the value of the inductance necessary to obtain a given value of ΔQ . In addition to these considerations it must be remembered that the arrangement of the bridge is not sensitive when large inductances are used with small condensers, and that the sensitiveness of the balance of the Q arm depends on the relative values of Q and pL . With such a large number of variables it is only a matter of trial to obtain a satisfactory arrangement for the case in hand.

Some trouble is met when the temperature of the copper wire of the inductance is changing, especially if the coil be of fairly high resistance. If, however, several measurements be made on the two condensers alternately, this error is eliminated.

7. THE DETERMINATION OF THE ABSOLUTE VALUE OF THE POWER FACTOR.

If a condenser is at hand whose power factor is known at a number of frequencies and temperatures, so that the value of θ corresponding to any particular working conditions can be obtained, then it is easy to determine θ for any other condenser of the same nominal value. By one of the four methods just described, the difference $(\theta_1' - \theta_1'')$ is determined. Knowing, then, the resistance of the leads of the condenser which is being tested, the observed value can be corrected, and the value $\theta_1 - \theta_1''$, corresponding to the condenser alone, is obtained. This quantity is the phase difference between the condenser and the standard plus its leads. From the known value, θ_0 , of the angle of the standard, and from the resistance of its leads, the value θ_1'' , corresponding to the standard plus its leads, is obtained, and combining with this the value of $(\theta_1 - \theta_1'')$ we obtain the angle θ_1 for the condenser in question. The correction ψ for a lead resistance, a , is calculated by the relation

$$\tan \psi = pCa$$

For condensers of large capacity at high frequency a resistance of only 0.001 ohm in the leads is appreciable. For a condenser of 1 mf capacity at 1,000 cycles the correction for a lead resistance of 0.001 ohm is 1''.3. This illustrates how important it is to know the resistance of the leads.

To avoid changes in the apparent power factor of a condenser, plug contacts on the top of its containing case must be kept in good condition, and to obtain a correct value of the power factor of the condenser itself the lead wires from the condenser plates to the terminal blocks or binding posts should be of known or negligible resistance.

Power factor measured by stepping up from an air condenser.—A well-insulated air condenser, the resistance of whose leads is too

small to appreciably affect the phase, may be considered as having a zero power factor.¹² The power factor due to an insulation resistance R may be calculated by the formula $\tan \theta = \frac{1}{pCR}$ in order to determine, in any case, whether the effect of leakage is negligible.

Such an air condenser may evidently be employed to determine the absolute value of the power factor of a condenser of the same nominal value. Air condensers, however, become bulky when their capacity exceeds 0.02 or 0.03 mf. The Bureau of Standards possesses two air condensers of 0.02 mf each, and two of 0.03 mf each, which, when joined in parallel, allow the power factors of condensers as large as 0.1 mf to be directly determined. To obtain the power factors of condensers of larger capacity, to be used as standards in condenser comparisons, a stepping-up process has been used. Knowing the value of θ for two or more condensers, we may calculate their joint power factor when connected in parallel.

Let the capacities be C_1 , C_2 , C_3 , etc., and their known phase differences θ_1 , θ_2 , θ_3 , etc.

Then

$$\begin{aligned}\tan \theta_1 &= pC_1 r_1 \\ \tan \theta_2 &= pC_2 r_2 \\ \tan \theta_3 &= pC_3 r_3, \text{ etc.}\end{aligned}$$

and the impedances Z_1 , Z_2 , Z_3 , etc., are

$$\begin{aligned}Z_1 &= r_1 - \frac{i}{pC_1} = r_1 + a_1 i & \tan \theta_1 &= \frac{r_1}{a_1} \\ Z_2 &= r_2 - \frac{i}{pC_2} = r_2 + a_2 i & \tan \theta_2 &= \frac{r_2}{a_2} \\ Z_3 &= r_3 - \frac{i}{pC_3} = r_3 + a_3 i & \tan \theta_3 &= \frac{r_3}{a_3}\end{aligned}$$

Their joint impedance Z is then found from

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \dots$$

But

$$\frac{1}{Z_1} = \frac{1}{r_1 + a_1 i} = \frac{r_1 - a_1 i}{r_1^2 + a_1^2} = A_1 - B_1 i$$

¹² At 1,000 cycles a resistance of only 0.01 ohm in the leads shifts the phase of a 0.1-mf condenser by 1".25.

Similarly

$$\frac{1}{Z_2} = A_2 - B_2 i$$

$$\frac{1}{Z_3} = A_3 - B_3 i, \text{ etc.}$$

$$\therefore \frac{1}{Z} = (A_1 + A_2 + A_3 + \dots) - i(B_1 + B_2 + B_3 + \dots)$$

and

$$Z = \frac{(A_1 + A_2 + A_3 + \dots) + i(B_1 + B_2 + B_3 + \dots)}{(A_1 + A_2 + A_3 + \dots)^2 + (B_1 + B_2 + B_3 + \dots)^2}$$

and the joint value θ is given by

$$\begin{aligned} \tan \theta &= \frac{A_1 + A_2 + A_3 + \dots}{B_1 + B_2 + B_3 + \dots} \\ &= \frac{\frac{r_1}{r_1^2 + \frac{1}{p^2 C_1^2}} + \frac{r_2}{r_2^2 + \frac{1}{p^2 C_2^2}} + \frac{r_3}{r_3^2 + \frac{1}{p^2 C_3^2}} + \dots}{\frac{\frac{1}{p C_1}}{r_1^2 + \frac{1}{p^2 C_1^2}} + \frac{\frac{1}{p C_2}}{r_2^2 + \frac{1}{p^2 C_2^2}} + \frac{\frac{1}{p C_3}}{r_3^2 + \frac{1}{p^2 C_3^2}} + \dots} \\ &= \frac{\frac{r_1 p^2 C_1^2}{1 + \tan^2 \theta_1} + \frac{r_2 p^2 C_2^2}{1 + \tan^2 \theta_2} + \frac{r_3 p^2 C_3^2}{1 + \tan^2 \theta_3} + \dots}{\frac{p C_1}{1 + \tan^2 \theta_1} + \frac{p C_2}{1 + \tan^2 \theta_2} + \frac{p C_3}{1 + \tan^2 \theta_3} + \dots} \end{aligned}$$

Now $(1 + \tan^2 \theta_1)$, $(1 + \tan^2 \theta_2)$, etc., differ but very little from unity ($= 1.0003$ for $\theta = 1^\circ$) and, further, they occur symmetrically in both numerator and denominator. The error from neglecting these small differences from unity is negligible in measurements of the power factor, and we may write

$$\begin{aligned} \tan \theta &= \frac{p^2 C_1^2 r_1 + p^2 C_2^2 r_2 + p^2 C_3^2 r_3 + \dots}{p C_1 + p C_2 + p C_3 + \dots} \\ &= \frac{C_1 \tan \theta_1 + C_2 \tan \theta_2 + C_3 \tan \theta_3 + \dots}{C_1 + C_2 + C_3 + \dots} \quad (39) \end{aligned}$$

For good mica condensers we may substitute the angles for their tangents, and we have the simple rule, that the *joint angle* θ is equal to the *weighted mean of the angles* θ_1, θ_2 , etc. Thus

$$\theta = \frac{C_1 \theta_1 + C_2 \theta_2 + C_3 \theta_3 + \dots}{C_1 + C_2 + C_3 + \dots} \quad (39a)$$

The joint value for two or more condensers in *series* may be found by a similar procedure, but the formula is not simple, except for the case of n condensers of equal capacity C .

For this case

$$\tan \theta = \frac{\tan \theta_1 + \tan \theta_2 + \tan \theta_3 + \dots}{n} \quad (40)$$

If the condensers are not exactly equal, but are of the same nominal value, a rather complicated correction term must be applied to (40). For the special case of two mica condensers of capacities $C(1+a_1)$, $C(1+a_2)$ where C is their nominal value, (40) becomes

$$\tan \theta = \frac{\tan \theta_1 + \tan \theta_2}{2} \left(1 + \frac{a_1 + a_2 + a_1 a_2}{2} \right) \quad (41)$$

In the supposed case a_1 and a_2 will rarely be greater than 1 per cent, and we may write the angles for their tangents. We have, therefore, to take the arithmetical mean of their angles and correct this value by the arithmetical mean of the percentage differences of the capacities from their nominal values, taken with their proper algebraic signs. Since θ may be as small as 1', the correction will usually be negligible, when

$$\theta = \frac{\theta_1 + \theta_2}{2} \quad (41a)$$

These formulæ have been applied to several condensers of which the capacities, temperature coefficients, and change of capacity with the frequency had been previously determined. These condensers have been used as standards in testing the methods and in obtaining the power factors and capacities of a large number of other condensers under various conditions. Since, however, so much depended on the determinations of the constants of the standards it was deemed advisable to obtain a check on this stepping-up process. The power factor of a 1 microfarad standard was therefore determined inde-

pends by means of formula (35), and the value obtained by this method was compared with that derived by stepping up from the 0.1-mf air condenser. As will be seen from Tables IX and X, the agreement was close enough to give confidence in the results of the latter method. Owing to considerations to be presented directly, the results derived from the air condenser will be given the greater weight.

Independent determination of power factor using formula (35).—

In an Anderson bridge, in which the noninductive resistances are free from residual inductance or capacity, and in which the inductance and resistance of the coil to be measured do not depend on the frequency, the power factor of the condenser may be calculated from formula (35). Generally, the deviations from these simple conditions give rise to important corrections, which, at high frequencies, may be larger than the quantity to be measured. While these effects are eliminated in the *comparison* of the power factors of two condensers, their values must be quite accurately calculated if reliable values of the angle θ are to be independently obtained.

If Q_p = observed resistance of the Q arm with alternating current

Q_0 = observed resistance of the Q arm with direct current

ΔW = increase in the resistance of the inductance with alternating current, over its resistance with direct current

$\Delta_r Q$ = change (+ or -) in the resistance of the Q arm with alternating current, due to the residual inductances or capacities of the resistance coils of the bridge

$\Delta Q_0 = Q_p - Q_0$ = measured change in the resistance of the Q arm.

Then, from (35), remembering that $Q_0 = S$, we have

$$\Delta Q_0 = pL \tan \theta + \Delta_r Q \quad (42)$$

The observations are, however, made on the resistances Q_p' and Q_0' , which are included in the Q arm with the inductance.

If W_p = resistance of the coil with alternating current

W_0 = resistance with direct current

and $\therefore \Delta W = W_p - W_0$, then, putting $\Delta Q_0' = Q_p' - Q_0'$

$$Q_p = Q_p' + W_p$$

$$Q_0 = Q_0' + W_0$$

$$\therefore \Delta Q_0 = \Delta Q_0' + \Delta W$$

also

$$\Delta_r Q' = \Delta_r Q.$$

Substituting these relations in (42)

$$\Delta Q_0' + \Delta W = pL \tan \theta + \Delta_r Q' \quad (42a)$$

$$\text{or} \quad \tan \theta = \frac{\Delta Q_0' + (\Delta W - \Delta_r Q')}{pL} \quad (43)$$

where $\Delta Q_0'$ is the measured change in the resistance, exclusive of the coil itself.

If l_1, l_2, l_3, l_4, l_5 are the residual inductances or capacities of the P, Q, R, S, r arms of the bridge, respectively, then ¹³

$$\begin{aligned} PS - RQ = & p^2 [l_1 l_4 - l_3 (l_2 + L)] \\ & + p^2 C [PR l_4 + SR (l_1 + l_5) + PS (l_3 + l_6) \\ & + r (Sl_1 + Sl_3 + Pl_4 + Rl_4)] \\ & - p^4 C (l_1 l_3 + l_3 l_5 + l_5 l_1) l_4 \end{aligned}$$

When $P = R$, this reduces to

$$\begin{aligned} \Delta_r Q_0 = & \frac{p^2}{P} [(l_2 l_3 - l_1 l_4) + l_3 L] \\ & - p^2 C \left[S \left(1 + \frac{r}{P} \right) (l_1 + l_3) + l_4 (2r + P) + 2Sl_5 \right] \\ & + \frac{p^4}{P} C (l_1 l_3 + l_3 l_5 + l_5 l_1) l_4 \end{aligned}$$

Usually the expression $(l_2 l_3 - l_1 l_4)$ is very small, and may be neglected with respect to $l_3 L$; also terms of the third degree in l_1, l_3 , etc., need not be considered. The following more simple formula is, therefore, obtained:

$$\Delta_r Q_0 = \frac{p^2 L l_3}{P} - p^2 C \left[S \left(1 + \frac{r}{P} \right) (l_1 + l_3) + l_4 (2r + P) + 2Sl_5 \right] \quad (44)$$

It is not easy to obtain accurately, ΔW , the change of resistance with the frequency. The resistance of standards of inductance of the larger values, wound with unstranded wire as large as No. 24 B. and S., is appreciably different, even at 100 cycles, from the steady current value. The increase of resistance of a 1-henry coil wound with wire of about this size was found to be 0.02 ohm, or two ten-

¹³ Rosa and Grover, this Bulletin, 1, p. 307 (equation 15); 1905.

thousandths of the resistance of the coil. This is about one-tenth of the calculated change (at 100 cycles) of the resistance Q , due to the power factor of the standard condenser, an inductance of 1 henry being used in the bridge. ΔW can be made smaller by using lower frequencies in the measurements, and by stranding the wire.

If the frequency is reduced, then by (35) the value of L must be increased to give the same value of ΔQ , and increasing L , other things being equal, increases the total resistance of the coil used. But since the effect of the frequency increases as the square, while the value of ΔQ in (35) is proportional to the first power, it is possible to decrease the value of ΔW very considerably without unduly decreasing ΔQ .

ΔW depends, however, on the electrostatic capacity of the coil itself, and on the capacity of the leads used. The increase of resistance, due to these causes, is proportional to the capacity, to the square of the frequency, and to the inductance of the coil, and although smaller than the effect of eddy currents, is, in the coil mentioned above, appreciable. It seems, therefore, impracticable to make ΔW negligible, and its value must be determined and allowed for. This may be done approximately by one of the following methods:

(1) By assuming the value of θ , determined by stepping up from the air condenser, and by working at a higher frequency, where the term $pL \tan \theta$ is not so large as $\Delta_r Q'$ (see equations 42a and 44) an approximate value of ΔW is obtained. Reducing this value in the ratio of the square of the frequencies, a value for ΔW at the lower frequency is obtained, which is sufficiently accurate when the correction is small.

(2) We may avoid thus assuming a value for the power factor by making the coil to be tested one of the coils of a mutual induction in Maxwell's method for comparing a mutual induction with the self-induction of one of its coils. In this case the term $pL \tan \theta$ drops out of (42a).

(3) Let a coil be wound with stranded wire, with the separate strands insulated, and let the value of ΔW , in this case very small, be found by one of the methods just described. Then this coil may be used as a standard, and the value of ΔW for any other coil of about the same inductance, may be found by comparing the two

coils directly in a bridge, or by balancing them successively against a third coil, both with direct current and with alternating current of the desired frequency.

8. MEASUREMENT OF SMALL CONDENSERS.

In comparing condensers of a few thousandths of a microfarad or less, the error arising from the electrostatic capacity of the bridge becomes relatively so important that direct comparisons by these methods give, at best, only approximate results, and a substitution method must, of necessity, be used.

For these measurements the Parallel Resistance Method is quite inapplicable if correct capacity ratios are desired; the power factor measurements are, however, not thus affected. The conditions are not favorable for using Anderson's method, owing to the insensitiveness of the bridge when large inductances are compared with small capacities. It is possible that better determinations of the power factor by this method might be obtained if the variable resistance r were inserted in the supply circuit, as suggested by Stroud and Oates.¹⁴ The Series Resistance Method gives good results if the corrections given in (13) and (14) be applied. The Series Inductance Method, however, seems most free from objection. The following procedure has been found satisfactory. The condenser to be tested is first balanced against an auxiliary condenser of the same, or nearly the same, nominal value, the balance being obtained by alternately varying the resistance R_s and the inductance L_s . Then, without changing the resistances, R_s , R_t , a calibrated variable air condenser of suitable range is substituted for the condenser to be tested, and a balance is again obtained by varying the capacity of the standard and the inductance in L_s . The required capacity is then equal to the capacity corresponding to the setting of the air condenser, and the power factor is calculated from the observed values of the inductance by equation (21), the phase difference, θ_1'' , of the air condenser being taken equal to zero. It sometimes happens that the inductance necessary for balance is beyond the range of the variable. In this case coils of fixed and convenient values will be inserted. When, now, the air condenser is inserted,

¹⁴ Phil. Mag., 6, p. 707; 1903.

these must be removed and R_3 must be increased by an amount equal to the resistance subtracted

The whole process may be accelerated, by using for the auxiliary condenser, C_3 , another variable air condenser. Then, making R_3 and R_4 nearly equal, the bridge is balanced, first with the condenser to be tested, by varying the capacity C_1 and the inductance L_3 ; second, with the standard air condenser (the capacity of C_1 being left unchanged) by varying the capacity of the standard and the inductance of L_3 , having first compensated any change in R_3 due to the removal of inductances.

The advantage arising from the use of a *variable standard* lies not so much in the quickness with which the balance may be found, but in the fact that nothing is changed in the bridge, when one condenser is substituted for the other, except the value of the inductance in the ratio arms. This insures that the electrostatic effects, which are so important, are as nearly as possible the same in the two cases, and are, therefore, without any considerable effect on the result.

The method just described may be used with advantage in the testing of cables. If the cable is measured immersed, care must be taken that the resistance of the water (which is in series with the capacity of the cable) is not so large as to cause an appreciable error in the measurement of the power factor. Of course, with such small capacities, a resistance of 50 or 100 ohms is not very important, but, in any case, the resistance should be measured and a correction applied if necessary.

9. THE APPARATUS.

The resistances R_3 and R_4 were ordinarily about 1,000 or 2,000 ohms. It is convenient to use for the adjustable part of R_3 a dial box reading to 0.1 ohm directly. The closer adjustment, necessary when capacities of 0.1 mf and greater were compared, was obtained by using a slide wire of 0.14 ohm resistance, which allowed direct readings to 0.001 ohm to be made. With 1 mf in each condenser arm, a change in R_3 of 0.02 ohm in a total of 1,000 ohms could be detected.

The variable inductance (of the Ayrton-Perry type, constructed by the Leeds & Northrup Company) had three different ranges, viz:

0.5 to 3 millihenry,
4.0 to 24 millihenry,
5.0 to 42 millihenry.

The middle range was ordinarily used when two condensers of nearly equal power factor were to be compared at a frequency of 100. For a frequency of 1,000 cycles the smallest range was employed. A fixed coil of 5 to 10 millihenrys, or less, was inserted in the arm adjacent to the auxiliary condenser, to avoid settings below the range of the variable inductance. In the comparison of condensers of 1 microfarad capacity, the inductance could easily be set to 0.01 millihenry. For making this fine adjustment a slow-motion attachment was found very convenient.

The current for measurements at 30 to 100 cycles was taken from the alternator of a small motor-generator set driven from a storage battery. This machine is rated at 2.5 amperes at 120 volts and 120 cycles. The electromotive force is far from being sinusoidal.

For higher frequencies up to 1,000 cycles, the various machines of a harmonic alternator set, constructed especially for the Bureau of Standards by the General Electric Company, was used. From this set frequencies, which are the odd multiples of 60 cycles up to 900 cycles, are available, and by increasing the speed of the driving motor, 1,000 cycles can be obtained. These machines generate electromotive forces which approximate to sine waves.

In the work at the lower frequencies, and for the most part at the higher frequencies also, vibration galvanometers were used as detectors. The galvanometers for frequencies up to 200 cycles were of the Rubens¹⁵ form, made by W. Oehmke, of Berlin, while the instruments responding to the higher frequencies were of the Wien¹⁶ type, constructed by E. Feldhausen, Aachen.

There is little to choose between the two forms, both being sensitive and satisfactory.

The frequency for which the galvanometer suspension was tuned was determined and maintained by an auxiliary bridge, as described in a previous paper.¹⁷ A condenser of known capacity, placed in

¹⁵ Wied. Annalen, 56, p. 27; 1895.

¹⁶ Wied. Annalen, 4, p. 425; 1901.

¹⁷ Rosa and Grover, this Bulletin, 1, p. 298; 1905.

the fourth arm of a Wheatstone bridge, is charged and discharged by a rotating commutator coupled to the shaft of the alternator. From the known capacity of the condenser and the resistances of the bridge, the frequency at which the bridge is balanced can be calculated. To determine the frequency with which the galvanometer suspension is in resonance, the galvanometer is inserted in a Wheatstone bridge supplied with current from the alternator. The balance of this bridge having been slightly disturbed, the frequency is adjusted by a resistance in the armature circuit of the motor until the deflection of the galvanometer is a maximum. At the higher frequencies, where the machine was inaccessible, an ammeter in the field circuit of the alternator gave a fairly satisfactory indication of the point where the proper speed was attained, the adjustment being made by a resistance in the field circuit.

The advantages connected with the use of a vibration galvanometer, which are discussed on pages 296 and 297 of the article just referred to,¹⁷ are especially emphasized here. The value of the power factor of a poor condenser is extremely sensitive to changes of frequency, as has been pointed out by Monasch, and, further, the effect of each harmonic in the electromotive force wave is magnified in the condenser current in proportion to the frequency of the harmonic. It is therefore impossible to make satisfactory setting with a telephone unless the overtones are so weak as not to confuse the balance point corresponding to the fundamental. This condition is pretty well realized with the high-frequency currents used in these experiments, and it was found possible to make fairly good settings with a telephone. To do so, however, it was necessary to eliminate a very important disturbance. It was at first found impossible, by any change of the ratio arms and the phase compensation, to bring the current due to the fundamental to zero, and the settings of the bridge were not only uncertain but inaccurate. It was finally discovered that by touching either terminal of the telephone circuit with the hand the resistance R , and the phase compensation could be so adjusted that all sound of the frequency of the fundamental disappeared, and only very high whistling noises due to the small harmonics remained, which were too weak to vitiate the determination of the balance. However, the setting obtained by touching one

¹⁷ Rosa and Grover, this Bulletin, 1, p. 298; 1905.

terminal was different from that found when the other terminal was touched, both in the value of the capacity ratio and in the value of the power factor. The differences with capacities of 0.1 mf were as large as 1 in 1,000 in the capacity ratio, and much more in the apparent value of the power factor. Experiment, however, showed that the values found by this procedure, using the substitution method, were in agreement with those obtained with the vibration galvanometer, and that the latter values were not sensibly different from the means of the two settings taken, touching the two terminals. Laying the hand upon one terminal, therefore, seems to shift the balance in one direction from the correct point as much as touching the other terminal shifts the balance in the other direction.

The following explanation of these facts seems to be in accord with the observations. The telephone consisted of two watch-case receivers, held against the ears by metal straps passing around the head of the observer. Each telephone winding, being insulated from the case, was at the potential of the middle of the bridge, while the case and the head of the observer were at a very different potential, and had, therefore, an appreciable capacity with respect to the telephone winding. The sound in the telephone which could not be removed by adjusting the bridge is, according to this hypothesis, due to the current which charges and discharges this capacity.

When, now, either junction of the telephone circuit and the bridge is touched by the hand, the observer and the case of the telephone are brought to the potential of the middle of the bridge, and no charging current flows. Touching this junction has, nevertheless, the effect of changing the electrostatic capacity of the bridge with respect to its surroundings, and thus shifts the balance point. When the hand is laid on the other terminal, the displacement of the balance point is, by symmetry, of sensibly the same value, but in the opposite direction.

The vibration galvanometer is free from this source of error. Since, in addition to those advantages already discussed, it is much more sensitive than the telephone at low frequencies, and about its equal at 1,000 cycles, the use of the vibration galvanometer, instead of a telephone, is to be recommended. To the author the former is much the pleasanter instrument to use, and when a large number of

settings are to be made it is much easier to maintain the initial precision of the settings than with the telephone.

A number of alternating-current galvanometers have been suggested in recent years, most of which consist, essentially, of a coil which carries the small current to be measured, suspended in the field of an electro-magnet, which is excited from the same supply as the current in the coil. There is, in general, an appreciable phase difference between the current in the field and in the coil, and when the galvanometer is used as a zero instrument in a bridge it is not feasible to keep this phase angle at a constant value. From page 379 and from Fig. 7 it is easy to see that when the bridge is nearly balanced the phase of the galvanometer current is very sensitive to changes in the bridge arms, and may vary through a wide angle. Such a galvanometer will give zero deflection whenever the phase difference of the galvanometer current and the current in the field is 90° , irrespective of the actual value of the galvanometer current; and since balance is reached by alternately adjusting two variables in the bridge, a large number of such points will be found. Further, such an alternating current galvanometer will give zero deflection when the *integral* current through its coil is zero, even though the instantaneous current may depart considerably from zero during the cycle. Such an instrument is, therefore, unfitted for a zero instrument when the conditions for balance presuppose that the galvanometer current is at every moment actually zero. On the contrary, both the vibration galvanometer and the telephone fulfill this requirement, since their indications depend only on the *amplitude* of the instantaneous current.

10. COMPARISON OF THE METHODS.

The foregoing methods have been used during the past two years and a half to measure the power factor of both mica and paraffined paper condensers by a number of the well-known manufacturers of England, France, Germany, and America. From the large mass of data thus accumulated it seems advisable to present only such results as shall suffice to illustrate the practical use of the methods, and to indicate the order of magnitude of the power factors which may be expected in condensers of different kinds.

In Tables I to VII, inclusive, are given the results of tests made to determine how closely the different methods can be relied upon

TABLE I.

Series Resistance Method.

March 15, 1907.

 $R_4=1,100$ and $1,600$ ohms. $n=99.42$ cycles per sec. Capacity of B= 1.0064 .

1	3	4	5	6	7	8	9	10	11
C_1	R_s	r_1	r_2	$\frac{C_1'}{C_1''} = \frac{R_s'}{R_s''}$	C_1' (mean)	pC_1r_1	$\tan(\theta_1' - \theta_1'')$	$(\theta_1' - \theta_1'')$	Temp.
	ohms	ohms	ohms		mf			' "	°
SH	1,075.135	14.47	30	1.02199		0.009298	0.00949	32 38	
B	1,098.78	29.89			1.02852	.018792			18.1
SH	1,563.57	14.45	30	1.02199		.009286	.00949	32 38	
B	1,597.96	29.87				.018780			
Mp	1,117.43	7.66	10	0.98330		.004733	.001516	5 13	
B	1,098.78	9.94			0.98958	.006249			18.1
Mp	1,625.12	7.62	10	0.98329		.004707	.001530	5 14	
B	1,597.97	9.92				.006237			
Wa	1,075.03	2.03	10	1.02206	1.02860	.001304	.004914	16 54	20.5
L	1,094.12	8.85		1.00422	1.01064	.005586	.000632	2 10.5	20.45
B	1,098.74	9.89				.006218			
Wa	1,563.37	1.99	10	1.02210		.001280	.004916	16 54	
L	1,591.19	8.80		1.00422		.005586	.000640	2 12	
B	1,597.91	9.86				.006196			

to give the same values for the capacity and power factor of a given condenser. Each method was compared directly with the Series Inductance Method at two frequencies. Substitution methods only were used. In Table I is given in some detail, the measurements and calculation of the capacity and power factor of four condensers by the Series Resistance Method (equations 15 and 16). Column 3 shows the observed values of R_s' and R_s'' , and column 4 the corresponding values of the resistances r_1' and r_1'' in series with the condensers C_1' and C_1'' when the bridge was balanced. Since these condensers were compared by substitution the values of C_s and r_s do not have to be known. In column 6 is calculated the ratio of the capacities to that of the standard condenser B. From the known capacity and temperature coefficient of this condenser, the capacities

of the other condensers could then be immediately calculated and are given in column 7. Column 8 contains the calculated values of $pC_1'r_1'$ and $pC_1''r_1''$, the difference of which (equation 16) is the tangent of the difference between the phase angles of the condensers and that of the standard, column 9.

In Table II are given measurements on the same condensers on the same day and at the same frequency, the Series Inductance

TABLE II.
Series Inductance Method.

March 15, 1907.

$R_4 = 1,102.8$ and $1,602.8$ ohms. Capacity of $B = 1.0064$ mf.

1	3	4	5	6	7	8	9	10	11
C_1	R_3	L_3	L_4	$C_1' = R_3''$ $C_1'' = R_3'$	C_1' (mean)	pL_3 R_3	$\tan$ $(\theta_1' - \theta_1'')$	$(\theta_1' - \theta_1'')$	Temp.
	ohms	milli-henrys	milli-henrys		mf			' "	°
SH	1,075.655	26.42	10	1.02208		0.015343	0.00949	32 37	
B	1,099.415	10.30			1.02860	.005853			18.1
SH	1,564.11	34.00		1.02206		.013579	.00959	32 58	
B	1,598.61	10.44				.003987			
Mp	1,120.31	13.115	10	0.98331	0.98959	.007313	.001520	5 13	18.1
Wa	1,077.81	18.46		1.02208	1.02860	.010700	.004907	16 52	20.5
L	1,096.97	11.30		1.00423	1.01064	.006435	.000642	2 12	20.45
B	1,101.615					.005793			
Mp	1,627.995	14.475	10	0.98330		.005554	.001525	5 14.5	
Wa	1,566.25	19.89		1.02206		.007933	.004904	16 51.5	
L	1,594.07	11.865		1.00422		.004650	.000621	2 8	
B	1,600.805					.004029			

Method being used. The comparisons were made by substitution as in the preceding table, and in columns 8 and 9 are given the results of the calculation of the phase angle differences by formula (21). The capacities are calculated by the simple formula (3).

The measurements of Tables I and II together with those made on the same day at a lower frequency are collected in Table III. In column 4 are given the capacities calculated by formula (3) and in column 5 the same values corrected by using (22.) It will be seen that in the case of the condensers with the smaller power factors this correction is less than one part in 100,000, but reaches four parts in

TABLE IV.
Parallel Resistance Method.

March 5, 1907.

Capacity of B 0.5 at 53 cycles, 0.50443 mf B at 53 cycles, 1.0067 mf.
" " " " " 100 " 0.50434 " " " " " 1.0065 "

1	2	3	4	5	6	7	8	9	10	11
C_1	C_2	R_1	r_1	r_2	$C_1' = R_1''$ $C_1'' = R_1'$	C_1' by (25)	C_1' by (27)	$(\theta_1' - \theta_1'')$ calculated	Frequency	Temp.
		ohms	ohms	ohms		mf	mf	" "		°
L	X_4	1,094.195	102,220	100,000	1.00446	1.01120	1.01120	2 22	53.43	19.95
B		1,099.075	100,290							19.5
L		1,591.32	102,250		1.00448			2 22		
B		1,598.42	100,320							
L 0.5	X_4'	1,097.08	101,330	100,000	1.00047	0.50467	0.50467	2 44.5	53.43	19.7
B 0.5		1,097.60	100,010							
L 0.5		1,595.475	101,330		1.00048			2 44.5		
B 0.5		1,596.25	100,010							
Wa	X_4	1,074.25	114,970	100,000	1.02312	1.02999	1.03001	14 55	53.43	19.5
B		1,099.09	100,320							
Wa		1,562.265	115,040		1.02314			14 59		
B		1,598.425	100,310							
SH	X_4	1,071.89	160,350	100,000	1.02536	1.03224	1.03233	39 33	53.43	17.3
B		1,099.08	100,290							
SH		1,558.86	160,350		1.02537			39 33		
B		1,598.42	100,320							
L	X_4	1,093.705	103,630	100,000	1.00409	1.01064	1.01064	2 12	95.25	19.95
Wa		1,075.03	138,750		1.02153	1.02819	1.02847	16 47		19.5
SH		1,074.935	228,400		1.02162	1.02829	1.02865	32 31		17.3
B		1,098.175	100,080							19.5
L		1,590.595	103,700		1.00409			2 8		
Wa		1,563.435	138,910		1.02153			16 42		
SH		1,563.28	228,800		1.02163			32 32		
B		1,597.10	100,150							
L 0.5	X_4'	1,596.27	102,200	100,000	1.00012	0.50440	0.50440	2 27	95.25	
B 0.5		1,596.46	100,000							

placed in parallel with the auxiliary condenser C_2 so that the resistances R_1 are of the same order of magnitude. The capacities calculated by the usual formula (25) and given in column 7 require correction owing to the appreciable capacity of these large resistances. Since these have been determined for a few of the resistance coils only, the corrections, which are calculated by (27), can be made only

approximately. It will be seen, however, from column 8 that they are by no means negligible with the condensers having the larger power factors.

From Table V it is evident that when this correction is negligible, as is the case with the condensers L and L 0.5, the two methods give

TABLE V.

Comparison of Results by the Parallel Resistance and Series Inductance Methods.

March 5, 1907.

Value of θ for B at 53 cycles 1' 30''
 100 " 1' 15''
 " " " B 0.5 at 53 " 1' 35''
 100 " 1' 17''

Condenser	Frequency	Capacity		Mean ($\theta_1' - \theta_1''$)		$\theta_1' - \theta_1''$ corrected for leads	θ
		by P. R.	by S. I.	by P. R.	by S. I.		
		<i>mf</i>	<i>mf</i>	' "	' "	' "	' "
L	53.43	1.01120	1.01119	2 22	2 22	2 21	3 51
	95.25	1.01064	1.01065	2 10	2 9	2 7	3 22
L 0.5	53.43	0.50467	0.50468	2 44.5	2 45	2 44	4 19
	95.25	0.50440	0.50440	2 27	2 27.5	2 25	3 42
Wa	53.43	1.03001	1.03006	14 57	14 54	14 55	16 25
	95.25	1.02847	1.02847	16 45	16 47	16 46	18 1
SH	53.43	1.03233	1.03247	39 33	39 44	39 37	41 7
	95.25	1.02865	1.02882	32 32	32 27	32 28	33 43

substantially the same values for the capacities. Apparently, however, the data from which the corrections to the capacities of Wa and SH have been calculated are not accurate. The larger the correction is the greater is the discrepancy between the values found by the two methods, indicating an appreciable error in the capacities assumed for the resistance coils. The phase differences, however, are in close agreement with those found by the Series Inductance Method.

In Table VI are given results obtained by means of an arrangement of the Anderson bridge found convenient for frequencies of 50 and 100. Since Q' could be set to 0.001 ohm the values of $\angle Q'$ are not in error by more than 0.002 ohm. The capacities, calculated by the simple formula (36a) appear in column 5, and the values corrected

by (36) in column 7. This correction is by no means negligible in the case of SH.

TABLE VI.
Capacities and Phase Differences by Anderson's Method.

February 26, 1907.

$P=R=350$ ohms. Inductance, Coil C=1.0145 henrys.
S=700 " Capacity of B at 53 cycles=1.00665 mf.
100 " =1.0064 "

1	2	3	4	5	6	7	8	9	10	11
Con- denser	Fre- quency	r	$2r+P$	C_2 C_1 by (36a)	C_2	C_2 cor- rected	Q'	$\Delta Q'$	$(\theta_2-\theta_1)$ calcu- lated	Temp.
		ohms	ohms	mf	mf	mf	mean	ohms	" "	"
L	53.72	542.125	1,434.250	1.00450	1.01119	1.01119	605.455	0.229	2 18	20.0
B		545.349	1,440.698				605.226			19.85
Wa	53.72	528.535	1,407.070	1.02394	1.03075	1.03077	606.762	1.492	14 58	20.0
B		545.377	1,440.754				605.270			
SH	53.72	527.560	1,405.120	1.02536	1.03218	1.03231	609.119	3.778	37 56	16.5
B		545.375	1,440.750				605.341			
L	98.11	542.457	1,434.914	1.00423	1.01069	1.01069	605.506	0.378	2 5	20.1
B		545.495	1,440.990				605.128			20.1
Wa	98.11	529.632	1,409.264	1.02252	1.02909	1.02912	608.146	3.067	16 51.5	20.0
B		545.496	1,440.992				605.079			
SH	98.11	529.986	1,409.972	1.02200	1.02857	1.02865	610.690	5.620	30 54	16.5
B		545.495	1,440.990				605.070			

Table VII shows that the results by this method are in close agreement with those by the Series Inductance Method. In a few cases the differences are larger than can be explained by the accidental errors of the bridge. It is, however, only with the condensers of larger power factor that these discrepancies occur, and they are to be expected, since with these condensers the capacity changes appreciably with only a very small change in the frequency or temperature. With SH, for instance, a change of one cycle in the frequency means a change of about 1 in 10,000 in the capacity.

From these results it is established that the measured values of the capacity and power factor of a condenser are very closely the same whichever of the four methods is used.

TABLE VII.

Comparison of Results by Anderson's Method and by the Series Inductance Method.

Date	Con- denser	Pre- quency	Capacity		$(\theta_1' - \theta_1'')$		$(\theta_1' - \theta_1'')$ cor- rected for leads	θ	Temp.
			by A. M.	by S. I.	by A. M.	by S. I.			
1907			<i>mf</i>	<i>mf</i>	' "	' "	' "	' "	°
Feb. 8	Wa	100.66	1.03077	1.03082	16 45	16 47.5	16 47	18 2	21.0
	Wb		1.03170	1.03169	13 33	13 36	13 36	14 51	
	Wc		1.11937	1.11950	16 54	16 44	16 44	17 59	
	Wd		1.11757	1.11763	15 22	15 33	15 33	16 48	
Feb. 28	L	53.72	1.01118	1.01119	2 18	2 17.5	2 17	3 47	20.0
		98.11	1.01069	1.01068	2 5	2 4	2 2	3 17	
	Wa	53.72	1.03077	1.03073	14 58	14 58	14 58	16 28	20.0
		98.11	1.02912	1.02912	16 51.5	16 50	16 51	18 6	
	SH	53.72	1.03231	1.03239	37 56	37 49	37 51	39 21	16.5
		98.11	1.02865	1.02867	30 54	31 2	30 56	32 11	

11. RESULTS OF ABSOLUTE DETERMINATIONS.

Table VIII illustrates the determination of the absolute values of the power factors of the various sections of the condenser B, which was used as a standard in this work. The 0.1 mf section and the two 0.2 mf sections in series were separately compared with the 0.1 mf air condenser of the Bureau. Then it was only necessary to compare with one another the two 0.2 mf sections; and the 0.5 mf section with the 0.2, 0.2', and 0.1 sections in parallel to obtain the value of θ for any single section, or for any combination of the separate sections.

For, from (41a) which is applicable to this case, the observed value of θ for the 0.2 and 0.2' sections in series is the arithmetical mean of the values θ (0.2), θ (0.2') of the separate sections, and since we also know, by observation, the small difference of these angles we may easily find their values individually. Having, therefore, obtained θ (0.1) by observation and θ (0.2) and θ (0.2') by observation and calculation, θ (0.2+0.2'+0.1) follows directly from these separate values by equation (39a). From this last obtained value is found by means of the observed difference the value of θ for the 0.5 section. The mean of this value, θ (0.5), and of θ (0.2+0.2'+0.1) will be the value of θ for the whole microfarad.

TABLE VIII.

Absolute Determination of Phase Differences by Stepping Up from Air Condenser.

Values for the various sections of condenser B by the Series Inductance Method.

Date	Con- denser	Fre- quency	L_3	R_3	pL_3 R_3	$\tan$ $(\theta_1' - \theta_1'')$	$(\theta_1' - \theta_1'')$	$(\theta_1' - \theta_1'')$ cor- rected for leads	Temp.
			milli- henrys	ohms			' "	' "	°
1907 Jan. 29	0.1 Air	52.49	9.575 8.22	1109.79 1106.65	0.002846 .002450	0.000396	1 22	1 22	18.8
"	0.2 and 0.2' in series	52.49	9.605 9.40	1111.52 1615.08	.002850 .001920	.000406 .000402	1 24 1 23	1 23.5	19.4
	Air		8.20 7.40	1106.60 1607.80	.002444 .001518	-			
"	0.2'	52.49	9.345 9.00	1111.76 1615.50	.002772 .001837	-.000002 +.000019	-0 0.5 +0 4	0 2	19.4
	0.2	52.49	9.39 8.94	1116.27 1622.03	.002774 .001818				
"	0.5	52.49	7.67 6.64	1104.34 1604.71	.002291 .001365	.000055 .000043	0 11.5 0 9	0 10	19.4
	0.2+0.2' +0.1	52.49	7.52 6.41	1109.06 1611.57	.002236 .001312				
Jan. 29	0.1	104.25	9.73 9.62	1109.50 1612.09	.005744 .003909	.000344 .000317	1 11 1 5	1 8	19.6
	0.2 and 0.2' in series	104.25	9.75 9.70	1111.31 1614.75	.005746 .003935	.000346 .000343	1 11.5 1 11	1 11	
	Air	104.25	9.12 8.815	1106.16 1607.25	.005400 .003592				
"	0.2'	104.25	9.72 9.58	1111.41 1615.14	.005728 .004463	.000014 .000025	0 3 0 5	0 4	19.6
	0.2	104.25	9.735 9.57	1116.02 1621.67	.005714 .004438				
"	0.5	104.25	9.01 8.57	1103.92 1604.10	.005346 .003499	.000032 .000031	0 6.5 0 6.5	0 6.5	19.6
	0.2+0.2' +0.1	104.25	8.995 8.53	1108.61 1610.92	.005314 .003468				
Mar. 18	0.1	970	1.900 1.9125	1107.37 1109.23	.010457 .010508	.000170 .000203	0 35 0 42	0 32 0 39	19.7
	0.2 and 0.2' in series		1.8525 1.857	1097.94 1097.81	.010283 .010309				
	Air								
"	0.2'	970	1.916 1.9175	1113.96 1109.48	.010525 .010491	.000034	0 7	0 7	19.8
	0.2								
"	0.5	970	1.859 1.870	1101.25 1105.915	.010288 .010306	-.000018	-0 4	-0 4	19.9
	0.2+0.2' +0.1	970							

Table IX illustrates the measurement and calculation of the absolute value of the phase angle of the condenser B (= 1 mf) by

TABLE IX.

Absolute Determination of Phase Difference by Anderson's Method.

P=R=350 ohms. Coils used H=0.9980 henry.
S = 700 ohms. C=1.0145 henry.

Date	Con- denser	Fre- quency	Q'	$\Delta Q'$	ΔW	$\Delta_1 Q'$	(5)+(6) -(7)	Coil	θ cal- culated	θ cor- rected for leads	Temp.
1907 Feb. 26	B		ohms	mean ohms	ohms	ohms	ohms		' "	' "	°
		D. C.	605.058	0.148	0.016	0.007	0.157	C	1 34	1 33	19.8
		53.72	605.204								
		D. C.	605.056								
		53.72	605.203								
		D. C.	605.052								
		53.72	605.199								
Feb. 26	B	98.11	605.209	0.192	0.065	0.024	0.233	C	1 17	1 15	19.8
		D. C.	605.012								
		98.11	605.200								
		D. C.	605.003								
		98.11	605.192								
		D. C.	604.998								
		98.11	605.187								
Mar. 16	B	99.75	604.657	0.252	0.015	0.024	0.243	H	1 20	1 18	20.0
		D. C.	604.396								
		99.75	604.638								
		D. C.	604.380								
		99.75	604.626								
Mar. 16	B	99.75	605.103	0.190	0.065	0.024	0.231	C	1 15	1 13	20.0
		D. C.	604.908								
		99.75	605.094								

Anderson's method, equation 43. The change in effective resistance of the coils used in the measurements was determined by the first of the methods described on page 406, and these values are probably correct to less than 0.01 ohm. To determine $\Delta_r Q'$, values for the residual inductances and capacities of the bridge arms were assumed which depend on only a few measurements, and the values for the capacities of the coils in the r arm were not measured, but were assumed as the same as those in a similar dial box. The

following were the values of the residual inductances and capacities used in formula 44:

$$l_1 = l_2 = -11.5 \text{ microhenry}$$

$$l_4 = -29.5 \text{ microhenry}$$

$$l_5 = -15 \text{ microhenry}$$

At the frequency of the measurements of Table IX the effect of the capacity of the leads on the apparent resistance of the coil is negligible, otherwise it would have to be included in the value of ΔW .

In Table X are collected the results of a number of determinations of the phase angles of the various sections of condenser B by stepping up from the air condenser, and also those values which have been found by Anderson's method. It will be seen that the

TABLE X.

Summary of Results of Absolute Determinations of Phase Differences of the Sections of Condenser B.

Date	Method	θ for 0.1 sec- tion	θ for 0.2 and 0.2' in series	θ 0.2' - θ 0.2	θ 0.5 - θ Σ 0.5	θ for 0.2	θ for 0.2'	θ for 0.2 + 0.2' + 0.1	θ for 0.5	θ for sum = 1.0 mf	Fre- quen- cy	Temp.
1907		" "	" "	" "	" "	" "	" "	" "	" "	" "		°
Jan. 29	Air con- denser	1 19	1 22	0 3	0 10	1 20.5	1 23.5	1 21.5	1 31.5	1 26.5	52.5	19.4
"	"	1 8	1 8.5	0 5	0 6	1 6	1 11	1 8.5	1 14.5	1 11.5	102.4	19.5
Feb. 2	"	1 29.5	1 31.5					1 31		1 35	52.7	19.5
"	"	1 14	1 16		0 6			1 15.5	1 21.5	1 18.5	102.2	19.0
Feb. 14	"	1 16	1 16	0 3	0 4.5	1 14.5	1 17.5	1 16	1 20.5	1 18	99.8	20.5
"	"	0 30	0 41	0 2	-0 2.5	0 40	0 42	0 39	0 36.5	0 38	920	
Feb. 20	"	1 9.5	1 13.5					1 12.5	1 18.5	1 15.5	98.9	18.5
"	"	0 33.5	0 42					0 40		0 39	920	
Feb. 26	Anderson's method									1 33	53.7	19.8
"	"									1 15	98.1	
Mar. 16	"									1 15.5	99.7	
" 18	Air con- denser	1 10	1 12	-0 2	0 3.5	1 12.5	1 11.5	1 11.5	1 15	1 13	99.8	
"	"	0 34	0 40.5		-0 1.5			0 39	0 37.5	0 38	970	

values obtained at the same frequency, but on different days, differ among themselves by as much as 3'' from the mean value. This is due to the fact that the sensitiveness of the settings, when capacities as small as 0.1 mf are being compared with 100 volts on the bridge,

is not quite large enough to give θ to the nearest second. It is probably safe, however, to use the following values for the power factor of the sum of the sections of condenser B:

Frequency	θ
	"
50	1 30
100	1 15
970	0 38

The determinations by Anderson's method are in pretty good agreement with these values, but since they are affected by the uncertainties in ΔW and $\Delta Q'$, they have not been used in finding the means. They are nevertheless of value as a check on the results found with the air condenser.

12. ILLUSTRATION OF METHODS FOR SMALL CAPACITIES.

Tables XI and XII illustrate the methods suggested for the measurement of capacities of a few thousandths of a microfarad. The 0.002 and 0.005 mf condensers, Table XI, were balanced against the mica auxiliary condenser by adjusting R_s and L_s . With 100 volts on the bridge, the resistance R_s could not be adjusted better than to the nearest ohm in 1,500, and L_s could not be set closer than about one-half of a millihenry. A 0.005 mf air condenser was then substituted for the condenser to be tested, and its capacity varied together with L_s to get a second balance. In the case of the 0.001 mf sections the adjustment was made with still less difficulty. The condenser to be tested was balanced against the 0.005 mf air condenser by varying the capacity of the latter and by adjusting L_s , the equal ratio arms being left untouched. A 0.001 mf air condenser was next substituted, and its capacity varied to find a second balance, the 0.005 mf air condenser being left at the setting just obtained. Results of measurements on the same condensers using the Series Resistance Method, Table XII, are in good agreement with those of Table XI by the Series Inductance Method. On calculating the corrections to be applied to the capacity ratios and the differences of the power factors due to the capacity of the resistance coils r_1' , r_1'' , introduced into the condenser arms, it was found that they were smaller than the errors of setting of the bridge. Cases will, how-

ever, occur where the corrections as calculated by formulæ (13) and (14) will not be negligible. The smaller the capacity to be measured, the more will this be likely to occur. In such an event,

TABLE XI.

Results on Small Capacities by Series Inductance Method.

Frequency 99.7.

Date	Condenser	Observed capacity	L_2	R_2	ΔL_2	$\theta_1' - \theta_1'' = \theta$
1907 April 4	M .005	<i>mf</i>	<i>millihenrys</i>		<i>millihenrys</i>	° ' "
			28.6	1158	16.3	0 30 20
			37.3	1604	25.2	32 15
	Air	.004838	12.3	1158		
		.004841	12.1	1604		
"	N .005		13.9	1116	2.6	0 5 1
			15.6	1622	3.3	4 23
	Air	.005010	11.3	1116		
		.005005	12.3	1622		
"	S .005		13.9	1046	3.15	0 6 29
			16.0	1519	3.9	5 32
	Air	.005183	10.75	1046		
		.005188	12.1	1519		
"	S .002'		26.25	1138	4.25	0 8 3
			23.6	1645	5.1	6 41
	Air	.001910	23.6	1138		
		.001910	18.5	1645		
"	S .002		110.3	1550	91.8	2 7 0
	Air	.002030	18.5	1550		
"	M .002'		36.8	1077	27.3	0 54 35
	Air	.002017	9.5	1077		
"	N .002'		32.2	1014	17.1	0 36 20
	Air	.002015	15.1	1014		
	N .002		36.0	1023	20.9	0 44 0
	Air	.001997	15.1	1023		
"	M .001		91.4	1100	65.2	2 7 0
	Air	.001009	26.2			
	N .001		38.0	1100	17.4	0 34 5
	Air	.000998	20.6			
	S .001		26.25	1100	3.65	0 7 10
	Air	.001110	22.6			

unless the capacities of the resistances are known, it will be better to use the Series Inductance Method. As examples of the importance, when measuring such small capacities, of the electrostatic effects due

to the capacity of the bridge with respect to its surroundings, the following may be cited. In Table XI, in the measurement of No.001 and So.001 the inductance associated with the auxiliary air condenser was 5 millihenrys. If, therefore, the bridge was free from electrostatic errors, the value of L_3 when the 0.001 mf air condenser was introduced should also be 5 millihenrys, since in that case C_1 and C_2 are both air condensers.

TABLE XII.

Results on Small Capacities by Series Resistance Method.

Frequency, 99.7.

Date	Condenser	Observed capacity	r_1	Δr_1	$\theta_1' - \theta_1'' = \theta$
1907 April 5	M .005 Air	.004812	ohms 17400 20400	ohms 3000	0 31 5
	N .005 Air	.004996	20900 21300	400	0 4 20
	S .005 Air	.005148	20200 20800	600	0 6 40
"	M .002' Air	.002015	13000 25500	12500	0 54 15
	N .002' Air	.002005	7000 26000	19000	1 22 0
	N .002 Air	.001994	16000 26000	10000	0 43 0
	S .002' Air	.001900	38500 40500	2000	0 8 10
	S .002 Air	.001930	81000 92000	11000	0 45 45
"	M .001 Air	.001005	8000 68000	60000	2 10 0
	N .001 Air	.000997	11000 27000	16000	0 34 25
	S .001 Air	.001110	6000 9000	3000	0 7 10

The observed values were, however, $L_3 = 26.2$ millihenrys in one case and $L_3 = 22.6$ in the other. That is, the apparent difference in phase angle of the two air condensers was in one case about 50' and in the other 43'.

Similarly in Table XII, when the two air condensers were directly compared, the difference of the resistances r_1 and r_2 , instead of being zero, was in one case 13,000 ohms and in the other 11,000 giving

angles of 30' and 26', respectively. When, on the other hand, the two air condensers were compared by the substitution method, no difference in their power factors could be detected. The electrostatic errors were, therefore, in this case satisfactorily eliminated. From this we easily see that the only practicable procedure is to compare the condensers by substitution.

Having now shown that the different methods are in good agreement among themselves, and that the power factors of the reference condensers can be accurately determined, examples will be given to show what values of θ may be looked for in such condensers as are met in practice.

13. EXAMPLES OF POWER FACTORS IN MICA AND PAPER CONDENSERS.

Values found for mica condensers at 100 cycles and 20° C. are given in Table XIII. From these it is evident that for a good mica

TABLE XIII.

Examples of Phase Differences in Mica Condensers.

Frequency 100 cycles. Temperature 20°.

Con- denser	Nominal Capacity	θ	Con- denser	Nominal Capacity	θ	Con- denser	Nominal Capacity	θ
		° ' "			° ' "			° ' "
E	$\frac{1}{2}$	0 0 30	M ₁	$\frac{1}{2}$	0 3 51	MM ₂	0.001	0 17 40
X ₆	0.5	0 0 50	W ₁	0.1	0 4 30	NN ₂	.001	0 18 0
Q	.5	0 1 0	M ₂	$\frac{1}{2}$	0 4 33	FE	.2	0 20 15
X ₆₁	.5	0 1 2	N	.005	0 4 40	MM ₁	.001	0 21 50
Q	.2'	0 1 12	L	.05'	0 5 0	NN ₂	.001	0 23 20
B	.5	0 1 15	W ₂	.1	0 5 40	H	.2	0 24 50
LL	.1'	0 1 24	H	.2	0 5 57	FE	.2	0 30 30
Q	.05	0 1 32	LL	.005	0 6 15	M	.005	0 31 20
W ₁	.1	0 1 53	MM ₁	.001	0 7 20	N	.001	0 34 15
X ₁₃	.5	0 2 3	H	.2'	0 7 20	N	.002	0 44 0
E	.1	0 2 25	L	.05	0 8 0	M	.002	0 54 30
LL	.05	0 2 29	N	.002'	0 8 0	MM ₁	.001	1 10 0
Ct	.5	0 2 31	H	.2"	0 9 10	NN ₁	.001	1 30 0
W ₂	.1	0 2 38	MM ₂	.001	0 10 45	MM ₂	.001	2 10 0
W ₂	.1	0 3 8	FE	.005	0 11 40	NN ₂	.001	3 30 0
X ₆	.5	0 3 16	NN ₁	.001	0 12 30	P ₁	.001	5 30 0
X ₁₁	.5	0 3 20	LL	.002'	0 14 35	P ₂	.001	11 30 0
L	.5	0 3 42	H	.1	0 16 30	P ₂	.001	32 15 0

condenser the angle θ lies between about 0' 30" and 5'. The larger values in the table were inserted to show how poor mica con-

condensers by well-known makers may occasionally be. It is noticeable that the very largest values of the phase difference occur in condensers of small capacity. The small condensers which have been tested are few in number and confined to a very limited number of makers. Of this number, however, only one condenser of less than 0.01 mf was found with a phase difference of less than 5'.

Only a few silvered mica condensers have been measured. Of these the sections marked W in Table XIII are typical. They exhibit no superiority, as far as power factor is concerned, over the ordinary tinfoil and mica condensers. In testing subdivided condensers one is very often struck by the great lack of uniformity in the quality of the various sections. For instance, in one box of six sections, ranging from 0.05 mf to 0.5 mf, values ranging from 4' to 30' were found; in another, values from 5' to 25'; while Tables XI and XII show how extremely large this variation may be in the case of the very small sections of such a condenser. This variability seems to be characteristic of paper condensers also.

In Table XIV those condensers which are marked W, T, and t are telephone condensers. It is among condensers made for this

TABLE XIV.

Examples of Phase Differences in Paper Condensers.

Frequency 100 cycles. Temperature 20°.

Con- denser	Nominal capacity	θ			Con- denser	Nominal capacity	θ			Con- denser	Nominal capacity	θ		
	mf	°	'	"		mf	°	'	"		mf	°	'	"
Mp	1.0	0	6	30	SH	0.5'	0	24	0	t ₁₉	2.0	2	20	0
S ₁₀	3.0	0	7	58	SH	.5	0	33	0	t ₁₇	2.0	3	30	0
S ₃	1.7	0	11	0	Be ₃₄	.6	0	33	15	t ₇	0.6	4	0	0
Wb	1.0	0	14	50	Be ₁₅	.6	0	41	10	t ₁	.6	4	45	0
Wa	1.0	0	18	5	S ₆	3.2	0	44	30	t ₆	.6	7	50	0
Lp	1.0	0	19	45	T ₂	2.2	1	5	0	t ₃	.6	8	30	0
Cp	0.5	0	20	20	t ₁₈	2.0	1	30	0					

class of work that the largest power factors have been found. Excepting the condensers W, included in this table, none of the telephone condensers thus far tested have been found with a value of θ below 45'. So great is the power factor of some of these poor paper condensers that the testing current appreciably heats them, and

this rise of temperature, owing to the large temperature coefficient of both the capacity and the power factor, causes the balance of the bridge to change, progressively, so rapidly that snap readings have to be made. The condensers Be have a dielectric of beeswax and rosin (all the rest of the condensers in Table XIV are of paraffined paper). As these are the only condensers of this kind so far tested no general conclusions can be drawn as to the quality of such a dielectric.

14. CONNECTION BETWEEN THE POWER FACTOR AND THE MAGNITUDE OF ABSORPTION EFFECTS.

In Tables XV and XVI the results of some tests with steady current are given in order to show how closely the magnitude of the absorption effects observed in ballistic galvanometer measurements

TABLE XV.

Comparison of Results with Steady Current with Value of Power Factor.

1	2	3	4	5	6	7	8
Condenser	Nominal capacity	θ	Time of charge	Time of discharge	Relative capacity	Residual charges	
						1 sec. charge	10 sec. charge
	<i>mf</i>	<i>o' "</i>	<i>sec.</i>	<i>sec.</i>		<i>per cent</i>	<i>per cent</i>
B	0.5	0 1 18	0.1 1.0	0.1 8.5	1.0016 1.0073	0.32	0.71
B	0.2'	0 1 10	0.1 1.0	0.1 8.5	1.0019 1.0066	.27	.62
B	0.2'+0.2+0.1	0 1 12	0.1 1.0	0.1 8.5	1.0020 1.0060	.28	.65
Q	0.5	0 1 0	0.1 1.0	0.1 8.5	1.0017 1.0054	.22	.56
Q	0.2'	0 2 51	0.1 1.0	0.1 8.5	1.0043 1.0100	.37	.96
X ₆	0.5	0 0 50	1.0	8.5	1.0056	.21	
X ₁₁	0.5	0 3 20	1.0	8.5	1.0177	.60	
E	0.1	0 2 25	1.0	8.5	1.0082		
X ₁	1.0	0 2 40	1.0	8.5	1.0080		

of condensers are connected with the magnitude of the power factor. In column 6 of Table XV are given the relative capacities, with different times of charge and discharge, referred to the capacity with

alternating current at 100 cycles. In columns 7 and 8 are given the sums of the residual charges observed with 1 sec. and 10 sec. charge, respectively, and a certain arbitrary schedule of subsequent successive periods of discharge and insulation. In Table XVI, column 5, is shown the ratio of the capacity with 1 sec. charge to that with 10

TABLE XVI.

Variation of Residual Charge with Value of the Power Factor.

Con- denser	Nominal capacity	Dielectric	θ	Ratio of capacities, 1 sec. and 10 sec. charges	Residual charges	
					1 sec. charge	10 sec. charge
	<i>mf</i>		<i>° ' "</i>		<i>per cent</i>	<i>per cent</i>
X_{40}	0.5	Mica	0 0 50		0.21	
Q	.5	"	0 1 0	0.9976	.22	0.56
B	.2'	"	0 1 10	.9974	.27	.62
X_{40}	.5	"	0 1 2		.23	
B	.5	"	0 1 18	.9974	.32	.71
Q	.2'	"	0 2 51	.9955	.37	.96
E	.1	"	0 2 25	.9969	.38	.92
Ct	.5	"	0 2 31	.9978		.76
E	$\frac{1}{2}$	"	0 0 30	.9976		.60
M_1	$\frac{1}{2}$	"	0 3 51	.9967		1.03
M_2	$\frac{1}{2}$	"	0 4 33	.9970		0.82
L	.5	"	0 3 42	.9977	.26	.76
L	.05'	"	0 5 0	.9961	.40	1.00
L	.05	"	0 8 0	.9914	1.49	6.93
Mp	1.0	Paraffined paper	0 6 30	.9984	0.25	0.70
S_{10}	3.0	"	0 7 58	.9960	.54	1.62
SH	.5'	"	0 24 0	.9888	1.52	3.70
SH	.5	"	0 33 0	.9848	1.88	4.63
T_2	2.0	"	1 5 0	.8540	11.4	23.8
S	.002'	Mica	0 6 40		2.08	5.38
S	.001	"	0 7 10		1.67	3.44
MM	.01	"	0 33 0		4.88	13.12
NN	.01	"	1 50 0		10.5	26.3

sec. charge, the length of discharge being, in both cases equal to the period of the first swing of the galvanometer, in this case $8\frac{1}{2}$ sec.

From these two tables the conclusion may be drawn that, qualitatively the magnitude of the residual charges and the dependence of the apparent capacity on the time of charge and discharge, both of which depend on the absorption of the condenser, are proportional to the value of the power factor. Cases occur, it is true, where

this seems to be contradicted, but this is most often the case where condensers of nearly equal power factor are being considered. It may be that such cases can be distinguished by something abnormal in the behavior of their power factor with change of frequency or temperature. Ordinarily, therefore, a measurement of the power factor may be depended upon, not only to distinguish between a good condenser and a poor one, but to show which is the better of two condensers of nearly equal grade.

15. SUMMARY.

(1) Four bridge methods (one of them due to Wien) for simultaneously obtaining the ratio of two capacities and the difference of their power factors are described, and the equations for calculating these quantities from the observations are derived.

(2) Various sources of error and methods of eliminating them are discussed. By the use of an auxiliary condenser, two condensers may be compared by substitution. This procedure has the advantage that electrostatic effects are eliminated, and only the relative values of the resistances and inductances of the bridge arms need be accurately determined.

(3) These four methods are compared with one another, and are shown to give results in good agreement. When used as substitution methods, the capacity ratios of condensers of 0.1 mf and greater may be measured to a few parts in 100,000 or better, and a phase difference of only a few seconds of arc, due to a difference in the absorption, may be detected with certainty. The small temperature coefficients of good mica condensers may readily be determined with accuracy.

(4) By the use of a variable air condenser, reliable comparisons of the capacities and power factors of condensers of 0.001 mf or less, such as short lengths of cable, may be easily made.

(5) Methods are given by which the power factor of the condensers to be used as standards may be measured. The Bureau of Standards makes such determinations when requested.

(6) Examples are given to show what values of the power factor may be expected in condensers by leading manufacturers.

(7) A knowledge of the power factor of a condenser gives reliable information as to the order of magnitude of its absorption effects. The determination of the power factor is the best single test of the quality of a condenser which can be made.

Experiments are being made at the Bureau of Standards on the change of the capacity and power factor with changes of temperature and frequency.

I am indebted to Prof. E. B. Rosa for many invaluable suggestions and for the constant interest he has shown in this research.

WASHINGTON, May 23, 1907.

A NEW DETERMINATION OF THE RATIO OF THE ELECTROMAGNETIC TO THE ELECTROSTATIC UNIT OF ELECTRICITY.

By E. B. Rosa and N. E. Dorsey.

“Etant donné l'intérêt qui s'attache à la détermination de la *vitesse v*, il paraît désirable que de nouvelles expériences soient entreprises. La précision des anciennes mesures peut être dépassée: *toutes les méthodes s'y prêtent*. Il y a encore à réduire quelques corrections trop incertaines; il y a à simplifier quelques mesures auxiliaires trop complexes, et par ce nouvel effort on pourra, sans aucun doute, apporter dans la mesure de *v* une précision supérieure à celle aujourd'hui acquise pour la vitesse de la lumière.” (H. Abraham, Rapport présenté au Congrès international de Physique, Paris, 1900.)

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I. INTRODUCTION.

1. THE METHOD.

Although many determinations of this important quantity have been made, even the best results hitherto found differ considerably from one another, and none are sufficiently accurate to be entirely satisfactory. The earlier results were rough, due largely to imperfect apparatus. Some of the more recent determinations have been made by methods that are interesting and important in themselves, but not capable of yielding results of high accuracy. Few, if any, even of the best determinations are exact enough to justify a claim of definitely fixing the value of ν to within one part in a thousand. Abraham, in his report to the Paris Congress of 1900, expresses a doubt whether as a result of all the determinations that have been made we can be sure of the value of ν to within 1 part in 1,000.

One of us published¹ a determination of ν in 1889, made at the Johns Hopkins University, using the spherical condenser which had been employed by Rowland ten years before, deriving ν from the ratio of the electrostatic to the electromagnetic capacity of the condenser. This is probably the best of all the methods yet employed for determining the ratio of the units. By this method ν is determined in terms of a resistance, but inasmuch as ν is equal to the square root of the ratio of the capacities, the uncertainty due to an error in the value of the ohm is divided by two, so that if we admit an uncertainty of two parts in five thousand in the value of the international ohm, that would involve an uncertainty of only one part in five thousand in ν . All of the methods of determining ν which do not involve the ohm are subject to even larger uncertainties in other directions. Hence it seemed to us desirable to make a new determination of the value of ν , using the method of capacities, and to undertake to attain a higher order of accuracy than had heretofore been realized.

¹ Phil. Mag., vol. 28, p. 315, 1889, and Am. Jour. Sci., vol. 38, p. 298, 1889. In this article I expressed the hope of repeating the determination of ν the following year in order to ascertain the cause of certain systematic differences appearing in the results. A favorable opportunity, however, did not arise until fifteen years afterwards, when the superior facilities of the Bureau of Standards suggested that the time had arrived for a new and more accurate determination of ν than was possible in the former work.—E. B. R.

2. SPHERICAL CONDENSERS.

Through the courtesy of Prof. Joseph S. Ames, of the Johns Hopkins University, we secured the same spherical condenser that had been employed in the experiments of 1879 and 1889, already referred to. The two balls were repolished and the two halves of the shell were reground so that they fitted together water-tight. The two electrostatic capacities of the condenser were redetermined, using first the larger ball and then the smaller ball within the shell. The electromagnetic capacity was found by means of the Wheatstone bridge, using a rotating commutator. The experiments were begun early in the summer of 1904, but owing to the difficulty of obtaining sufficiently high insulation at that time they were deferred until the following winter. Meanwhile we designed a new rotating commutator better fitted for this experiment than the one used in the preliminary work and had it constructed in the instrument shop of the Bureau.

On resuming the work in the winter of 1904-5, we were able to obtain much better measurements of the electromagnetic capacity than before, but we were unable to get a sufficiently exact determination of the capacity of the charging wire which passed through the small hole at the pole of the outer hollow sphere, touching the inner sphere. When this wire is lifted far enough to break the contact with the ball its capacity changes for two separate reasons—first, because its length within the shell has changed, and, second, because its potential is then not the same as that of the ball; and hence the field is different after the contact is broken. This difficulty we afterwards overcame completely by getting the capacity with two charging wires, one at the pole and one at the equator. First withdrawing the polar charging wire, the difference in electromagnetic capacity gave the capacity of the charging wire itself, the ball meanwhile being charged by a wire at the equator. Then replacing the polar charging wire and withdrawing that at the equator, the difference gave the capacity of the latter. This gives a determination of the correction for the capacity of the charging wire that is free from theoretical objection, and which in practice gave remarkably accurate results.

3. CYLINDRICAL CONDENSERS.

Before this difficulty due to the charging wire was overcome, however, we designed and had constructed in the instrument shop of the Bureau (early in 1905) a set of cylindrical condensers¹ which could be measured without the necessity of determining the capacity of the charging wire, in the way since described by Lord Rayleigh.² A pair of coaxial cylinders was mounted on a suitable base with their axes vertical, and so joined to the Maxwell bridge that the electromagnetic capacity C_1 of the inner cylinder could be measured. A second pair of cylinders of the same radii was then added to the first and the increased capacity C_2 determined. Then a third pair was added and the total capacity C_3 determined. The difference $C_2 - C_1$ is then the capacity of the second pair of cylinders, while $C_3 - C_2$ is the capacity of the third pair, assuming that the unknown end corrections and the capacity of the charging wire (which has remained undisturbed) have not changed during the process of building up the cylinders by adding successive sections.

4. GUARD CYLINDERS.

This method gave as good results as we had anticipated, but in the meantime we had so much improved the work with the spherical condensers that the results with the cylinders were not satisfactory. We then added a fourth pair of cylinders to the set and insulated the two end sections of the inner cylinders so that they formed guard cylinders, after the manner of the guard rings of a plate condenser. The upper guard cylinder was provided with a micrometer screw, so that the breadth of the air gap could be varied and the computed correction for the gap checked by experimental determinations with varying widths. The use of the guard cylinders necessitated a modification of the rotating commutator for charging and discharging the condensers, by adding three brushes on the opposite side of the ring, so that the middle cylinders and the guard cylinders could be charged and discharged simultaneously. Using the guard cylinders in this way we obtained results of much greater uniformity and greater apparent accuracy than we had expected.

¹ We are under obligations to Mr. H. B. Brooks for assistance in designing this condenser. It and some other apparatus employed was built by Mr. Joseph Ludwig in the instrument shop of this Bureau.

² Phil. Mag., 12, p. 97; 1906.

5. DIFFERENTIAL METHOD.

We therefore thought it wise to check the work done by the Maxwell bridge method by independent measurements, using a differential method and a differential galvanometer. The Weston Electrical Instrument Company built us for this work a very delicate differential moving coil galvanometer, with a bifilar suspension (both above and below) of fine silver wire. As this instrument could be used not only in the differential method but also in the bridge method (with the two coils in series), we were able to compare the results by the two methods very critically, as we could change over from one to the other in a moment, using the same resistances. The results agreed with great exactness. In the differential method we measured sometimes the charge and sometimes the discharge. No difference whatever could be observed.

We had now obtained a considerable number of determinations of the ratio v , using two different spherical condensers (the same shell in each case, but two balls of different sizes), two different capacities of cylindrical condensers, both with and without guard cylinders, and had made the measurements (both on spheres and cylinders) by two very different means, namely, the Maxwell bridge and the differential galvanometer methods. We had, moreover, varied the conditions in many ways, such as changing the speed of the commutator and so the frequency of the charge and discharge; varying the voltage of the battery; interchanging resistances; varying the galvanometers and other accessory apparatus; measuring both the charge and the discharge; and working in both summer and winter, at higher and lower temperatures (always of course correcting results to a standard temperature of 20°) and with considerable changes in atmospheric humidity. In summer this was kept low enough to get sufficiently high insulation by the use of coils of iron pipe in the room, through which cold calcium chloride brine was circulated, which condensed the surplus moisture from the atmosphere and enabled us to maintain a nearly constant temperature and a moderate humidity. In the exceptional cases when this could not be done observations were discontinued.

6. PLATE CONDENSERS.

In order to make the series of measurements more complete and to have one more chance of detecting any constant error that might possibly still remain in the mean value of ν which we were obtaining, we designed and had built in the instrument shop of the Bureau a parallel plate, guard ring condenser.⁴ This is not as favorable a form of condenser as the others, and possesses some disadvantages that we did not appreciate when it was designed. We have, however, obtained fair results, which confirm the value for ν given by the other condensers, although the values found by the plate condenser are entitled to less weight than those obtained by the spherical condensers and the cylindrical condensers with end guard cylinders.

The value which we have found for the ratio of the electrostatic to the electromagnetic unit, resulting from all our work (carried on without interruption since November, 1904) is

$$\nu = 2.9963 \times 10^{10}$$

taking the dielectric constant of air as unity. Referring to vacuum this becomes

$$\nu = 2.9971 \times 10^{10}$$

We believe this result is correct to within 1 part in 10,000, barring any uncertainty in the value of the international ohm. This is more fully discussed below.

II. DESCRIPTION OF THE CONDENSERS.

7. SPHERICAL CONDENSER.

As already stated, the spherical condenser was loaned to the Bureau by the Johns Hopkins University. It was made in 1879 from the designs and specifications of the late Prof. H. A. Rowland. The construction of the condenser is shown in Figs. 1 and 2. It consists of a brass shell provided with leveling screws and divided along its equator so that the upper hemisphere can be removed. The interior of the shell is accurately worked to a spherical surface

⁴ We are under obligations to Mr. F. S. Durston for assistance in designing this condenser and the direct reading chronograph described below. The plate condenser was built by Mr. Oscar Lange, chief mechanic of this Bureau.

and is nickel-plated. At the pole of the upper hemisphere is a hole through which passes one of the charging wires and also the silk

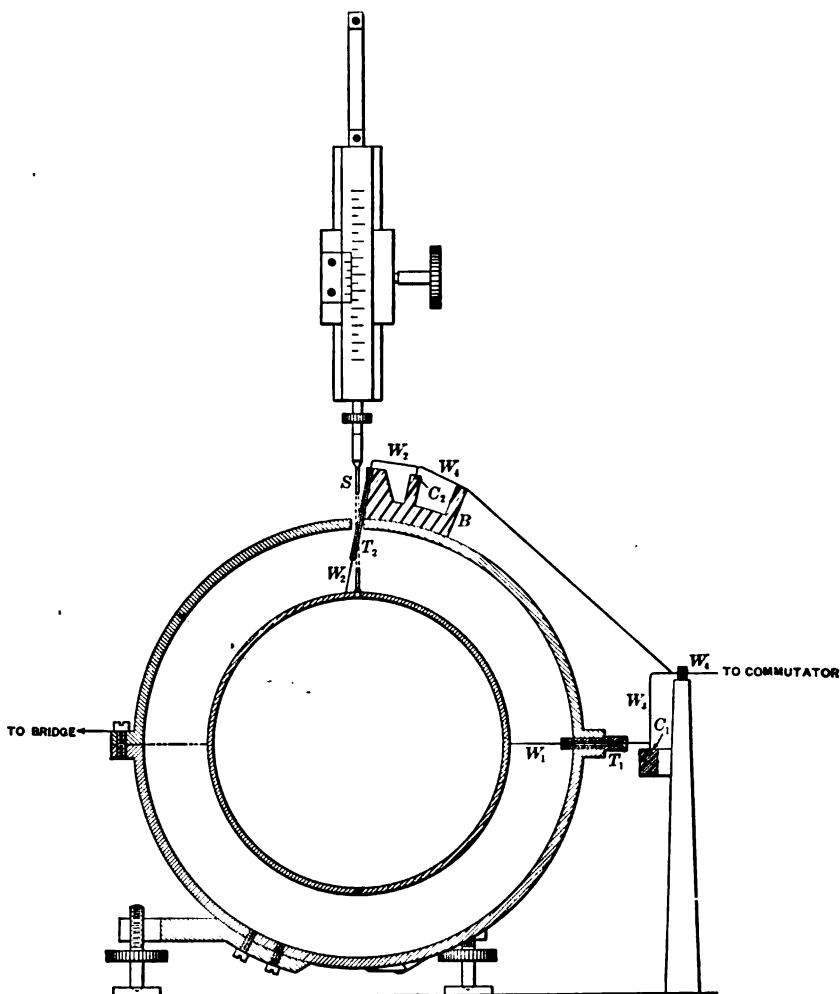


Fig. 2.—Section of Spherical Condenser.

The inner sphere is suspended by the silk cord *S* at the center of the shell. The two charging wires *W*₁ and *W*₂ are guided by the fixed tubes *T*₁ and *T*₂. The outside ends of *W*₁ and *W*₂ dip into the small mercury cups *C*₁ and *C*₂. These mercury cups were joined by the wires *W*₃ and *W*₄ to the rotating commutator.

cord by which the inner ball is suspended. In the equatorial plane of the shell is a second hole, through which we passed the second



Fig. 1.—*Spherical Condenser and Larger Ball, A.*

The ebonite block which supports the tube that serves as guide for the top charging wire is shown in place, and the upper end of the bent charging wire is shown lying in place along the top of the block and projecting out over the first notch in the ebonite; to this end is soldered a fine wire, not visible in the photograph, which is of such a length as just to dip into a small mercury cup in the top of the middle projection of the ebonite. The end of the bushing that serves as a guide for the side charging wire can be seen on the left.

[3607—07. To face page 440.]

charging wire. The inner ball is either one of two carefully spun nickel-plated balls of different radii. These with the shell form two condensers of different capacities. The silk cord supporting the ball, after passing through the hole in the upper hemisphere of the shell, is attached to the lower end of a scale which can be moved vertically in guides with a rack and pinion adjustment. Fastened to one of the guides is a vernier, by means of which the position of the scale can be directly read to a tenth of a millimeter. From the upper end of the scale a strip of thin spring steel passes to one end of a lever to the other end of which is attached a counterweight. The end of the lever to which the spring steel is attached is cut to the arc of a circle with its center at the pivot of the lever. The radius of the circle being equal to the distance of the plane of the scale from the pivot of the lever, the tension upon the scale will always be in the direction of its length whatever the position of the lever, within working range.

8. ADJUSTING THE CONDENSER.

In assembling the condenser the upper rim of the lower hemisphere is first carefully leveled. The ball and the upper hemisphere are then put in place and the ball adjusted by the rack and pinion movement until its center is approximately in the plane of the rim of the lower hemisphere. The upper hemisphere is now raised and propped on blocks so that the ball can be reached along the horizontal plane between the hemispheres, care being taken that the silk suspension of the ball hangs freely through the hole in the top of the hemisphere. The horizontal position of the ball is now tested at four points by means of a brass distance piece resting upon the broad rim of the lower hemisphere. Then by carefully sliding the lower hemisphere one way or another the horizontal centering of the ball can be attained with an error probably not greater than 0.1 mm. The contact between the ball and the distance piece is determined electrically.

This adjustment being completed the upper hemisphere is lowered and fastened in place on the lower one. Then by alternately raising and lowering the ball, by the rack and pinion adjustment, so as to make contact with the top or with the bottom of the shell, contact being determined electrically, it is easy to determine the scale read-

ing to $\frac{1}{10}$ mm corresponding to the central position of the ball. Setting the scale to this reading the condenser is ready for use.

This centering was later tested by measuring the capacity of the condenser in its adjusted position, and then displacing the ball first up and then down by known amounts. The capacity is a minimum with the ball in the center, and increases approximately as the square of the displacement in either direction. This gave a very delicate method of verifying the centering.

The warping of the floor and of the table, occasioned by variations in the humidity, causes the level of the instrument to change more or less. This of course displaces the ball from its central position slightly and so varies the capacity. In the early portion of the work the importance of this source of error was not fully appreciated, but later a delicate level was kept on the brass arm supporting the guides for the movable scale and the level of the instrument was readjusted as often as need be. The magnitude of the error that may be thus introduced is small but appreciable.

The radius of the shell is approximately 12.671 cm and of the balls 10.118 cm and 8.874 cm approximately.

9. THE CHARGING WIRES.

The method of charging the condenser is of great importance and involves the determination of a relatively large correction term. The charging can be done only by means of a wire reaching the ball through a hole in the shell. This wire produces two distinct effects, both of which must be eliminated: (1) It has a relatively large capacity which is thus added to that of the ball; (2) its presence distorts the field of force within the shell. These effects can not be eliminated by any process involving the mere withdrawal in whole or in part of the wire, or by a lowering of the ball, but must be definitely measured and allowed for. Furthermore, the capacity of the wire is greatly affected by slight changes in its position with respect to the shell, especially with respect to the edges of the hole through which it passes. Hence we must rigidly fix the position of the charging wire and then accurately determine its capacity when in this position.

In order to accomplish this we have employed two charging wires (Fig. 2). One extended diagonally through the hole in the

top of the shell through which hung the silk thread supporting the ball; the second extended horizontally through a hole in the equatorial plane of the shell. In the latter hole was tightly fitted an ebonite bushing extending 12 mm outside the outer surface of the shell and 5 mm inside the inner surface. The hole through the center of this was wide at the two ends of the bushing, but for about 2 centimeters in its central portion was of such a size as just to allow a copper wire (1.3 mm in diameter) to slide with gentle friction. The enlarging of the hole at the ends of the bushing, as well as the extension of the bushing beyond the surfaces of the shell, were of course intended to improve the insulation of the charging wire. The charging wire used for this side hole consisted of the above-mentioned piece of copper wire, to the inner end of which were soldered a couple of very fine phosphor bronze wires 2 or 3 mm in length. To the outer end of the charging wire was soldered a section of fine copper wire (0.13 mm diameter) bent so that by rotating the charging wire when in position the end of this fine wire could be dipped into a minute mercury cup just large enough to receive it and a similar wire connecting permanently with the rest of the charging system. The phosphor bronze tips allow good contact to be made with the ball without appreciably displacing it from its undisturbed position. The position of the charging wire is fixed by a mark on the wire in the plane of the end of the ebonite bushing.

To the top of the shell was fastened with hard wax a block of ebonite carrying a slender ebonite tube, which extended through the hole in the shell, just clearing the cord supporting the ball and reaching within a few millimeters of the surface of the larger ball when the latter is in position. This tube was made long, on account of the necessity of using a much finer wire, than in the case of the hole in the side of the shell; the hole through this tube was of such a size that a copper wire (0.72 mm diameter) could just pass without danger of jamming. This wire also had fine phosphor bronze tips soldered to its lower end; its upper end was so bent that when the wire was pushed in the tube so that the phosphor bronze tips made good contact with the ball the bent portion was in contact with the top of the ebonite block. To the outer end of this also was soldered a section of fine copper wire of such a length as just to dip into a diminutive mercury cup when the charging wire was in place.

The two small mercury cups were permanently connected with the rest of the charging system by fine wires stretched taut between rigid supports, so as to prevent accidental variations in capacity. The position of the upper charging wire was thus uniquely determined, and its capacity could vary only by a slipping of the ebonite block waxed to the shell; this block did slowly slip as a result of the tension of the wire running from it to the rest of the apparatus, but the variation was not appreciable in a single day, and the capacity of the charging wire was redetermined every time the capacity of the condenser was measured. There was no chance for such a change in the capacity of the side charging wire.

Having definitely fixed the position of the charging wires the next thing was to measure their capacities. This was done by a method of differences. The side wire w_1 being in position and the top one w_2 removed, the electro-magnetic capacity K_1 of the commutator, leads, and ball, was measured as accurately as possible. Then w_2 was placed in position, everything else remaining as before, and the total capacity K was measured. The difference between these two capacities, $K - K_1$, consists of three terms: (1) The capacity of w_2 ; (2) the change in the capacity of the ball due to the redistribution of charge upon it, resulting from the presence of w_2 ; (3) the change in capacity of the mercury cup A and of the wire leading to it, produced by the presence of w_2 . Call the sum of these three effects c_2 , the effective capacity of w_2 .

If we now remove w_1 , leaving everything else unchanged and measure the resulting total capacity K_2 , the difference between this capacity and the capacity (K) with *both* w_1 and w_2 in position will give us c_1 , the effective capacity of w_1 . This is made up of three terms corresponding exactly with those composing the effective capacity of w_2 .

If we now remove both w_1 and w_2 and measure the remaining capacity k , which is made up of the capacity of the commutator and of the wires leading to (and including) the mercury cups C_1 and C_2 , then put w_1 in place, everything else remaining unchanged, and again measure the total resulting capacity K_1 , the difference of these two capacities, $K_1 - k$, will be exactly equal to the capacity of the ball increased by what we have called c_1 the effective capacity of w_1 and which we have already measured. Thus $K_1 - k - c_1$ is the electromag-

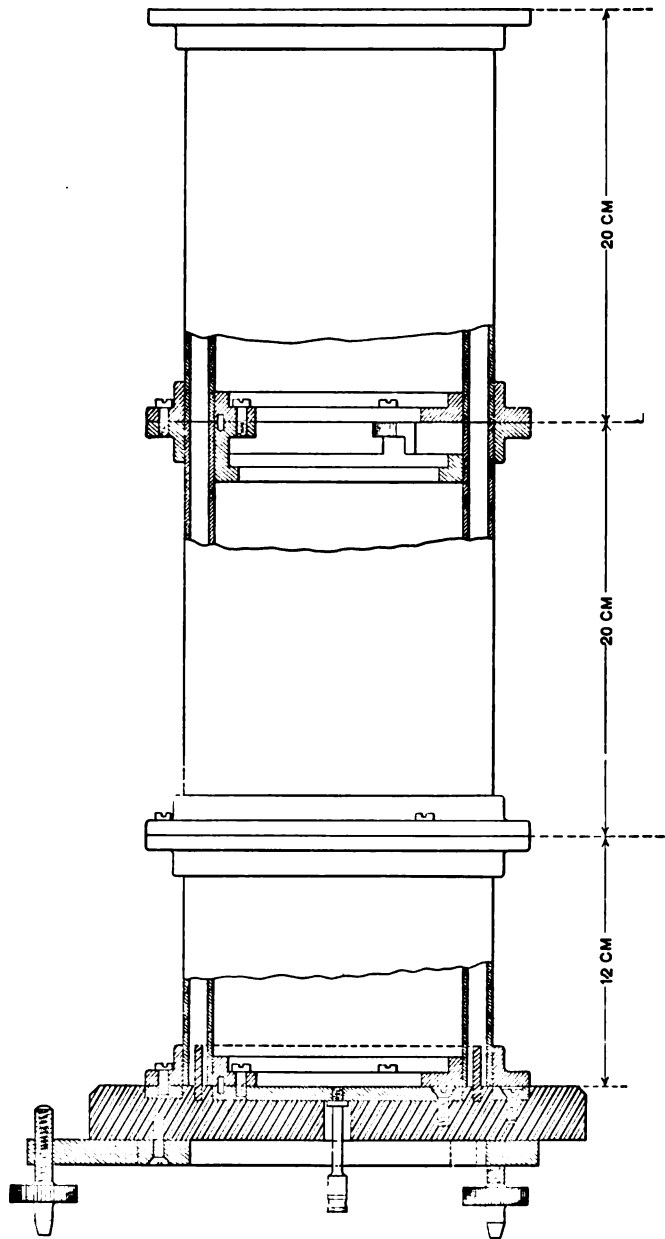


Fig. 3.—Cylindrical Condenser of Three Sections. No Guard Cylinder.

netic capacity C_a , or C_b , of the ball *entirely free* from all effects produced by the charging wire.⁵

10. CYLINDRICAL CONDENSER.

As stated above, the cylindrical condenser was built in the instrument shop of the Bureau. The condenser, as first constructed, consisted of three sections—a base section 12 cm long and two other sections each 20 cm long. Later a fourth section, 12 cm long, was made to serve as an upper guard cylinder. The radii of the inner cylinders are, approximately, 6.257 cm, those of the outer ones 7.241 cm. The base of the instrument is a heavy ebonite block mounted on leveling screws. In the top of the block is set a circular brass plate of slightly smaller diameter than the inner cylinder. This plate, to which the lower section of the inner cylinder is attached by screws, entirely closes the bottom of the inner cylinder. In the ebonite base, midway between the two cylinders, is set an ebonite ring, 2 mm thick and 1 cm high, thus making the leakage path between the cylinders about three times as long as it would otherwise have been. Beyond this a brass ring of a slightly larger inner diameter than the outer cylinders is set in the ebonite, and the lower section of the outer cylinder is attached to this ring by screws. Connection to the outer cylinder is made by means of one of the screws fastening it to the base. Connection to the inner cylinder is made by means of a binding post set in the middle of the under side of the ebonite base and connecting with the plate which closes the bottom of the inner cylinder. The various sections of the cylinders are interchangeable and are fastened together by three screws and three steadying pins at each joint. A brass plate of the internal diameter of the inner cylinders, provided with steadying pins and

⁵ It has been stated (Abraham, Rapport, Congrès International de Physique, 1900) that Rowland, who used this condenser in his determination of the ratio of the units in 1879, failed to eliminate the capacity of the charging wire. This is only partially correct. As we have seen, the charging wire produces two effects. The portion due to its own capacity was eliminated in Professor Rowland's work, for, after charging the condenser, the charging wire was withdrawn and a second wire was introduced to discharge it. (See p. 267, Physical Papers.) It was the quantity discharged that he measured electromagnetically. On the other hand, the effect of the distortion of the field by the wire was not eliminated. This quantity, however, was very small and need not be considered when an accuracy of not more than 1 in 1,000 is sought.



Fig. 3a.—*Guard Cylinder Condenser.*

[3607-07. To face page 446.]

with short projections to rest upon the upper rim of the cylinder, is used for closing the upper end of the inner cylinder when desired.

The cylinders are of brass, silver plated both inside and out. Sections of tubing of the required diameter were cut to the proper

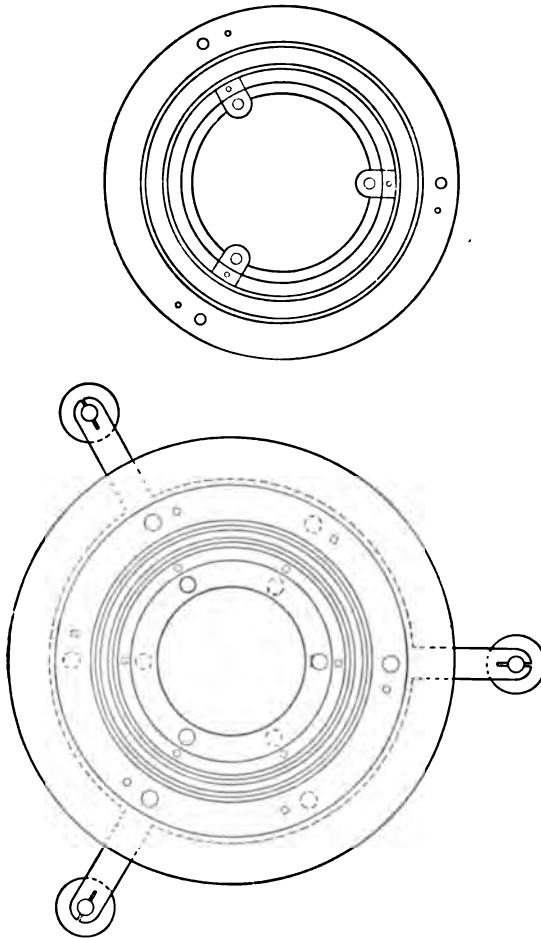


Fig. 4.—Plan of Base of Cylindrical Condenser, showing Method of Attachment of Cylinders.
Plan of End of a Section of the Condenser, showing Location of Screws and of Steadying Pins.

lengths, and each section was then stiffened by having heavy rings soldered to it near each end, the rings being, of course, placed inside the inner cylinders and outside the outer ones; the screws and steadying pins for connecting the sections pass through the rings.

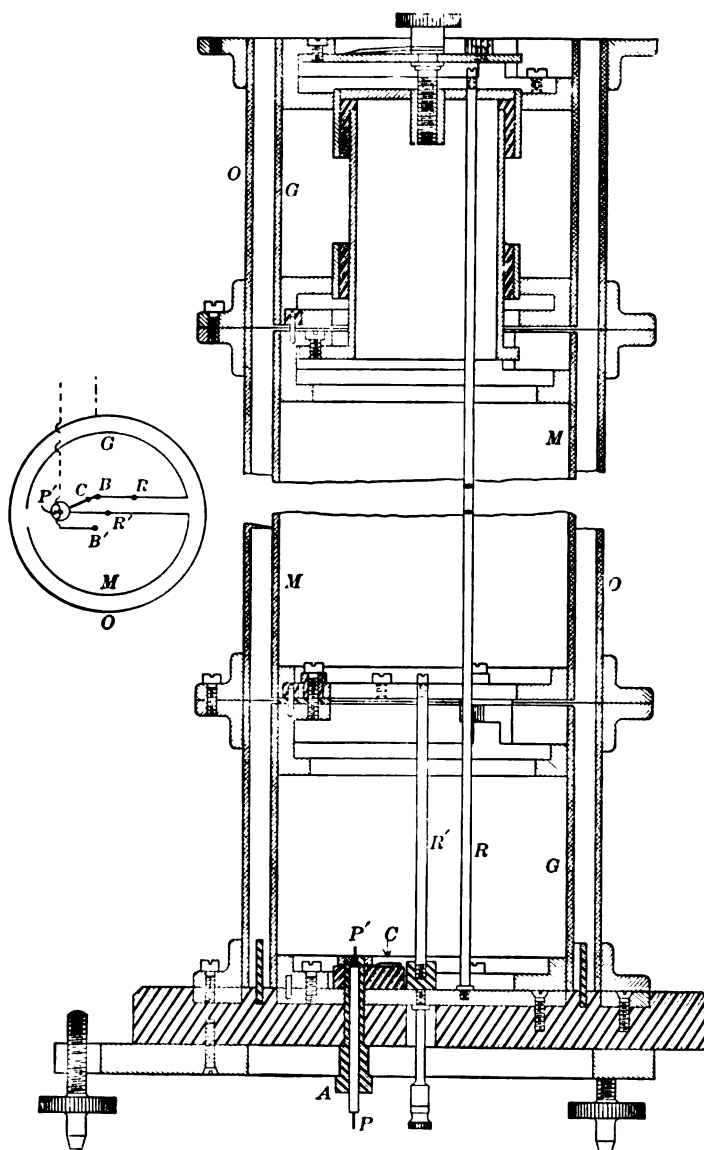


Fig. 5.—Section of Guard Cylinder Condenser.

The guard sections *G* are connected by the rod *R*, the upper section can be moved by the micrometer screw so as to regulate the width of the gap. The middle section *M* is connected to the rod *R'*, which in turn is connected to the spring arm *C* (small figure), which is rotated by the knob *A* so as to make contact with either *B* or *B'*, thus connecting the middle section either to the guard section or to the lead from the commutator.

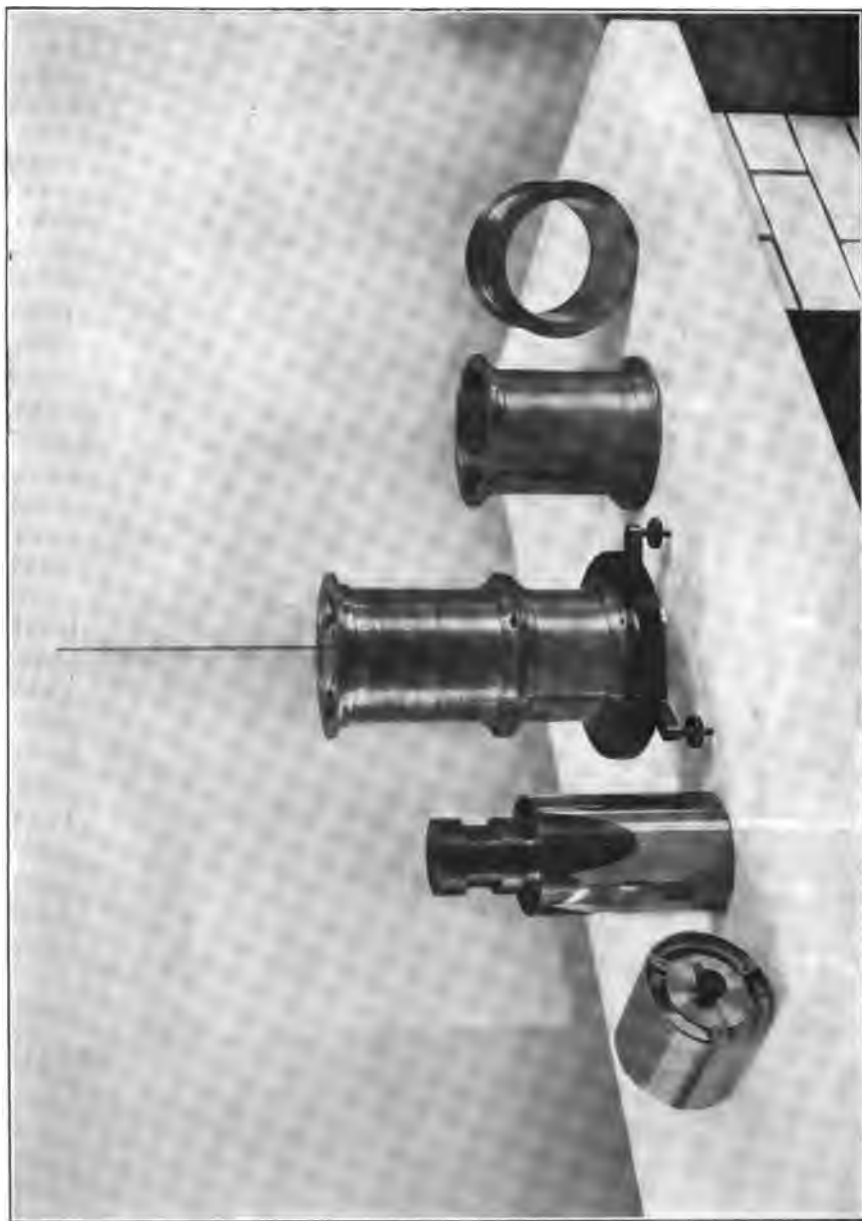


Fig. 4a.—*Cylindrical Condenser Partly Knocked Down.*

On the extreme left is shown the upper cylinder; next to it is seen inner cylinder No. 3 with the attachment which serves to guide the guard cylinder as it is raised or lowered. The corresponding outer cylinders are shown on the right. The rod extending above the lower sections serve to connect the upper and lower cylinders.

[3607-07. To face page 418.]

The cylinders were then carefully turned to size in a precision lathe, after which they were silvered and buffed down. Each inner cylinder was then placed coaxial with its corresponding outer one, and the ends of the combinations thus formed were ground until the two ends could be closed water-tight by means of flat ground glass plates.

These cylinders were intended to be built up to different heights so that the end effects could be eliminated by a method of differences, according to the plan since outlined by Lord Rayleigh.⁸

11. THE GUARD CYLINDERS.

When it was decided to use guard cylinders, a fourth section was added. These cylinders are of course of the same style as those just described, but the attachment of the inner cylinder is peculiar. To the top of the adjoining section is rigidly fastened a smaller brass tube, attached to which, but insulated from it, is a broad, carefully turned brass ring (Fig. 5). To the top of this tube, but insulated from it, is a second broad ring closed at the top with a heavy plate, in the center of which is a hole 5 mm in diameter threaded to fit a well-made screw of $\frac{1}{2}$ mm pitch. Attached to the inside of the inner guard cylinder are a pair of rings, which pass with almost a piston fit over the broad rings mentioned. This enables the guard cylinder to be moved along its axis without lateral displacement. The top of the guard cylinder is closed by a brass plate having a divided circular scale; through the center of this plate and attached to it by a collar passes the screw of $\frac{1}{2}$ mm pitch of which we have spoken. In this way the cylinder can be raised or lowered by the screw and its exact position can be determined from the position of a pointer attached to the screw and moving over the scale. This is for the purpose of studying the variation of the capacity with the width of the gap between the cylinder and its guard cylinder. The guard cylinder is prevented from turning with the screw, as it is raised and lowered, by means of a steadying pin attached to the adjoining cylinder and passing through an ebonite block. The lower guard cylinder was insulated from its neighbor by means of three ebonite plates 1.6 mm thick placed between the rings attached to the inside of the cylinders. The rings were turned down until when the blocks are in place the gap between the two sections of the cylinders is 0.6 mm. The screws and steadying pins for holding these sections together are insulated from the upper section by means of ebonite bushings. The ebonite plates do not come within 3 mm of the outer

face of the cylinders; the bushings for the screws and pins are still farther back.

For charging the guard-ring condenser a key of the form shown in Fig. 5 was employed.

12. SWITCH AND CHARGING WIRES.

The lead from the commutator is stretched taut and attached to an ebonite post, screwed to the table and vertically below the point P (Fig. 5). From this post it is stretched tightly to the point P, to which it is soldered, forming thus a prolongation of PP', so that when PP' is rotated by means of the ebonite head A the wire will be twisted about its axis, but will not be displaced with reference to other objects. P' is connected to the button B'. The spring arm C which rotates with PP', but is insulated from it, is connected with the rod R', which is connected with the middle section M of the inner cylinder. The other button, B, with which C can be brought into contact, is connected with the rod R, which connects the two guard cylinders G.

Hence when C rests upon B' the middle sections of the inner cylinder are connected through PP' with the lead to the commutator, and the capacity measured is that of the commutator, the lead, and the condenser. On the other hand, when C rests on B the button B' is insulated, and the middle sections of the inner cylinder are connected with the guard cylinders, and so are charged to the same potential as before, but by way of the guard-ring section of the commutator. Hence the capacity now measured is that of the commutator and leads only. The capacity of the latter must be the same as it was in the former case, for the only change that has been made is the rotation of C from B' to B, and these portions of the system are entirely inclosed by the inner cylinder, all of which is at the same potential in both cases. They are, therefore, always uncharged, and so contribute nothing to the capacity in either case. The leakage from B' to B and to C was frequently tested, and in all cases was found to be negligible.

13. PLATE CONDENSER.

This condenser was of the guard-ring type, and, as already stated, was built in the instrument shop of the Bureau. The details of its construction are shown in Fig. 7. The top of the instrument con-

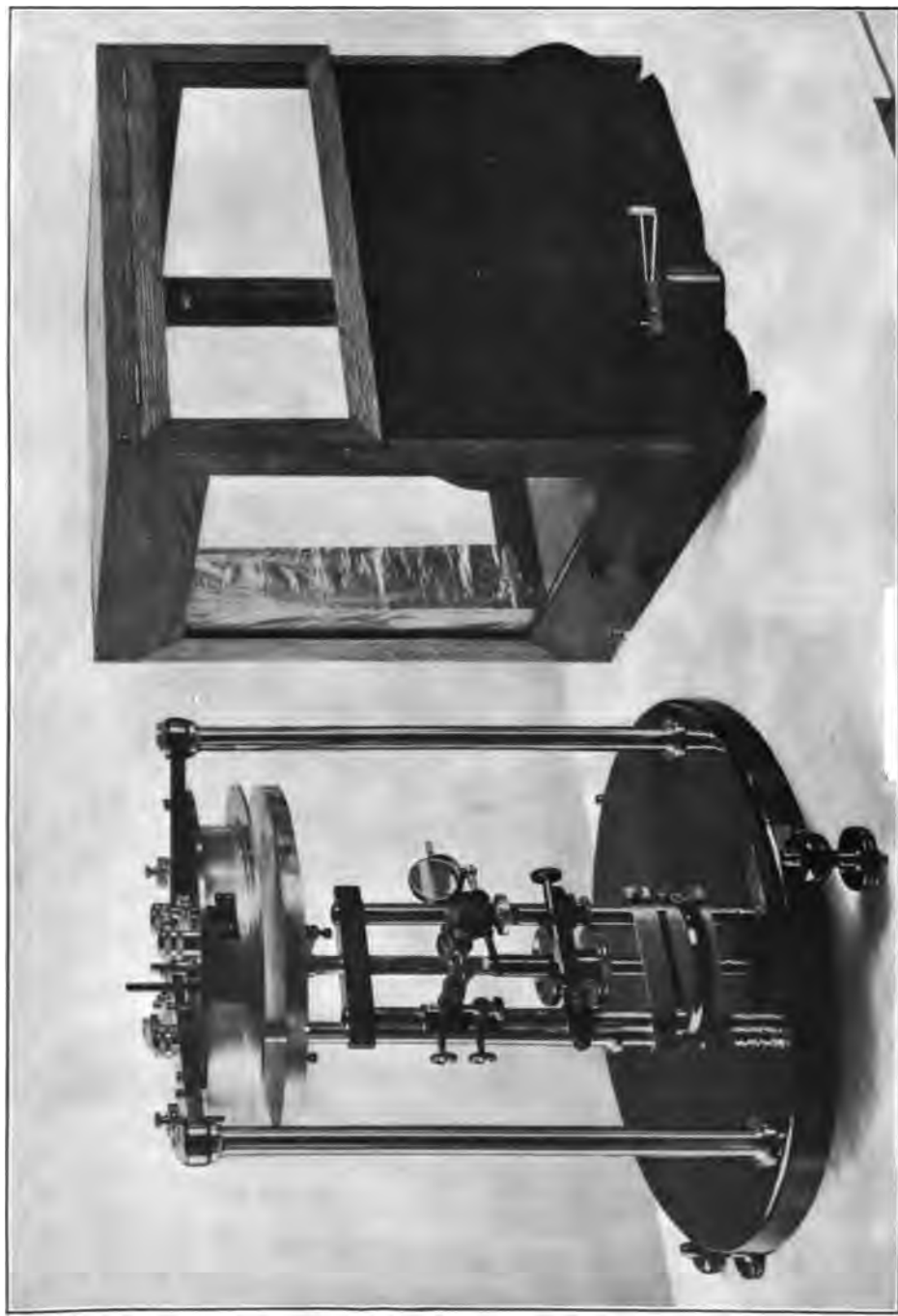


Fig. 6.—*Plate Condenser.*

To the right of the condenser is the case which fits over the condenser and within which the temperature can be maintained constant by electrical heating. Part of the front of the case is closed by a black felt curtain through which projects the eye piece of the reading microscope, and in which are two slits through which the hands can be passed for the purpose of moving the lower plate of the instrument. The back left-hand corner of the case is covered on the inside with tin foil, which is earthed so as to make the field around the leads (which pass up this corner of the case) as constant as possible. In front of the case may be seen the lever for adjusting the plates.

sists of a brass plate 6 mm thick and 30 cm in diameter; this is supported on three steel columns by means of heavy brass blocks 3.5 cm wide and 1.2 cm thick, screwed securely to the top of the plate and extending nearly to its center. The blocks are insulated

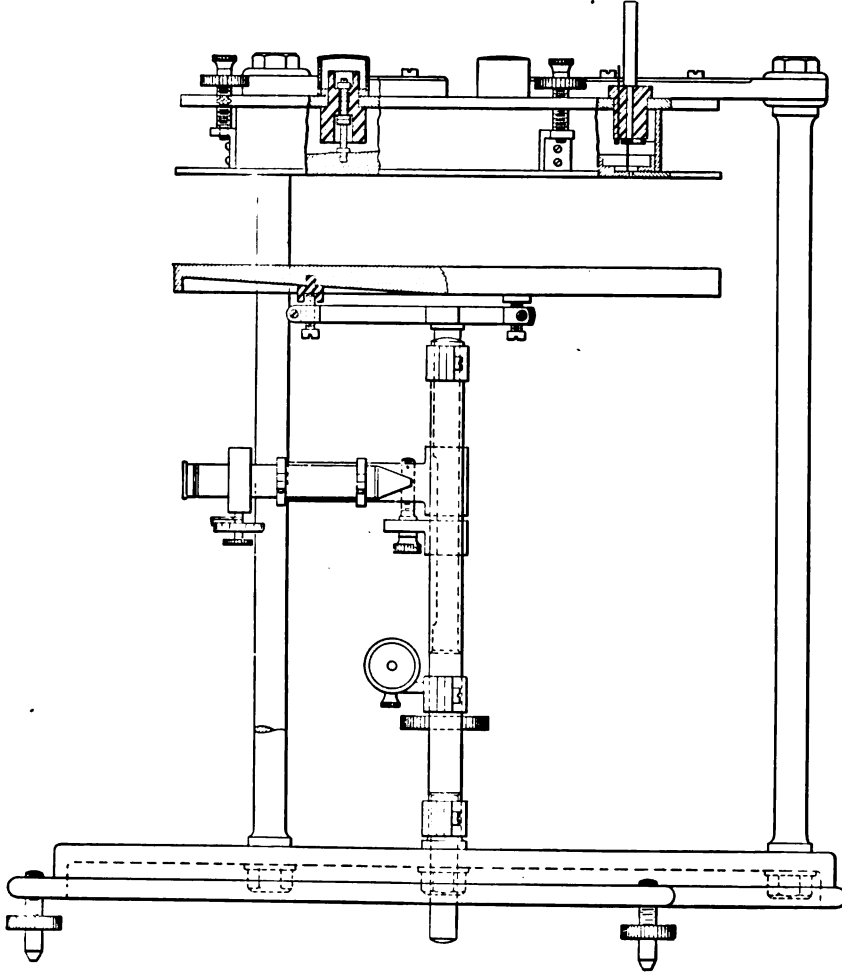


Fig. 7.—Plate Condenser showing Manner in which the Guard Ring and Plates are Supported and Adjusted, Switch for Connecting the Collector Plate to either Guard Ring or Charging Wire, and the Microscope for Determining the Position of the Lower Plate.

from the steel pillars by ebonite bushings. By means of three screws passing through ebonite bushings in this plate, the insulated plate of the condenser is supported below it. Lock nuts make the

position of this plate definite. Three other lock-nut screws passing through the top plate support the guard ring and enable us to adjust the latter to the plane of the insulated plate. Attached to the top of the guard ring is a short cylinder having a radius of about 3 cm less than that of the ring and of such a height as to reach nearly to the top plate of the instrument; it falls short of reaching the top plate by an amount just sufficient to allow the proper adjustment of the guard ring. This cylinder, with the top plate of the instrument, completely screens the top face of the insulated plate.

14. VARYING THE CAPACITY.

The other plate is supported under these on ebonite blocks on the three leveling screws of an inverted tripod. The tripod is screwed rigidly to a steel rod 1.5 cm in diameter. This rod has at its lower end a screw of 0.5 mm pitch and passes through guides so that it can be moved slowly and accurately along its own axis. To it is fastened a fine silver scale which can be read by means of a micrometer microscope rigidly attached to the base of the instrument.

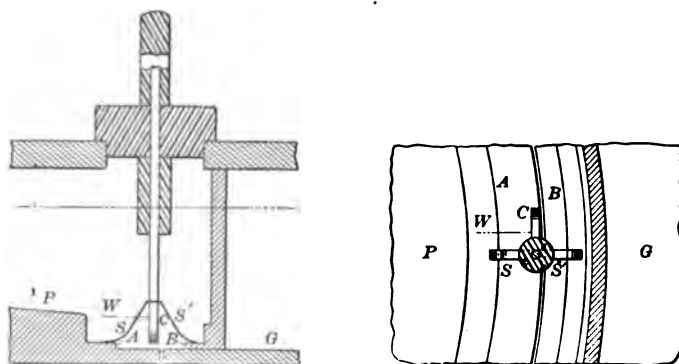


Fig. 8.—Switch for Connecting Collector Plate *P* with either the Guard Ring *G*, or with the Wire *W* Connected with the Commutator.

As shown the plate and ring are connected; by rotating the switch 90° , *C* makes contact with *A* and the plate is connected to the commutator.

By this means we can vary the distance between the plates by accurately known amounts. The entire instrument is placed inside a glass case in which the temperature can be controlled. The plates are of silvered brass and are ground plane; their backs are heavily ribbed so as to give them great rigidity. The instrument is provided with two collector plates of slightly different radii, so that two widths of guard ring gap can be employed.

15. SWITCH AND CHARGING WIRES.

The condenser is charged by means of a special key shown in Fig. 8. The wire from the commutator is stretched tightly to a stiff wire passing through an ebonite bushing set in the center of the top plate of the instrument. From the lower end of this wire, which is within the space entirely inclosed by the top plate of the condenser, the collector plate, and the guard ring, a spiral W of fine double-silk covered copper wire goes to the point C of the key. The key is situated immediately over the gap between the ring and the plate; it passes through an ebonite bushing in the top plate and is actuated by means of an ebonite handle. The guard ring and the collector plate are cut away near their edges, as shown in the figure, so that when the key is turned in the proper way and depressed the spring C makes contact with the plate, while the ends of the spring SS' lie in the gap between A and B. In this position the plate and the guard ring are separately charged. If now the key be turned by 90° , the spring C will lie in the groove between A and B, while the other spring SS' will touch A and B, thus connecting the plate and the guard ring; both will now be charged by means of the guard ring lead. The only portion of the lead to the collector plate which differs in the two cases is the portion of the key which is entirely surrounded by the collector plate, guard ring, and top plate of the condenser. Being in a region of uniform potential, this portion is uncharged. Hence the capacity of the main lead is the same in the two positions, and the difference in the capacities measured with the key in the two positions is exactly equal to the capacity of the collector plate of the condenser. Later a key similar to that employed for the cylinders was used.

16. ADJUSTING THE CONDENSER.

To adjust the condenser, the collector plate and guard ring are first adjusted, so as to lie approximately in a plane parallel to the top plate of the instrument and with the center of the collector plate coinciding with the center of the guard ring. When they are parallel to the top plate of the instrument they will be normal to the direction of motion of the rod carrying the lower plate. The centering of the plate is tested by measuring the width of the gap

between the plate and the ring at three points by means of a micrometer microscope, which can be placed in any one of three holes in the top plate and over the gap. The coincidence of the planes of the plate and of the ring was tested by a delicate lever, whose base rested on the lower plate of the condenser and could be slid so that the shorter arm of the lever touched either the plate or the ring. In this way the planes of the plate and of the ring could be brought into coincidence to within 0.005 mm., which suffices to reduce the error due to this cause to 3 parts in 10,000 in the most unfavorable case. Having adjusted the plate and the ring, the nuts on the screws were tightly locked and the upper plate with the collector plate and guard ring attached was removed from the steel pillars supporting it, turned upside down, and the width of the gap between the ring and the plate in the plane of the face of the condenser was measured at several equidistant points. The top plate with its attachments is then replaced and the lower plate of the condenser is raised to a suitable position, and the short arm of an accurate steel square is rested on it, and the distance between its supporting rod and the pendant long arm of the square is measured at two points as far apart as possible. This is repeated with the short arm of the square in a position at right angles to its former position. These measurements afford sufficient data to enable us to determine the angle the rod makes with the plane of the plate and to correct our results for it, if necessary. The coincidence of the planes of the ring and of the plate was occasionally tested.

When in use, the various holes in the top plate of the condenser, whether closed with ebonite bushings or not, are closed with metal caps having only the openings necessary to admit the leads.

No attempt is made to measure directly the true distance between the plates. Instead of this the plates are placed close together, say, 1 mm apart, and the capacity is measured. From this and an approximate value of ϵ the true electrical distance between the plates is calculated. The distance between the plates is now increased by a measured amount and the capacity is again measured. From this capacity and the total distance between the plates (the small electrical distance calculated plus the measured increase in the distance) a more approximate value of the ratio is calculated. If necessary, this better value of ϵ may be used to recalculate the

initial distance, and so a second approximation may be obtained. The capacity is actually measured for many different distances between the plates.

III. ELECTROSTATIC CAPACITIES.

17. SPHERICAL CONDENSER—GRAVIMETRIC METHOD.

The electrostatic capacity of a condenser bounded by two concentric spherical surfaces of radii R and r , taking the dielectric constant of the medium as unity, is

$$C = \frac{Rr}{R-r}$$

It is necessary, therefore, to determine accurately the mean interior radius of the hollow spherical shell, and the mean radii of the two spherical balls that are used in turn within the shell. The determination of these radii was made by two different methods. In the first, or gravimetric method, the volume of the shell is determined by ascertaining the mass of water it contains at a particular temperature, and the volume of the balls by finding the mass of water they displace when submerged. In the second method measurements of their diameters are made directly on the balls. On the shell accurate direct measurements can not be made. We have a check, however, on the gravimetric determination of the radius of the shell in the fact that if it were appreciably in error the results by the two balls would differ, since the error would affect the capacity when using the large ball more than when using the small one. The close agreement of the results by the two balls shows that there could be no serious error in the radius of the shell as determined by the gravimetric method.

a. The Shell.—The two hemispheres of the shell were screwed together in the position in which they were to be used in the determination of the electromagnetic capacity, and the joint, which was practically water-tight, was made perfect by means of a very little white lead around the joint on the outside of the shell (1904) or by means of a very little vaseline spread on the outermost edges of the flanges before the hemispheres were put together (1905). The shell thus prepared was weighed by the method of substitution. It was then filled with distilled water, care being taken to remove as completely as possible the last traces of air bubbles; this was done by

tipping the shell rather violently from side to side and by scraping the inner surface of the filled shell with a bent copper wire. After the shell had been filled level with the top of the hole in the upper hemisphere, it was weighed by the substitution method. A little of the water was then removed by means of a pipette, and the shell was refilled as before and again weighed. Finally the shell was emptied and dried and the process repeated. The temperature of the water was determined just before the completion of each filling, and again after each weighing. From these weighings the volume is computed and from this the radius is determined. The shell and weights both being of brass the correction to vacuo in the case of the empty shell is zero. A specimen determination of the volume of the shell is given below.

June 18, 1904.

Dry Bulb 22°4	Barometer 76.36 cm; at 25°9
Wet " 20°6	-.79 " correction for temperature, humidity, etc.
	75.57 " corrected pressure.
Shell filled, $t = 20^{\circ}77$, weighs	26 lbs. + 8596.631 g
" empty	" 26 lbs. + 98.869 "
Apparent weight of water =	8497.762 "
Approximate volume of water =	8520 cc
" " of weights =	1010 "
Resultant volume air displaced =	7510 "
Log dens. air at 22°4, 1 mm press. =	6.1965
Log 755.7	= 2.8783
" 7510	= 3.8756
Log weight w of air displaced	= 0.9504 $\therefore w = 8.920$ g
$\therefore$ Mass of water	= 8506.68 g
Log 8506.68	= 3.9297601
" D at 20°77	= 1.9991627
" Vol. at 20°77	= 3.9305974 $\therefore$ Volume = 8523.097 cc
" $\frac{4}{3}\pi$	= 0.6220886
" R^3	= 3.3085088
" R	= 1.1028363 $\therefore R = 12.67174$ cm at 20°77
	0.00018 cor. to 20°0
	$R_{20} = 12.67156$ cm

As a further check one determination of the volume of the inter-space between the larger ball (ball *A*) and the shell was made by an analogous method. This gave a value very closely agreeing with the other results.

b. The Balls.—A similar method was adopted for determining the radius of the balls. From the difference in their apparent weights when in air and when suspended in water, their volumes are determined and from these their mean radii are obtained. Owing to the fact that the balls are hollow they float in water; hence a sinker has to be attached in order to submerge them. In 1904 the sinker was attached to a wire harness inclosing the ball, but in 1905 it was fastened to a small hook screwed into the bottom of the ball. The weight of the ball in air was determined by the method of double weighings, and the arms of the balance were found sufficiently equal to necessitate no correction for the weighings in water. The method of reduction is shown below.

(1) *Weight of Ball B in Water.*

April 11, 1905.

Air temperature 21°1 C	Barometer 74.30 cm
Hydrograph 50 per cent	<u>-.38</u> " correction
= 9.3 mm press.	73.92 " corrected pressure.
(ball <i>B</i> + sinker, both in water at 21°05) + wire	= 54.853 g
(0.041 cc more wire + sinker) in water + wire	= <u>651.979</u> "
(ball <i>B</i> - 0.041 cc) in water at 21°05	= -597.126 "
Correction for 0.041 cc	= - 0.041 "
Correction for air displaced by 597 g of brass	= + 0.083 "
True weight of ball <i>B</i> in water at 21°05	= -597.084 "

(2) *Weight of Ball B in Air.*

Air temperature 21°0	Barometer = 74.10 cm
Hydrograph 50 per cent	<u>-.37</u> " correction
= 9.2 mm press.	73.73 " corrected pressure
	Approximate volume = 2927 cc
	Vol. Weights = 275 "
	Difference = 2652 "

Weight of ball <i>B</i> in air	=	2321.144 g	
Weight of 2652 cc of air	=	3.089 "	
Weight in vacuo of ball <i>B</i>	=	2324.233 "	
" " water at 21°05	=	597.084 "	
Loss of weight in water	=	-2921.317 "	
Log 2921.317	=	3.4655787	
" <i>D</i> at 21°05	=	1.9991360	
" Vol. at 21°05	=	3.4664427	∴ Volume at 21°05 = 2927.134 cc
" $\frac{4}{3}\pi$	=	0.6220886	
" r^3	=	2.8443541	
" r	=	0.9481180	∴ $r = 8.87397$ cm at 21°05
			0.00017 correction to 20°0
			$r = 8.87380$ cm at 20°0

The weights used were compared with the standards of this Bureau and the necessary corrections have been applied. The thermometer used in 1904 had its corrections determined just before it was used, but owing to an oversight that used in 1905 was not standardized until 14 months after it was used. This may explain the slight difference between the two sets of results shown in the following summary; a rise of zero of 0°11 C between the time the thermometer was used and the time it was standardized will account for the difference.

18. SUMMARY OF DIMENSIONS.

Shell; radius at 20°		Ball A; radius at 20°		Ball B; radius at 20°
1904	1905	1904	1905	1905
12.67156 cm	12.67134 cm	10.11799 cm	10.11794 cm	8.87380 cm
12.67151	12.67118	10.11811	10.11787	8.87378
12.67169	12.67149	10.11814	10.11790	8.87381
12.67159 ⁶	12.67140	10.11808		8.87380
	12.67151			
	12.67143			
	12.67143			
	12.67140			

It will be noticed that the 1905 values are a little lower than the values of 1904, and by nearly the same relative amount for both ball *A* and the shell.

If we assume the thermometer used in 1905 changed $0^{\circ}.11$ between the time it was used and the time it was standardized the values for 1905 become:

Shell = 12.67158 cm; ball *A* = 10.11806 cm; ball *B* = 8.87391 cm.

From the above values of the radii we obtain the following values for the capacities:

Ball <i>A</i> and shell as determined in 1904	C = 50.2102 cm
" " " " " " 1905	C = 50.2087 "
Mean	= 50.2094 "
" " " " " " (Assuming change in thermometer)	C = 50.2098 "
Ball <i>B</i> " " " " " "	C = 29.6091 "
" " " " " " (Assuming change in thermometer)	C = 29.6093 "
" as determined in 1905; shell as determined in 1904	C = 29.6080 "

The last is evidently an unfair combination, involving as it does a differential error in the thermometers; but even this value of the capacity differs from the others by only 4 parts in 100,000, which corresponds to an error of 2 parts in 100,000 in ∇ . Hence, if we assume that the capacities of the condenser with the two balls are as follows:

With ball *A*, $C = 50.2095$ cm at 20° C
 " " *B*, $C = 29.6092$ " " "

we shall probably not be in error by more than 2 in 100,000 in the electrostatic capacity, and this error would affect the value of ∇ by only 1 in 100,000.

The possible effect of the two small holes in the shell (through which the charging wires pass, and through one of which the cord for suspending the ball passes) will be discussed below.

* The following values of the radii were found in 1889 (E. B. R., loc. cit.):

Ball *A* = 10.1180 cm at $17^{\circ}.0$ C = 10.1185 cm at $20^{\circ}.0$ C

Ball *B* = 8.8735 " " $16^{\circ}.5$ C = 8.8741 " " "

Shell = 12.6805 " " $18^{\circ}.4$ C = 12.6811 " " "

These differ from those values just given by 5μ for ball *A*, 3μ for ball *B*, and 96μ for the shell. The differences in the case of the balls are less than the error allowed in the former work; the large difference in the case of the shell is due to the fact that before beginning the present work a few elevations on the flanges of the hemispheres were ground down so that the two hemispheres should make good contact all around the circumference, thus reducing the mean radius of the shell.

In the above reduction D is the specific gravity of water referred to water at 4°C as unity as given by Chappuis.⁷ If we assume that the density of water at 4°C is 0.999955 as given by Guillaume,⁸ then the volumes as given above are too small by 45 parts in 1,000,000, the electrostatic capacities are too small by 15 parts in 1,000,000, and the value found for ν by using the assumed capacities will be too small by 7 parts in 1,000,000.

19. SPHERICAL CONDENSERS—DIRECT MEASUREMENTS.

In order to obtain an idea as to the figuring of the balls, and to get an independent check on the dimensions determined gravimetrically each ball was measured along 50 diameters. For this pur-

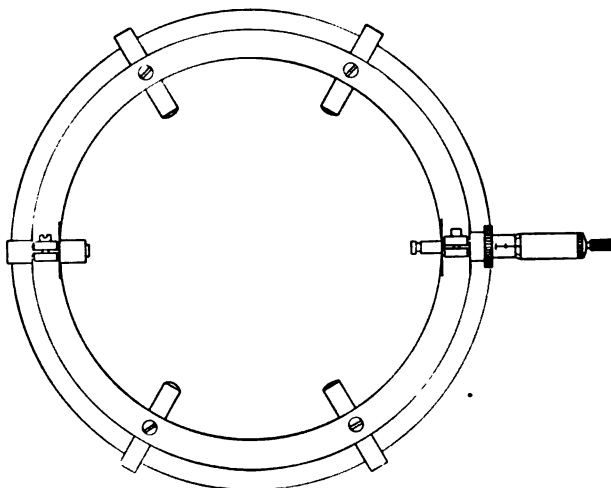


Fig. 9.—Ring Caliper for Measuring Spheres.

The four guides are adjusted so that the micrometer screw and its anvil lie along a diameter of the ball and the ball is just free to turn.

pose a ring caliper such as is shown in Fig. 9 was used. It was so constructed that when it and the ball to be measured were in position and resting upon a flat surface the micrometer screw and its

⁷ *Dilatation de l'Eau, Travaux et Memoires du Bureau International*, vol. 13, pp. 1-40; 1904.

⁸ *Rapports Int. Cong. Paris, 1900*, vol. 1, p. 99.

anvil lay in the equatorial plane, and the four guides lay along a circle of latitude of the ball. The guides were so adjusted that the ball could just turn freely, and so that the screw and its opposing anvil lay along a diameter. Diameters were measured every 18° around the equator, giving 20 measurements of 10 different diameters. Similar measurements were taken along four meridian circles differing in longitude by about 45° . Taken in this manner the diameters are not equally spaced over the surface of the sphere but are crowded at the poles and spread out at the equator, the spacing being inversely proportional to the sine of the polar angle. Hence in obtaining the mean diameter we must give each diameter a weight proportional to the sine of its polar angle. No measurement is taken nearer than 9° to the pole. In the tables below are given the differences "Diameter — End Standard" (in millimeters) for the various diameters measured; these have been corrected for a slight variation in temperature during the measurement.

The mean diameter of ball *A* at 24.6° is equal to the length of the end standard plus the difference between it and the diameter of the ball, this difference being (as shown on the next page) 0.0609 mm the mean diameter of ball *A* is

$$= 20.23132 + 0.00609 = 20.23741 \text{ cm}$$

$$\frac{0.00167}{\text{Mean diameter ball } A} = \text{correction to } 20.0^\circ$$

$$\text{Mean diameter ball } A = 20.23574 \text{ " at } 20.0^\circ$$

$$\text{" radius " = } 10.11787 \text{ " " "}$$

This differs from the value found gravimetrically by about 2μ , which is very fair agreement. For this ball the range is from 0.122 to $-0.031 = 0.153$ mm. This is a rather large quantity, but will not affect the final result appreciably.

TABLE I.

Measurements of Diameters of Ball A at 24°6; Temperature of Standard 25° C.

Meridian position	Polar distances ϕ					
	9° 171° 189° 351°	27° 153° 207° 333°	45° 135° 225° 315°	63° 117° 243° 297°	81° 99° 261° 279°	Equator
0°	0.107	0.043	0.069	0.032	0.028	0.047
	0.111	0.109	0.102	0.073	0.051	0.068
	0.096	0.031	0.045	0.041	0.072	0.090
	0.110	0.115	0.114	0.115	0.098	0.100
						0.101
45°	0.087	0.103	0.076	0.070	0.074	0.086
	0.120	0.118	0.122	0.107	0.099	0.030
	0.049	0.100	0.071	0.072	0.100	0.001
	0.038	0.119	0.115	0.113	0.110	-0.028
						-0.020
90°	0.041	0.118	0.071	0.054	0.040	0.021
	0.032	0.103	0.080	0.078	0.062	0.067
	0.108	0.059	0.040	0.055	0.058	0.098
	0.020	0.108	0.078	0.079	0.070	0.101
						0.053
135°	-0.010	0.102	0.106	0.069	0.020	0.012
	0.027	0.060	0.022	-0.020	-0.019	-0.015
	0.012	0.101	0.067	0.049	-0.005	-0.031
	-0.025	0.064	0.029	-0.010	-0.024	0.021
						0.040
Mean δ	0.0577	0.0908	0.0754	0.0611	0.0521	0.0421
$\sin \phi$	0.1564	0.4540	0.7071	0.8910	0.9877	1.0000
$\delta \sin \phi$	0.0090	0.0412	0.0533	0.0544	0.0515	0.0421

$$\Sigma \sin \phi = 4.1962; \Sigma \delta \sin \phi = 0.2515$$

$$\therefore \frac{\Sigma \delta \sin \phi}{\Sigma \sin \phi} = \frac{0.2515}{4.1962} = 0.0600 \text{ mm}$$

$$\frac{0.0009}{0.0609} \text{ " } = \text{cor. to bring standard to } 24^{\circ}6$$

TABLE II.

Measurements of diameters of Ball B, at 23°4 C. The values given in the table are the differences δ between the diameters and the length of the end standard.

Meridian positions	Polar distances ϕ					
	9° 171° 189° 351°	27° 153° 207° 333°	45° 135° 225° 315°	63° 117° 243° 297°	81° 99° 261° 279°	Equator
0°	+0.010	+0.007	+0.012	+0.003	—0.020	—0.022
	—0.014	—0.034	—0.053	—0.090	—0.043	—0.035
	+0.002	+0.011	+0.014	+0.017	—0.010	—0.047
	+0.022	—0.009	—0.051	—0.074	—0.045	—0.053
						—0.054
45°	+0.013	+0.020	+0.018	—0.004	0.000	—0.055
	+0.009	+0.019	—0.020	—0.022	—0.017	—0.061
	+0.018	+0.018	—0.007	+0.001	—0.018	—0.073
	+0.010	+0.019	+0.008	—0.020	—0.027	—0.073
						—0.054
90°	+0.008	+0.013	+0.015	—0.006	—0.005	—0.031
	+0.013	—0.008	—0.016	—0.040	—0.026	—0.029
	+0.016	+0.010	+0.012	—0.007	—0.005	—0.052
	+0.018	+0.005	—0.013	—0.032	—0.026	—0.062
						—0.052
135°	+0.009	+0.003	+0.010	—0.009	—0.027	—0.072
	+0.014	—0.027	—0.039	—0.054	—0.025	—0.091
	+0.009	+0.008	+0.010	—0.017	—0.022	—0.057
	+0.010	+0.026	—0.022	—0.039	—0.054	—0.035
						—0.028
Mean δ	+0.0104	+0.0050	—0.0076	—0.0246	—0.0231	—0.0518
$\sin \phi$	0.1564	0.4540	0.7071	0.8910	0.9877	1.0000
$\delta \sin \phi$	+0.0016	+0.0023	—0.0054	—0.0219	—0.0228	—0.0518

$$\Sigma \sin \phi = 4.1962; \Sigma \delta \sin \phi = -0.0980$$

$$\therefore \frac{\Sigma \delta \sin \phi}{\Sigma \sin \phi} = \frac{-0.0980}{4.1962} = -0.0234 \text{ mm.}$$

Hence mean diameter of ball B at $23^{\circ}.4$ C
 $= 177.5$ mm end standard -0.0234 mm
 $= 177.5116 - 0.00234 = 17.74882$ cm
 $\quad\quad\quad 0.00109$ " correction to $20^{\circ}.0$ C
Mean diameter of ball $B = 17.74773$ " at $20^{\circ}.0$
 $\therefore$ " radius " " " = 8.87386 " " "

This agrees with the value found by the gravimetric method to within 0.6μ , which is as close an agreement as can be expected. The extreme range of the settings is from -0.091 to $+0.026 = 0.117$ mm. This is not large enough to affect our results appreciably.

We regard the results by the gravimetric method as much more reliable than the direct measurements, and hence take the latter only as a check on the others.

It is not possible to study the figuring of the shell with the same thoroughness as we have the balls, but by bringing the ball into contact with the top of the inner surface of the shell and then lowering it until it touches the bottom of the shell we can measure the difference between the polar diameter of the ball and the vertical diameter of the shell. By this method we find—

Vertical diameter of shell = 25.326 cm
Mean " " " = 25.343 "
Difference = 0.017 cm

We shall see later (page 476) that this error in figuring will not affect the final result.

20. CYLINDRICAL CONDENSERS—GRAVIMETRIC METHOD.

The electrostatic capacity of a condenser, consisting of two coaxial circular cylinders of radii R and r and length l , taking the dielectric constant of the medium as unity, is

$$C = \frac{l + \delta l}{2 \log_e \frac{R}{r}} \quad (1)$$

where δl is the correction to the length due to the end effect. If

end cylinders are used as guard cylinders the value of δl can be calculated, and by varying the gap between the middle cylinder and the guard cylinders the calculated value of δl can be verified experimentally. This we have done. If there are no guard cylinders, δl can be eliminated by varying the length l , *provided care is taken to make the end conditions identical for the two cases*, so that δl will have the same value in each case. This is a very important matter, and involves many precautions.

In order to calculate the capacity C with high accuracy it is of course necessary, in addition to ascertaining or eliminating δl , to determine l , R and r with corresponding accuracy. Since

$$C = \frac{l}{2 \log_e \frac{R}{r}} = \frac{lr}{2(R-r)} \text{ approximately}$$

it is evidently necessary to get the difference of the radii $R-r$ with the same precision that is required for r . This makes it undesirable to have this difference very small. If it is 1 cm, this difference must be known to 0.5 micron to give C to within 1 in 20,000. As in the case of the spherical condenser, the radii were determined by two independent methods, (1) by gravimetric observations and (2) by direct measurements.

It ought to be stated that at the time these cylindrical condensers were built the Bureau instrument shop possessed no grinding machine, and the cylinders were not as accurately turned as could be wished. We had not at that time set so narrow a limit of tolerance as we came to later, and hence what was satisfactory then was not entirely satisfactory a year afterwards. We have, however, taken so large a number of measurements on the cylinders that we believe the mean values obtained are thoroughly trustworthy.

The cylinders were ground on their ends in pairs, so that when placed one within the other the two ends of the space between the cylinders could be closed water-tight by a pair of slightly greased, flat, ground-glass plates. By a procedure exactly analogous to that employed in the case of the shell the volume of the outer cylinder and the volume of the interspace between the two cylinders were determined. In every case the cylinders were closed at the top with a flat, ground-glass plate fitting water-tight and containing two holes

to facilitate the filling. The filling was always such that the meniscus rising from these holes was estimated to be just sufficient to fill the holes level full. During the weighing metal caps were placed over the holes in order to reduce the evaporation. As the space to be filled was closed with glass at both top and bottom, the entrapped air bubbles could be readily seen and (with patience) removed.

By means of an end standard comparator the length of each cylinder along four generating lines was measured by the division of weights and measures of this Bureau. Though the ends of the cylinders were ground in pairs with the cylinders approximately coaxial, the inner cylinder of each pair was found to be about 7μ longer than the corresponding outer one. This is not sufficient to cause trouble from leakage, but needs to be considered in the determination of the radii. Since the capacity of concentric cylinders depends upon the *ratio* of their radii, it will be unaffected by any error in the absolute density of water, or in the thermometer, provided these errors are the same in the determination of the volume of the interspace between the cylinders as in the determination of the volume of the outer cylinder. The dimensions found were as follows:

Volume, outer cylinder No. 2 at 20°0 C

$$= 3297.033 \text{ cc; length} = 19.99875 \text{ cm}$$

Volume, interspace, cylinders No. 2 at 20°0 C

$$\left. \begin{array}{l} = 836.897 \text{ cc, weight 2} \\ = 836.822 \text{ " " 1} \\ \hline 836.872 \text{ " weighted mean} \end{array} \right\}; \text{ length} = 19.99946 \text{ cm}$$

Volume, outer cylinder No. 3 at 20°0 C

$$\left. \begin{array}{l} = 3293.167 \text{ cc} \\ = 3293.114 \text{ " } \\ \hline 3293.140 \text{ " } \end{array} \right\}; \text{ length} = 20.00718 \text{ cm}$$

Volume interspace, cylinders No. 3 at 20°0 C

$$\left. \begin{array}{l} = 832.110 \text{ cc} \\ = 832.083 \text{ " } \\ \hline 832.096 \text{ " } \end{array} \right\}; \text{ length} = 20.00768 \text{ cm.}$$

From these we find:

Mean radius outer cylinder No. 2 at 20° C = 7.24411 cm
 " " inner " " " " = 6.25760 "
 " " outer " No. 3 " " = 7.23831 "
 " " inner " " " " = 6.25740 "
 Capacity per unit of length of No. 2 = 3.41549 cm
 " " " " " No. 3 = 3.43350 cm.

The charge measured being that on the *inner* cylinder, the effective capacities of the cylinders are the capacities per unit length multiplied by the lengths of the *inner* cylinders. Hence we have for the capacities from formula (1), not including any corrections for guard-ring gap or end or edge effects, the following values:

Capacity of cylinders No. 2 = 68.3080 cm at 20° C
 " " " No. 3 = 68.6965 " " "

21. CYLINDRICAL CONDENSERS—DIRECT MEASUREMENTS.

As in the case of the spheres, the direct measurements were taken partly to obtain a check on the more accurate gravimetric determinations and partly to determine the irregularities in the cylinders. The diameters of the outer cylinders were measured at angular intervals of 22° 5' at eight or ten sections of the cylinders. The results are as follows:

TABLE III.

Measurements of Diameters of Outer Cylinder No. 2.

Angular position	Difference; Diameter—Standard. 1 unit = 0.4617 μ									Temperature	Correction to 20° C	Difference at 20° C
	0°	22° 5'	45°	67° 5'	90°	112° 5'	135°	157° 5'	Mean			
1. At top	82.7	85.8	104.0	103.8	100.4	90.4	81.1	81.8	91.2	25° 6'	-5.7 μ	36.4 μ
2. 2.5 cm from top ..	222.9	232.4	222.6	232.1	235.4	227.9	213.0	206.4	224.1	27° 8'	-7.9 μ	95.6 μ
3. 5.0 " " "	218.4	223.2	223.2	224.8	230.2	219.0	214.4	213.3	220.8	25° 7' 5"	-5.8 μ	96.2 μ
4. 7.5 " " "	89.3	111.3	97.8	93.5	84.0	83.2	75.0	76.0	88.8	27° 8'	-7.9 μ	33.1 μ
5. 10.0 " " "	60.8	68.6	68.8	55.6	49.8	44.8	41.8	49.2	54.9	26° 2'	-6.3 μ	19.0 μ
6. 10 cm from bottom	62.0	68.2	61.6	48.0	45.0	43.5	44.0	49.2	52.7	27° 25'	-7.4 μ	16.9 μ
7. 7.5 cm from bottom	51.6	54.5	41.9	24.4	14.7	19.6	32.8	34.0	34.2	27° 7'	-7.8 μ	8.0 μ
8. 5.0 cm from bottom	66.1	69.6	51.2	41.6	27.3	16.6	21.6	47.8	42.7	28° 4'	-8.5 μ	11.2 μ
9. 2.5 cm from bottom	157.2	168.6	146.6	120.2	103.2	106.0	113.6	149.8	133.2	27° 6'	-7.7 μ	53.8 μ
10. At bottom	267.8	258.8	223.8	204.4	186.2	191.1	208.1	227.9	221.0	29° 25'	-9.4 μ	92.6 μ

Weighting the top, bottom, and two middle values half as heavily as the others we find for the mean difference 47.5μ

Standard at $20^{\circ}0$	= 14.48324 cm
Diameter—Standard	= 0.00475 "
Mean diameter No. 2	= 14.48799 "
" radius "	= 7.24400 "

This differs from the value found gravimetrically by 1.1μ . Considering that the diameter varies through a range of 117μ , this agreement is all that could be expected.

TABLE IV.

Measurements of Diameters of Outer Cylinder No. 3.

Angular position	Difference; Diameter—Standard. 1 unit=0.4617 μ									Temperature	Correction to $20^{\circ}0$ C	Difference at $20^{\circ}0$ C
	0°	$22^{\circ}5$	45°	$67^{\circ}5$	90°	$112^{\circ}5$	135°	$157^{\circ}5$	Mean			
1. At top	+103.0	+111.2	+121.4	+123.4	+125.8	+115.8	+108.3	+107.2	+114.5	27 $^{\circ}$ 85	-8.0 μ	+ 44.9 μ
2. 2.5 cm from top	- 39.0	- 35.2	- 27.1	- 12.8	- 9.0	- 6.6	- 17.5	- 23.4	- 21.4	24 $^{\circ}$ 90	-5.0 μ	- 14.9 μ
3. 5.0 cm from top	-131.8	-120.0	-106.0	-100.6	-105.9	-117.5	-136.2	-144.2	-120.3	28 $^{\circ}$ 23	-8.4 μ	- 63.9 μ
4. 10 cm from top	-177.6	-201.6	-193.4	-177.9	-161.0	-152.8	-150.8	-162.0	-172.1	29 $^{\circ}$ 22	-9.4 μ	- 88.9 μ
5. 10 cm from bottom ..	-165.0	-180.4	-183.0	-158.0	-149.3	-133.9	-133.5	-145.1	-156.0	25 $^{\circ}$ 98	-5.9 μ	- 77.9 μ
6. 5.0 cm from bottom ..	-224.9	-246.3	-242.8	-233.8	-225.0	-211.6	-196.5	-206.4	-223.4	26 $^{\circ}$ 92	-6.3 μ	-109.4 μ
7. 2.5 cm from bottom ..	-219.6	-228.7	-241.0	-245.5	-221.8	-208.4	-196.6	-204.1	-220.7	25 $^{\circ}$ 91	-5.2 μ	-107.1 μ
8. At bottom	- 10.8	- 8.6	- 4.8	- 5.9	- 3.8	+ 6.4	+ 6.8	+ 4.8	- 2.0	26 $^{\circ}$ 94	-6.5 μ	- 7.4 μ

Weighting the end sections 1, the middle sections 3 each, the others 2, we find for the mean difference -65.8μ .

Standard at $20^{\circ}0$	= 14.48324 cm
Diameter—Standard	= -0.00658 "
Mean diameter No. 3	= 14.47666 cm at $20^{\circ}0$ C
" radius "	= 7.23833 " " "

This differs from the value found gravimetrically by only 0.2μ .

The diameter of this cylinder varies over a range of 172μ .

Having found that the sections of the inner cylinders are very nearly circular, three diameters of each of 20 sections were measured. The screw micrometer employed was graduated to 0.01 mm.

TABLE V.
Measurements of Diameters of Inner Cylinder No. 2.

Series 1.—Micrometer reading in mm					Series 2.—Micrometer reading in mm				
Diameter at—	0°	60°	120°	Mean	0°	60°	120°	Mean	Weight
Distance from top—									
0.5 cm	0.154	0.162	0.159	0.1583	0.158	0.155	0.154	0.1557	1
1.5 "	.139	.155	.144	.1460					
2.5 "	.140	.151	.140	.1437	.148	.140	.148	.1453	3
3.5 "	.140	.152	.141	.1443					
4.5 "	.141	.150	.141	.1440					
5.5 "	.139	.150	.139	.1427	.145	.138	.143	.1420	3
6.5 "	.140	.148	.138	.1420					
7.5 "	.141	.146	.134	.1403					
8.5 "	.144	.149	.140	.1443	.145	.139	.144	.1427	3
9.5 "	.148	.150	.149	.1490					
10.5 "	.142	.145	.146	.1443					
11.5 "	.144	.144	.144	.1440	.144	.132	.139	.1387	3
12.5 "	.140	.143	.140	.1410					
13.5 "	.139	.143	.140	.1407					
14.5 "	.138	.142	.140	.1400	.144	.135	.139	.1393	3
15.5 "	.140	.144	.147	.1437					
16.5 "	.140	.140	.139	.1397					
17.5 "	.130	.137	.137	.1347	.138	.130	.130	.1327	3
18.5 "	.137	.140	.140	.1390					
19.5 "	.169	.166	.168	.1677	.165	.159	.164	.1627	1
Mean				0.14447				0.14202	

RESULTS, SERIES 1.

Mean reading on diameter	+0.1445 mm
" " " end standard	—0.0061 "
" diameter—standard	+0.1506 "
Correction from 23°7 to 20°0 C	—0.0032 "
(Diameter—Standard) at 20°0	+0.1474 "

SERIES 2.

Mean reading on diameter	+0.1420 mm
" " " standard	—0.0084 "
" diameter—standard	+0.1504 "
Correction from 24°2 to 20°0 C	—0.0037 "
(Diameter—Standard) at 20°0	+0.1467 "

A third set gave (Diameter—Standard) at 20°0 = +0.1503 mm.

The mean of all three is (Diameter—Standard) at 20°0 = 0.1481 mm.

End-Standard = 124.9962 mm at 20°0

Diameter—Standard = 0.1481 “ “ “

Mean Diameter No. 2 = 125.1443 “ “ “

“ Radius No. 2 = 6.25722 cm “ “

This is 3.8 μ smaller than that found by the gravimetric method.

TABLE VI.

Measurements of Diameters of Inner Cylinder No. 3.

Diameter at—	Series 1 Temp. 24°0 Micrometer reading in mm				Series 2 Temp. 24°6 Micrometer reading in mm			
	0°	60°	120°	Mean	0°	60°	120°	Mean
Distance from top:								
0.5 cm	0.156	0.156	0.152	0.1547	0.159	0.160	0.155	0.1580
1.5 “	.138	.147	.133	.1393	.139	.138	.141	.1393
2.5 “	.133	.144	.130	.1357	.135	.137	.140	.1373
3.5 “	.139	.146	.138	.1410	.140	.141	.140	.1403
4.5 “	.137	.149	.141	.1423	.143	.145	.148	.1453
5.5 “	.134	.149	.140	.1410	.141	.146	.147	.1447
6.5 “	.137	.145	.140	.1407	.143	.143	.146	.1440
7.5 “	.138	.145	.140	.1410	.141	.141	.146	.1427
8.5 “	.136	.146	.141	.1410	.142	.141	.148	.1437
9.5 “	.139	.150	.144	.1443	.147	.145	.150	.1473
10.5 “	.141	.157	.149	.1490	.149	.149	.155	.1510
11.5 “	.145	.155	.151	.1503	.151	.150	.159	.1533
12.5 “	.140	.155	.150	.1483	.149	.149	.159	.1523
13.5 “	.140	.151	.149	.1467	.148	.146	.158	.1507
14.5 “	.139	.152	.147	.1460	.148	.142	.154	.1480
15.5 “	.140	.150	.144	.1447	.145	.144	.150	.1463
16.5 “	.139	.150	.148	.1457	.148	.148	.153	.1497
17.5 “	.140	.150	.147	.1457	.142	.140	.151	.1443
18.5 “	.140	.151	.148	.1463	.141	.143	.150	.1447
19.5 “	.150	.161	.157	.1560	.150	.150	.159	.1530
				Mean 0.1450				Mean 0.1468
				Correction to 20° —.0035				Correction to 20° .0040
				Mean reading at 20° 0.1415				Mean reading at 20° 0.1428
Mean of both series, 0.1422 mm.								

Micrometer reading on end-standard at 20°0 = -0.0077 mm

“ “ “ cylinder “ “ = +0.1422 “

Diameter—Standard = +0.1499 “

“ = 124.9962 “

Diameter of inner cylinder No. 3 = 125.1461 “

Radius “ = 6.25730 cm

This is 1.0μ smaller than the value found by the gravimetric method.

In the same manner several diameters at the upper end of the lowest section (No. 1) and at the lower end of the upper guard section (No. 4) were measured. The results of these measurements, as well as of those at the extremities of the other sections, are given in the following table:

TABLE VII.
Mean Radius at Ends of Sections.

Section	1	2	3	4
Radius at top, outer cylinder	7.24192	7.24344	7.24386	
“ “ bottom, “ “		7.24625	7.24125	7.23774
Radius at top, inner cylinder	6.25758	6.25785	6.25782	
“ “ bottom, “ “		6.25832	6.25774	6.25675

From these measurements it is evident that if the ends of the inner cylinders were truly circular and if the sections were always placed truly coaxially and with top end uppermost the offset at the junction of any two sections can in no case exceed 11.0μ . Under the same conditions the offset in the case of the outer cylinders may amount to 61.2μ . The effect of these offsets in the combinations actually used will be discussed below in Section V.

22. PLATE CONDENSER—DIMENSIONS OF THE PLATES.

The electrostatic capacity of a guard-ring condenser consisting of parallel circular plates is

$$C = \frac{A}{4\pi d} = \frac{(r + \delta r)^2}{4d} \quad (2)$$

if the dielectric constant of the medium between the plates is unity. Here r is the radius of the collector plate; δr is a term due to the effect of the gap between the collector plate and the guard ring, depending upon the width of this gap and the values of r and of d ; d is the distance between the plates. As already described, d is varied over wide limits and is determined at the time the condenser is used. On the other hand, the value of r is a constant and can be measured once for all. Similarly the width of the gap between the collector plate and the guard ring, depending only upon the radius

of the plate and the internal radius of the guard ring, is a constant of the instrument.

The diameters of the two plates were measured with an end standard comparator. Nine equally spaced diameters were measured for each plate; the diameters were measured in both direct and reversed directions and two settings were made in each position. The results obtained are given in Table VIII.

TABLE VIII.

Diameters of the Plates at 20°0 C.

Diameter	Plate A	Plate B
1	20.0080 cm	20.0354 cm
2	20.0086 "	20.0359 "
3	20.0094 "	20.0363 "
4	20.0094 "	20.0363 "
5	20.0092 "	20.0357 "
6	20.0086 "	20.0350 "
7	20.0082 "	20.0346 "
8	20.0076 "	20.0345 "
9	20.0076 "	20.0347 "
Mean	20.00851 "	20.03538 "

From these measurements we see that the plates are practically ellipses of semi-axes 10.00470, 10.00380 for plate A and 10.01824, 10.01725 for plate B. For plate A, $2\sqrt{ab} = 20.00850$ cm, for plate B, $2\sqrt{ab} = 20.03538$ cm. Hence the areas of the plates are practically the same as the areas of the circles with diameters equal to the mean diameters of the plates, so we may take these mean diameters as the values of r in the expression for the capacity of the condenser.

Plate B, having been adjusted as already described, the entire top of the condenser was removed, and turned upside down, and the width of the gap between the collector plate and the guard ring (in the plane of the face of the plate) was measured at six points around the circumference of the plate. The measurements were made by means of a micrometer microscope, one division of

the scale being equal to 1.708 microns. Six measurements were made at each point. The values found are given in Table IX.

TABLE IX.

Width of Guard Ring Gap, Plate B. One Unit = 1.708μ , $20^{\circ}2$ C.

Position	Width of gap						Mean
1	121.0	120.5	122.3	118.5	121.3	120.6	120.7
2	103.4	101.2	105.0	99.2	102.9	101.4	102.2
3	100.5	102.5	104.0	105.0	102.1	102.0	102.7
4	109.1	105.2	108.0	108.8	105.5	106.7	107.2
5	88.9	94.3	90.8	97.1	89.0	92.3	92.4
6	102.9	(110.2)	102.1	107.0	101.8	102.1	103.2
1	124.7	124.0	119.1	119.1	117.4	120.2	120.9

Hence the mean width of the gap is 104.7 divisions = 178.8 microns. It is evident that the opening in the guard ring is considerably more elliptical than the plates. This width of gap corresponds to a mean radius of 10.03557 cm for the inner boundary of the guard ring.

IV. CORRECTIONS AND SOURCES OF ERROR—SPHERICAL CONDENSERS.

23. EFFECT OF SHELL NOT BEING PERFECT SPHERE.

The capacity of the spherical condenser is calculated on the hypothesis that both ball and shell are truly spherical and accurately concentric. As these conditions are never exactly fulfilled, it is necessary to consider the possible error thus introduced. We shall begin with a study of the shell. We have determined the radius, R , of a sphere having the same volume as the shell, and by determining the distance through which we raise the ball in order to bring it from contact with the bottom of the shell to contact with the top of it we have measured the difference between the vertical diameters of the balls and of the shell. This was done for both balls, and the mean of these measurements showed that the vertical diameter of the shell is less than its mean diameter R by

0.017 cm. If we assume this error is due to the fact that the flanges where the hemispheres come together have been ground down too much, so that each half is somewhat less than a complete hemisphere, we shall not be far from the truth and shall obtain a correction that is probably greater than the true one. For a perfect spherical condenser the capacity is $\frac{Rr}{R-r} = \frac{Rr}{d}$ where R and r are the radii of shell and ball respectively and d is the difference in their radii. Now any small change in R or r affects Rr very slightly, but produces a relatively large change in d ; hence in the study of the effect of *small* errors, we may regard Rr as known and limit ourselves to an investigation of the factor $1/d$. We also know that

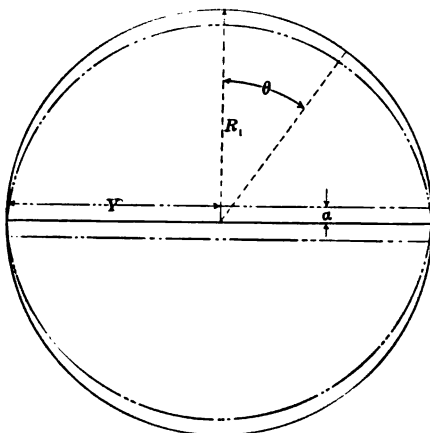


Fig. 10.

if the irregularities are slight the lines of force can depart but slightly from right lines normal to the surface. Hence from analogy with the parallel plate condenser we must give each element of $1/d$ a weight proportional to the area of the surface to which it applies. Now in the case in which d is the same for all points on a circular cone having its vertex at the center of the sphere, we may take $2 \pi R^2 \sin \theta \cdot d\theta$ for our element of area so that for the d corresponding to the angle θ , the weight of $1/d$ will be

$$\frac{2 \pi R^2 \sin \theta \, d\theta}{4 \pi R^2} = \frac{1}{2} \sin \theta \, d\theta$$

Hence the total capacity will be

$$C = Rr \int_0^{\frac{\pi}{2}} \frac{\sin \theta}{d_1} d\theta \quad (3)$$

If d_1 is independent of θ this becomes $C = \frac{Rr}{d}$ as before.

Let R_1 = true radius of curvature of the shell, assumed constant,

R = radius of a sphere *having the same volume*,

a = the amount each hemisphere has been ground off on its flange beyond the true equatorial plane,

ϵ = the amount by which the vertical radius $R_1 - a$ falls short of the mean radius R ; that is,

$$\epsilon = R - (R_1 - a) = R - R_1 + a$$

Write $\delta = R_1 - R$

Then $\epsilon = a - \delta$ or $a = \delta + \epsilon$

But the volume of the shell is

$$V = 2\pi \int_a^{R_1} (R_1^2 - x^2) dx = \frac{4}{3} \pi R_1^3 - 2\pi \left(R_1^2 a - \frac{a^3}{3} \right) = \frac{4}{3} \pi R^3$$

Substituting $R_1 = R + \delta$ we find, neglecting squares, products and higher powers of ϵ and δ

$$\frac{4}{3} \pi (R^3 + 3R^2\delta) - 2\pi(\delta + \epsilon)R^2 = \frac{4}{3} \pi R^3$$

$$\therefore 4R^2\delta = 2R^2(\delta + \epsilon)$$

$$\therefore \delta = \epsilon$$

$$\text{or, } a = \delta + \epsilon = 2\epsilon$$

The distance Y along the plane of union of the two hemispheres from the center of the ball to the shell is

$$Y = \sqrt{R_1^2 - a^2} = \sqrt{R^2 + 2R\delta + \delta^2 - a^2} = \sqrt{R^2 + 2R\epsilon - 3\epsilon^2}$$

$$= R \left\{ 1 + \frac{2\epsilon}{R} - \frac{3\epsilon^2}{R^2} \right\}^{\frac{1}{2}} = R \left\{ 1 + \frac{\epsilon}{R} \dots \right\}$$

$\therefore Y = R + \epsilon$ approximately.

That is, the horizontal radius is greater than the mean radius R by ϵ , while the vertical radius is less than the mean by the same

amount. Hence, if d is the mean distance between the shell and the ball ($R-r$) the distance between the shell and the ball along a radius of the latter making an angle θ with the vertical is:

$$d_1 = d + \epsilon - a \cos \theta$$

$$\text{or, } d_1 = d + \epsilon (1 - 2 \cos \theta)$$

Substituting this value of d_1 in equation (3), we have for the true capacity of the ball

$$C = Rr \int_0^\pi \frac{\sin \theta d\theta}{d + \epsilon (1 - 2 \cos \theta)} = Rr \int_0^1 \frac{dz}{d + \epsilon - 2\epsilon z}, \text{ where } z = \cos \theta$$

$$= Rr \left[\frac{1}{2\epsilon} \log (d + \epsilon - 2\epsilon z) \right]_1^0 = \frac{Rr}{2\epsilon} \log \frac{d + \epsilon}{d - \epsilon}$$

$$\therefore C = \frac{Rr}{2\epsilon} \left[\frac{\epsilon}{d} - \frac{1}{2} \left(\frac{\epsilon}{d} \right)^2 + \frac{1}{3} \left(\frac{\epsilon}{d} \right)^3 - \frac{1}{4} \left(\frac{\epsilon}{d} \right)^4 + \frac{1}{5} \left(\frac{\epsilon}{d} \right)^5 - \dots \right]$$

$$+ \left[\frac{\epsilon}{d} + \frac{1}{2} \left(\frac{\epsilon}{d} \right)^2 + \frac{1}{3} \left(\frac{\epsilon}{d} \right)^3 + \frac{1}{4} \left(\frac{\epsilon}{d} \right)^4 + \frac{1}{5} \left(\frac{\epsilon}{d} \right)^5 + \dots \right]$$

$$= \frac{Rr}{d} \left[1 + \frac{1}{3} \left(\frac{\epsilon}{d} \right)^2 + \frac{1}{5} \left(\frac{\epsilon}{d} \right)^4 + \dots \right] \quad (4)$$

But $\frac{Rr}{d} = C_0$ = the capacity calculated on the assumption that the figuring is perfect and that the dimensions are those of perfect spheres of the same volume. Writing $\frac{\Delta_1 C}{C_0} = \frac{C - C_0}{C_0}$ = relative increase in capacity produced by the irregularity, we have

$$\frac{\Delta_1 C}{C_0} = \frac{1}{3} \left(\frac{\epsilon}{d} \right)^2 + \frac{1}{5} \left(\frac{\epsilon}{d} \right)^4 + \dots$$

But

$$2\epsilon = 0.017 \text{ cm}$$

$$d = 2.554 \text{ cm for ball A}$$

$$d = 3.798 \text{ " " " B}$$

$$\therefore \frac{\Delta_1 C}{C_0} = 0.0000037 = 3.7 \times 10^{-6} \text{ for ball A}$$

$$= 0.0000017 = 1.7 \times 10^{-6} \text{ " " B.}$$

This increase in the capacity due to the shell not being perfectly spherical is entirely negligible. It has been computed on the

assumption that the center of the ball coincides with the center of figure of the shell. As this adjustment is never exact we shall proceed to investigate the effect of a slight displacement of the center of the ball from the center of figure of the shell.

24. EFFECT OF DISPLACEMENT OF THE BALL FROM THE CENTER.

Suppose the ball is raised by the amount β cm. Then for the upper hemisphere $d_1 = d + \epsilon - (2\epsilon + \beta) \cos \theta$ and for the lower hemisphere it is $d_2 = d + \epsilon - (2\epsilon - \beta) \cos \theta$.

The capacity then is

$$C = \frac{Rr}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\sin \theta d\theta}{d + \epsilon - (2\epsilon + \beta) \cos \theta} + \int_{\frac{\pi}{2}}^{\pi} \frac{\sin \theta d\theta}{d + \epsilon - (2\epsilon - \beta) \cos \theta} \right]$$

Putting $\cos \theta = z$, we have

$$C = \frac{Rr}{2} \left[\int_0^1 \frac{dz}{d + \epsilon - (2\epsilon + \beta)z} + \int_1^0 \frac{dz}{d + \epsilon - (2\epsilon - \beta)z} \right]$$

$$= \frac{Rr}{2} \left[-\frac{1}{2\epsilon + \beta} \log \{d + \epsilon - (2\epsilon + \beta)z\} - \frac{1}{2\epsilon - \beta} \log \{d + \epsilon - (2\epsilon - \beta)z\} \right]$$

$$= \frac{Rr}{2} \left[-\frac{1}{2\epsilon - \beta} \log \frac{d - \epsilon + \beta}{d + \epsilon} - \frac{1}{2\epsilon + \beta} \log \frac{d - \epsilon - \beta}{d + \epsilon} \right]$$

$$= \frac{Rr}{2} \left[\frac{1}{2\epsilon - \beta} \left[\frac{\epsilon - \beta}{d} + \frac{1}{2} \left(\frac{\epsilon - \beta}{d} \right)^2 + \frac{1}{3} \left(\frac{\epsilon - \beta}{d} \right)^3 + \frac{1}{4} \left(\frac{\epsilon - \beta}{d} \right)^4 + \dots \right] \right. \\ \left. + \frac{1}{2\epsilon + \beta} \left[\frac{\epsilon + \beta}{d} + \frac{1}{2} \left(\frac{\epsilon + \beta}{d} \right)^2 + \frac{1}{3} \left(\frac{\epsilon + \beta}{d} \right)^3 + \frac{1}{4} \left(\frac{\epsilon + \beta}{d} \right)^4 + \dots \right] \right]$$

$$= \frac{Rr}{2} \left[\frac{1}{2\epsilon - \beta} \left\{ \frac{2\epsilon - \beta}{d} - \frac{1}{2} \frac{\beta(2\epsilon - \beta)}{d^2} + \frac{1}{3} \frac{\epsilon^2 + (\epsilon - \beta)^2}{d^3} - \dots \right\} \right. \\ \left. + \frac{1}{2\epsilon + \beta} \left\{ \frac{2\epsilon + \beta}{d} + \frac{1}{2} \frac{\beta(2\epsilon + \beta)}{d^2} + \frac{1}{3} \frac{\epsilon^2 + (\epsilon + \beta)^2}{d^3} + \dots \right\} \right]$$

$$\begin{aligned}
&= \frac{Rr}{2} \left[\frac{2}{d} + \frac{1}{3d^3} \left\{ \frac{\epsilon^2 + (\epsilon - \beta)^2}{2\epsilon - \beta} + \frac{\epsilon^2 + (\epsilon + \beta)^2}{2\epsilon + \beta} \right\} + \frac{1}{4d^4} \left\{ \frac{(\epsilon - \beta)^4 - \epsilon^4}{2\epsilon - \beta} \right. \right. \\
&\quad \left. \left. + \frac{(\epsilon + \beta)^4 - \epsilon^4}{2\epsilon + \beta} \right\} + \frac{1}{5d^5} \left\{ \frac{(\epsilon - \beta)^6 + \epsilon^6}{2\epsilon - \beta} + \frac{(\epsilon + \beta)^6 + \epsilon^6}{2\epsilon + \beta} \right\} \dots \right] \\
&= \frac{Rr}{2} \left[\frac{2}{d} + \frac{2}{3d^3} (\epsilon^2 + \beta^2) + \frac{\epsilon\beta^2}{d^4} + \frac{2(\epsilon^4 + 4\epsilon^2\beta^2 + \beta^4)}{5d^5} + \dots \right] \quad (4a)
\end{aligned}$$

But $C_0 = \frac{Rr}{d}$ is the capacity for concentric spheres,

$$\therefore \frac{\mathcal{A}_3 C}{C_0} = \frac{1}{3} \frac{\epsilon^2 + \beta^2}{d^2} + \frac{\epsilon\beta^2}{2d^3} + \frac{\epsilon^4 + 4\epsilon^2\beta^2 + \beta^4}{5d^4} + \dots$$

If C_1 = capacity when the ball is in its central position, that is C_1 is the value of the expression (4a) when $\beta = 0$ (this is the same as expression 4), then

$$\begin{aligned}
C - C_1 &= \frac{Rr}{2} \left[\frac{2\beta^2}{3d^3} + \frac{\epsilon\beta^2}{d^4} + \frac{2\beta^2(4\epsilon^2 + \beta^2)}{5d^5} + \dots \right] \\
\therefore \frac{\mathcal{A}_3 C}{C_1} &= \frac{\beta^2}{d^2} \left[\frac{1}{3} + \frac{1}{2} \frac{\epsilon}{d} + \frac{1}{45} \frac{31\epsilon^2 + 9\beta^2}{d^2} + \dots \right]
\end{aligned}$$

This quantity $\frac{\mathcal{A}_3 C}{C_1}$ can be directly measured and in this way the validity of this method of treating the problem may be checked by experiment.

We know

$$\begin{aligned}
2\epsilon &= 0.017 \text{ cm} \\
d &= 2.554 \text{ cm for ball A} \\
d &= 3.798 \text{ cm for ball B} \\
\therefore \frac{\epsilon}{d} &= 0.00332 \text{ cm for ball A} \\
&= 0.00224 \text{ cm for ball B}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\mathcal{A}_3 C}{C_1} &= \frac{1}{3} \frac{\beta^2}{d^2} \left[1 + 0.00498 + \dots \right] \text{ for ball A} \\
&= \frac{1}{3} \frac{\beta^2}{d^2} \left[1 + 0.00336 + \dots \right] \text{ for ball B}
\end{aligned}$$

Whence we obtain the results given in Table X, which are represented by the curves on the accompanying plate, Fig. 11. The experimental values, which were obtained by displacing the ball

and observing the changes in the capacity, are marked on the same plate and it is evident that these two curves represent the facts within the limits of experimental error. The average difference between the calculated and observed values is only 2 or 3 parts in 100,000.

TABLE X.

Relative Changes in Capacity of Spherical Condenser due to Displacements β of Ball from Center.

Displacement β	Value of $\frac{\Delta_2 C}{C_1}$ for	
	Ball A	Ball B
0.01 cm	0.051×10^{-4}	0.023×10^{-4}
.02 "	0.205 "	0.093 "
.04 "	0.821 "	0.371 "
.06 "	1.849 "	0.835 "
.08 "	3.287 "	1.484 "
.10 "	5.135 "	2.319 "
.12 "	7.396 "	3.340 "
.16 "	13.147 "	5.936 "
.20 "	20.540 "	9.275 "
.24 "	29.582 "	13.356 "
.30 "	46.216 "	20.868 "
.36 "	66.560 "	30.050 "

When set up the balls are probably centered to within 0.1 mm, so that when first adjusted the capacity does not differ from that calculated by more than 1 in 100,000. Owing, however, to the slight warping of the stand, the stretching of the thread supporting the ball, etc., the ball may become excentric with the shell by possibly 0.2 mm; that is, the capacity may be in error by 2 in 100,000 for ball A; but even then for ball B it will be in error by less than 1 in 100,000. Wherever possible we have endeavored to determine this correction and to apply it. During the greater portion of the time a delicate level was kept on the arm carrying the vernier by which the vertical position of the ball was determined, and the leveling screws were adjusted as need be to keep this arm level. This

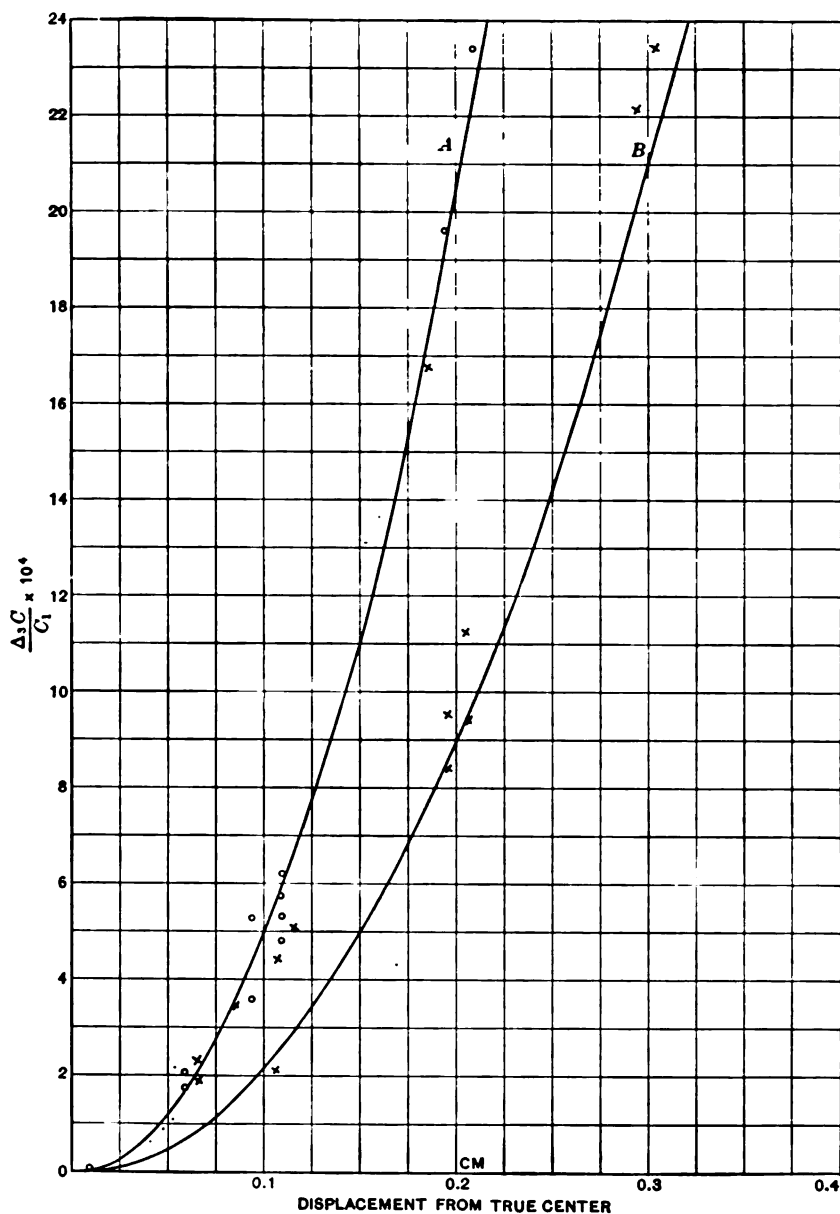


Fig. 11.—Curve Showing the Change in the Capacity of the Spherical Condenser Due to a Displacement of the Inner Sphere.

The upper curve applies to ball A; the lower to ball B.

largely eliminates displacement from the center due to the warping of the table and floor, but does not eliminate that due to a possible warping of the base of the instrument itself; this would tend to produce a very slight relative displacement of the shell and of the pillar from which the ball is suspended.

25. EFFECT OF HOLES IN SHELL, BUSHINGS, CORD, ETC.

There remains to be considered the effect of the holes in the shell, and of the ebonite bushings or tubes running through them, the small eye by which the silk cord is attached to the ball, and the effect of the silk cord.

The effect of the silk cord is surely less than that obtained by assuming that it is a rod passing radially from the ball to the shell (the cord does not touch the shell) without distorting the lines of force. Assuming its specific inductive capacity to be 2 and its diameter to be $\frac{1}{2}$ mm, which is greater than the truth, we have for the relative increase in capacity

$$\frac{\delta C}{C} = \frac{\text{area of section of cord}}{\text{area of ball}} = \frac{\pi(0.025)^2}{4\pi(10.1)^2} = \frac{1}{653,000}$$

= 1.5 parts in 1,000,000.

The leakage along the cord was found to be inappreciable.

The eye in the top of the ball is 2 mm in diameter and projects 2 mm above the surface of the ball; applying the method employed on page 474 we have for the capacity

$$C = \frac{Rr_1}{4\pi r_1^2} \left(\frac{4\pi r_1^2 - A}{d} + \frac{A}{d'} \right)$$

where r_1 is the *true radius* of the ball; d is the true radial distance from the surface of the ball to the shell; d' is the distance from the top of the eye to the shell; A is the sectional area of the eye.

$$\therefore d' = d - 0.2 \text{ cm}$$

$$A = 0.01\pi = 0.0314 \text{ cm}^2$$

In calculating the capacity we used the quantity r , which is the mean radius of the ball; that is, it is the radius of the perfect sphere having a volume equal to that of the ball plus the volume of the

eye, since the eye was in place when the volume was determined. Hence

$$\frac{4}{3}\pi r^3 = \frac{4\pi r_1^3}{3} + 0.2A$$

Putting $r_1 = r - \delta$ we find

$$\begin{aligned}\delta &= \frac{0.2A}{4\pi r^2} \\ \therefore C &= \frac{Rr_1}{d} \left\{ 1 - \frac{A}{4\pi r_1^3} + \frac{Ad}{4\pi r_1^3 d'} \right\} = \frac{Rr_1}{d} \left\{ 1 + \frac{\delta}{0.2} \frac{d-d'}{d'} \right\} \\ &= \frac{Rr_1}{d} \left\{ 1 + \frac{\delta}{d'} \right\}\end{aligned}\quad (5)$$

The capacity we have assumed is

$$C_0 = \frac{Rr}{d_0}$$

where $d_0 = R - r = d - \delta$; substituting in (5) for r_1 , d' and d their values in terms of r and d_0 , we find for the true capacity, since $r_1 = r - \delta$, $d = d_0 + \delta$ and $d' = d_0 - 0.2 + \delta$,

$$\begin{aligned}C &= \frac{Rr \left(1 - \frac{\delta}{r} \right)}{d_0 \left(1 + \frac{\delta}{d_0} \right)} \left\{ 1 + \frac{\delta}{d_0 \left(1 - \frac{0.2 - \delta}{d_0} \right)} \right\} \\ &= \frac{Rr}{d_0} \left\{ 1 - \frac{\delta}{r} + \frac{\delta(0.2 - \delta)}{d_0^2} \right\} \\ &= C_0 \left(1 - \frac{\delta}{r} + \frac{0.2\delta}{d_0^2} \right) \text{ approximately.}\end{aligned}$$

$$\begin{aligned}\text{But } \delta &= \frac{0.2A}{4\pi r^2} \therefore \frac{\delta}{r} = \frac{0.2A}{4\pi r^3} = \frac{0.00628}{4\pi(10.12)^3} \text{ for ball A.} \\ &= 0.0000048 \\ \frac{0.2\delta}{d_0^2} &= 0.0000015\end{aligned}$$

Hence the capacity we have assumed is too large by only 3.3 parts in 1,000,000. For ball B the correction is still less. Hence the correction for the eye is inappreciable.

The corrections for the ebonite were determined by measuring the change produced in the electromagnetic capacity when the ebonite was removed; and the effect of the side hole was determined by measuring the change in the electromagnetic capacity produced by plugging this hole with a brass rod. From this the effect of the top hole was calculated on the assumption that the relative effects are directly proportional to the areas of the two holes. This assumption was not tested experimentally, but is surely not far wrong.

Following is a summary of the various small corrections due to the holes and ebonite:

	Ball A	Ball B
Correction to capacity $\left(\frac{\Delta C}{C}\right)$ due to side hole	$- 11 \times 10^{-6}$	$- 8 \times 10^{-6}$
" " " " " " top "	$- 28 \times 10^{-6}$	$- 20 \times 10^{-6}$
" " " " " " side ebonite	$+ 8 \times 10^{-6}$	$+ 5 \times 10^{-6}$
" " " " " " top "	$+ 63 \times 10^{-6}$	$+ 32 \times 10^{-6}$
Total	$+ 32 \times 10^{-6}$	$+ 9 \times 10^{-6}$

Hence the total correction to be added to the calculated capacity is

	Ball A	Ball B
Correction $\left(\frac{\Delta C}{C}\right)$ for density of water	$+ 15 \times 10^{-6}$	$+ 15 \times 10^{-6}$
" " " error in shell	$+ 4 \times 10^{-6}$	$+ 2 \times 10^{-6}$
" " " holes and ebonite	$+ 32 \times 10^{-6}$	$+ 9 \times 10^{-6}$
" " " eye for suspending ball	$- 3 \times 10^{-6}$	$- 2 \times 10^{-6}$
Total	$+ 48 \times 10^{-6}$	$+ 24 \times 10^{-6}$
Correction to ∇	$+ 24 \times 10^{-6}$	$+ 12 \times 10^{-6}$
Correction to ∇ , in centimeters per second	$= 0.00007 \times 10^{10}$	$= 0.00004 \times 10^{10}$

V. CORRECTIONS AND SOURCES OF ERROR—CYLINDRICAL CONDENSER.

The corrections and sources of error that must be considered in the case of the cylindrical condensers are as follows:

1. Error due to a lateral displacement of the axis of the inner cylinder.
2. Error due to an inclination of the axis of the inner cylinder with respect to the outer one.
3. Errors due to the figuring of the cylinders.

4. Correction for the presence of offsets where two sections come together.

These corrections and errors apply to the cylinders however they may be used. If they are used without guard cylinders we must also determine the two following corrections:

5. Correction due to the irregular distribution of the charge upon the ends of the inner cylinder and to the manner in which the magnitude of this charge is affected by slight variations in the dimensions of the ends of the different sections, and in particular the effect of varying the length of the condenser upon this charge.

6. Correction for possible variation in the capacity between the caps closing the ends of the cylinders.

If the cylinders are used with guard cylinders we must determine one other correction:

7. Correction for the gap between the guard cylinder and the middle section of the condenser.

26. CORRECTION FOR LATERAL DISPLACEMENT OF THE AXIS OF THE INNER CYLINDER.

We shall proceed as in the case of the spheres, page 477. The distance between the surfaces at any point (Fig. 12) is

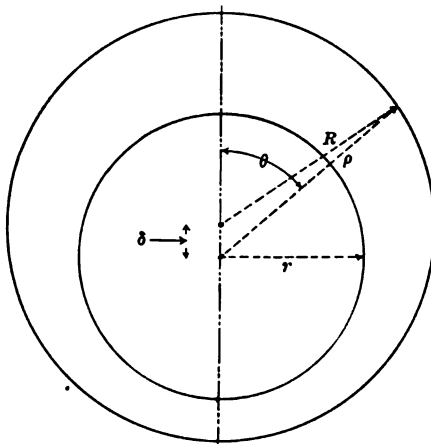


Fig. 12.

$$D_1 = \rho - r$$

$$\rho^2 + \delta^2 - 2\rho\delta \cos\theta = R^2$$

$$\therefore \rho \left\{ 1 - \frac{\delta \cos \theta}{\rho} + \frac{\delta^2}{2\rho^2} - \frac{\delta^2 \cos^2 \theta}{2\rho^2} \dots \right\} = R, \delta \text{ being very small.}$$

$$\begin{aligned} \therefore \rho &= R + \delta \cos \theta - \frac{\delta^2 \sin^2 \theta}{2\rho} \\ &= R + \delta \cos \theta - \frac{\delta^2 \sin^2 \theta}{2R}, \text{ approximately.} \end{aligned}$$

$$\therefore D_1 = R - r + \delta \cos \theta - \frac{\delta^2 \sin^2 \theta}{2R}$$

The distance for coaxial cylinders is

$$D_0 = R - r$$

Hence the relative increase in the capacity of the system is

$$\frac{A_1 C}{C_0} = \int \frac{D_0 - D_1}{D_0} \cdot \frac{d\theta}{2\pi} = \frac{\delta^2}{4R(R-r)} \quad (6)$$

Hence the effect varies as the square of the displacement, and is negligible when δ is very small.

In building up the condensers the distance between the inner and the outer cylinders of each section was tested at their tops, at four points around the circumference, by means of a strip of brass cut wedge shaped with a slope of 1 mm in 10 cm, and the cylinders were slightly tipped by a suitable tightening of the screws fastening them to the lower section until they were as nearly coaxial as possible. The relative displacement of the axes at the top of any section was surely never over 0.025 mm. This determines the maximum error due to this cause.

$$\text{Taking } R = 7.25$$

$$r = 6.25$$

$$\delta = 0.0025$$

we find from equation (6), $\frac{A_1 C}{C_0} = 0.22 \times 10^{-6}$, which is negligible.

27. CORRECTION FOR INCLINATION OF THE AXIS OF THE INNER CYLINDER.

The error due to want of parallelism of the axes of the cylinders will be practically equal to that due to the same inclination of the two plates of a plane condenser, having the same length and distance apart, and of breadth πr , r being the radius of the inner cyl-

inder, end effects being neglected in both cases. We shall therefore calculate the increase in the capacity of a plane condenser when the plates are inclined at an angle θ . The lines of force passing from plate A at potential V to B at potential zero (neglecting effect of edges) will be arcs of circles. The electric force f will be uniform

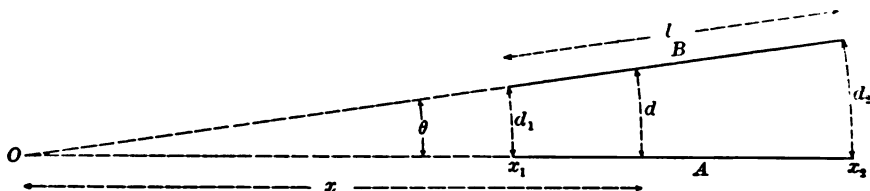


Fig. 13.

along any such line and equal to $V/\theta x$. If σ is the surface density, Q the total quantity of electricity on the plate of length l and breadth b , we have

$$f = \frac{V}{\theta x} \quad \sigma = \frac{V}{4\pi\theta x} \quad Q = b \int_{x_1}^{x_2} \sigma dx$$

$$\therefore Q = \frac{bV}{4\pi\theta} \int_{x_1}^{x_2} \frac{dx}{x} = \frac{bV}{4\pi\theta} \log \frac{x_2}{x_1}$$

The capacity then is

$$C_1 = \frac{Q}{V} = \frac{b}{4\pi\theta} \log \frac{x_2}{x_1} = \frac{bl}{4\pi\delta} \log \frac{x_2}{x_1}$$

since

$$\theta = \frac{d_2}{x_2} = \frac{d_1}{x_1} = \frac{d_2 - d_1}{x_2 - x_1} = \frac{\delta}{l}$$

Also, since

$$\frac{x_2}{x_1} = \frac{d_2}{d_1}$$

$$\begin{aligned} C_1 &= \frac{bl}{4\pi\delta} \log \left(1 + \frac{\delta}{d_1} \right) \\ &= \frac{bl}{4\pi d_1} \left(1 - \frac{1}{2} \frac{\delta}{d_1} + \frac{1}{3} \frac{\delta^2}{d_1^2} - \dots \right) \end{aligned}$$

If d_1 is the mean distance between the plates, the point x_1 will

correspond to the middle point of the plate, and the expression just obtained will be the capacity of that half of the plate for which the distance between the planes is greater than d_1 . For the other half of the plate, $\log \frac{x_2}{x_1}$ becomes $\log \frac{d_1}{d_1 - \delta} = -\log \left(1 - \frac{\delta}{d_1} \right)$ and the expression for the capacity of this half is

$$C_2 = \frac{bl}{4\pi d_1} \left(1 + \frac{1}{2} \frac{\delta}{d_1} + \frac{1}{3} \frac{\delta^2}{d_1^2} + \dots \right)$$

The sum of these, which is the capacity of the entire condenser, is

$$C = C_1 + C_2 = \frac{2bl}{4\pi d_1} \left(1 + \frac{1}{3} \frac{\delta^2}{d_1^2} + \frac{1}{5} \frac{\delta^4}{d_1^4} + \dots \right) \quad (7)$$

But, if the plates were parallel and at the same mean distance, their capacity would be

$$C_0 = \frac{2bl}{4\pi d_1} \\ \therefore \frac{A_2 C}{C_0} = \frac{C - C_0}{C_0} = \frac{1}{3} \frac{\delta^2}{d_1^2} + \frac{1}{5} \frac{\delta^4}{d_1^4} + \dots \quad (8)$$

Hence the increase in the capacity of the coaxial cylinders due to a slight tipping of the inner with respect to the outer can not be greater than the quantity given in (8), which is obviously negligible when δ/d_1 is very small. If δ/d_1 is 0.01, $\frac{A_2 C}{C_0}$ amounts to about 1 part in 30,000.

By the method of adjustment described on page 485 it is evident that the greatest possible tipping will occur when the bottom of a section is displaced 0.025 mm in one direction and the top an equal amount in the other direction. This would correspond to $\delta = 0.0025$ cm. Substituting this in the expression for $\frac{A_2 C}{C_0}$ for these cylinders, we find that the maximum error due to the relative tipping of the axis of the cylinders can not exceed about 2 in 1,000,000. Usually the tipping was much less than that here assumed.

28. CORRECTION FOR ERRORS IN FIGURING.

These errors are due (1) to conicality of the cylinders, (2) to ellipticity, (3) to other irregularities. The sections of all the cylinders

are quite accurately circular, and the smaller irregularities in the surface are too slight to cause an error as large as 1 part in 100,000. The outer cylinders are, however, slightly variable in section as we pass along their length, so that there is an appreciable conicality, the effect of which needs to be investigated.

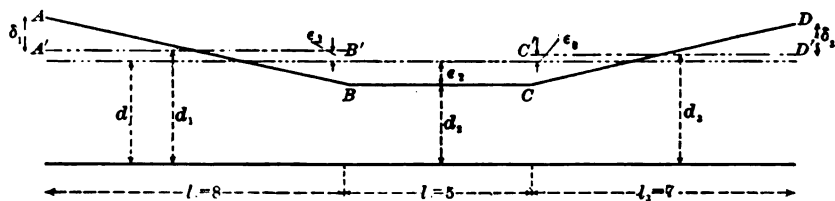


Fig. 14.

If d_1 is the mean distance between the cylinders for the first section, and δ_1 is the maximum variation of the distance from this mean for this conical portion, we can calculate the difference in the capacity between this conical section and one of uniform distance d_1 by the formula of the preceding section. This may be done for each section. Then we can calculate the difference in capacity between the several cylindrical sections of distances d_1 , d_2 , d_3 , etc., and the single cylinder having the same mean distance d . The sum of these differences will be the correction sought. Let us take a parallel plate condenser that is made up of three sections, two of which, AB and CD, are not parallel. $AA' = \delta_1$, $DD' = \delta_2$. The parallel plates $A'B'$ and $C'D'$, which are nearly equivalent to AB and CD, are distant $d + \epsilon_1$, and $d + \epsilon_2$; and BC is distant $d - \epsilon_3$ from the lower plate.

The first section of distance $d_1 = d + \epsilon_1$ has a capacity

$$C_1 = \frac{A_1}{4\pi d_1} = \frac{bl_1}{4\pi} \cdot \frac{1}{d + \epsilon_1} = \frac{bl_1}{4\pi d} \left(1 + \frac{\epsilon_1}{d}\right)^{-1}$$

$$\text{or, } C_1 = \frac{bl_1}{4\pi d} \left\{ 1 - \frac{\epsilon_1}{d} + \frac{\epsilon_1^2}{d^2} - \dots \right\}$$

$$\text{Similarly, } C_2 = \frac{bl_2}{4\pi d} \left\{ 1 + \frac{\epsilon_2}{d} + \frac{\epsilon_2^2}{d^2} + \dots \right\} \text{ assuming } \epsilon_2 \text{ is negative.}$$

$$C_3 = \frac{bl_3}{4\pi d} \left\{ 1 - \frac{\epsilon_3}{d} + \frac{\epsilon_3^2}{d^2} - \dots \right\}$$

Adding these equations for the total capacity of the three sections

we have, since the total area A is $bl_1 + bl_2 + bl_3$ and $l_1\epsilon_1 + l_2\epsilon_2 = l_3\epsilon_3$

$$C = \frac{A}{4\pi d} \left\{ 1 + \frac{l_1\epsilon_1^2 + l_2\epsilon_2^2 + l_3\epsilon_3^2}{ld^2} + \dots \right\}$$

$$\therefore \frac{A_3 C}{C_0} = \frac{l_1}{l} \frac{\epsilon_1^2}{d^2} + \frac{l_2}{l} \frac{\epsilon_2^2}{d^2} + \frac{l_3}{l} \frac{\epsilon_3^2}{d^2}$$

Thus, the increase in capacity is the weighted mean of $\left(\frac{\epsilon}{d}\right)^2$ for all the sections, while the difference between the conical sections and the corresponding parallel ones is $\frac{1}{3}$ the weighted mean of $\left(\frac{\delta}{d}\right)^2$

As an example, suppose $l_1 = 8$, $l_2 = 5$, $l_3 = 7$ cm

Take $\epsilon_1 = +0.01$ mm $\delta_1 = 0.054$ mm $d = 1$ cm

$\epsilon_2 = -0.044$ " $\delta_2 = 0$

$\epsilon_3 = +0.02$ " $\delta_3 = 0.064$ mm

These are small differences, but very appreciable, being of the same order of magnitude as those of our cylinders, as shown by Tables III, IV, V, and VI.

$\Sigma \left(\frac{\epsilon}{d}\right)^2$ will here be 0.0000066

$\frac{1}{3} \Sigma \left(\frac{\delta}{d}\right)^2$ " " " 0.0000087

Sum = 0.0000153 = 1.5 parts in 100,000.

Applying this method to the cases of the outer cylinders Nos. 2 and 3, we find:

Correction for No. 2 = +1.0 part in 100,000

" " No. 3 = +1.8 " " 100,000.

The mean of these is 1.4 parts in 100,000 in the electrostatic capacity, the latter being larger than if the cylinders were perfectly cylindrical. These cylinders could be reground with our present facilities so that the correction due to conicality would be not more than one-tenth as great, and hence wholly negligible in the most refined work.

29. CORRECTION FOR OFFSETS.

In the treatment of this and several succeeding problems we shall, as a first approximation, assume that we may consider each element of the cylindrical condenser as an element of a parallel plate condenser. The problem of obtaining the distribution of charge along the length of the condenser then becomes a two-dimensional problem and may be solved by the method employed by J. J. Thomson in the third chapter of "Recent Researches." The method consists in transforming the polygon defined by the section of the condenser by a plane through its axis (call this the z plane) into the real axis of what we shall call the t plane. The polygon, defined by this section, in the w plane ($w = \phi + i\psi$, where ϕ is the stream function and ψ is the potential function) is also to be transformed into the real axis of the t plane and made to correspond there point by point with the other polygon. By eliminating t from these two equations of transformation we get ϕ and ψ as function of x and y , ($x + iy = z$), and so can obtain the distribution.

The transformation to be employed is

$$\frac{dz}{dt} = C \left(t - t_1\right)^{\frac{a_1}{2}-1} \left(t - t_2\right)^{\frac{a_2}{2}-1} \left(t - t_3\right)^{\frac{a_3}{2}-1} \dots$$

where $a_1, a_2, a_3, \dots$ are the interior angles of the polygon at the vertices corresponding to $t = t_1, = t_2, = t_3, \dots$

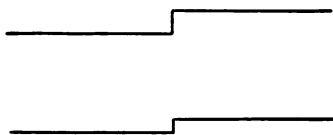


Fig. 15.

The general problem we desire to solve is that of two parallel broken planes, the two steps being in the same normal plane and of unequal height (Fig. 15). While this problem can be solved, it is easier to consider another problem, viz, the problem in which there is a step in but one of the planes; from this, by the method of images, we at once get the solution of the problem in which there are two equal steps. From these two problems we can obtain with sufficient accuracy the effect of the second step.

A condenser consisting of an infinite plane parallel to two semi-infinite planes joined by a vertical step.—The diagram in the z plane is as shown in Fig. 16. Let p and q be the distances of the infinite plane from the two semi-infinite planes. Then $q-p$ will be the height of the step.

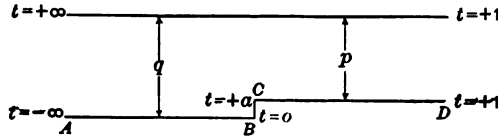


Fig. 16.

Take $t = +\infty$ at $x = -\infty$ on the infinite plate,

$t = +1$ at $x = +\infty$ on the infinite plate,

$t = 0$ at B, the foot of the step; B is taken as the origin. Then at the top of the step, t will have some value a , between 0 and +1, to be determined from the dimensions of the apparatus. The transformation is

$$\frac{dz}{dt} = \frac{C}{t-1} \sqrt{\frac{t-a}{t}}$$

$$\therefore z = C \left[\log \left(\sqrt{t(t-a)} + t - \frac{a}{2} \right) - \sqrt{1-a} \log \left(\frac{\sqrt{t(t-a)} + \sqrt{1-a}}{t-1} + \frac{2-a}{2\sqrt{1-a}} \right) \right]$$

$$\therefore z = -\frac{q}{\pi} \left[\log \frac{2\sqrt{t(t-a)} + 2t-a}{a} - \sqrt{1-a} \log \frac{2\sqrt{1-a}\sqrt{t(t-a)} - a + t(2-a)}{a(t-1)} \right] + iq$$

if the origin is taken at the point $t=0$. But, as t decreases through +1, z must decrease by iq ,

$$\therefore \sqrt{1-a} = \frac{p}{q}$$

$$\begin{aligned} \therefore z = & -\frac{q}{\pi} \log \frac{2q\sqrt{t} [tq^2 - (q^2 - p^2)] - (q^2 - p^2) + 2tq^2}{q^2 - p^2} \\ & - \frac{p}{\pi} \log \frac{2p\sqrt{t} [tq^2 - (q^2 - p^2)] - (q^2 - p^2) - (q^2 - p^2)t - iq^2}{(q^2 - p^2)(t-1)} \end{aligned}$$

The diagram in the w plane consists of two infinite lines (Fig. 17) and the transformation is

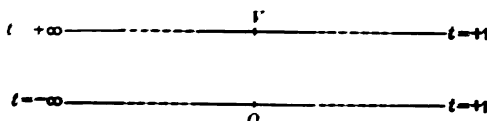


Fig. 17.

$$\frac{dw}{dt} = \frac{B}{t-1}$$

$$\therefore w = -\frac{V}{\pi} [\log(t-1) - i\pi]$$

if the potential of the upper plane is V and that of the lower one is zero. Hence the charge on a unit strip bounded by t_1 and t_2 , $t_1 < t_2$, is

$$\sigma = -\frac{1}{4\pi} \frac{\partial \psi}{\partial \nu} = -\frac{1}{4\pi} \frac{\partial \phi}{\partial s}$$

$$E = \int_{t_1}^{t_2} \sigma ds = -\frac{1}{4\pi} [\phi(t_2) - \phi(t_1)]$$

$$\therefore E = \frac{V}{4\pi^2} \log \frac{t_2 - 1}{t_1 - 1}$$

Hence, the charge on the semi-infinite plane at the top of the step, and extending from $t=a$ to $t=t_1$ (t_1 nearly = 1) is

$$E_1 = \frac{V}{4\pi^2} \log \frac{1-t_1}{1-a}$$

But, for t nearly equal to but less than 1, we have

$$x_1 = -\frac{q}{\pi} \log \frac{q+p}{q-p} + \frac{p}{\pi} \log \frac{4p^2}{q^2-p^2} - \frac{p}{\pi} \log(1-t_1)$$

$$\therefore E_1 = -\frac{V}{4\pi p} \left[x_1 + \frac{q}{\pi} \log \frac{q+p}{q-p} - \frac{p}{\pi} \log \frac{4q^2}{q^2-p^2} \right]$$

If q differs but slightly from p , this expression can be put in a form more suitable for computation by putting

$$q = p(1+a)$$

Then,

$$\frac{q+p}{q-p} = \frac{2+a}{a}$$

$$\frac{4q^2}{q^2-p^2} = \frac{4(1+a)^2}{a(2+a)}$$

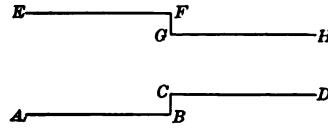


Fig. 18.

$$\begin{aligned} \therefore E_1 &= -\frac{V}{4\pi p} \left[x_1 + \frac{p}{\pi} \left\{ \log \frac{2+a}{a} \cdot \frac{a(2+a)}{4(1+a)^2} + a \log \frac{2+a}{a} \right\} \right] \\ &= -\frac{V}{4\pi p} \left[x_1 + \frac{p}{\pi} \left\{ a \log \frac{2+a}{a} - \log \left(1 + a - \frac{a^2}{4} \right) \right\} \right] \\ &= -\frac{V}{4\pi p} \left[x_1 + \frac{pa}{\pi} \left\{ \log \frac{2+a}{a} - 1 + \frac{3a}{4} \dots \right\} \right] \end{aligned}$$

By the method of images, we find that the corresponding expression for the case of two equal steps, as shown in Fig. 18, is

$$E_1' = -\frac{V}{4\pi p} \left[x_1 + \frac{pa}{\pi} \left\{ \log \frac{1+a}{a} - 1 + \frac{3a}{2} \right\} \right]$$

in which p (the distance between GH and CD) and the height of each step (ap) have the same numerical values as before:

$$\begin{aligned} \therefore E_1 - E_1' &= -\frac{V}{4\pi p} \cdot \frac{pa}{\pi} \left\{ \log \frac{2+a}{a} - \log \frac{1+a}{a} - \frac{3a}{4} \right\} \\ &= -\frac{Va}{4\pi^2} \left\{ \log \left(2 - \frac{a}{1+a} \right) - \frac{3a}{4} \right\} \\ &= -\frac{Va}{4\pi^2} \left\{ \log 2 - \frac{5a}{4} + \dots \right\} \\ &= -\frac{Va}{4\pi} (0.21) \text{ approximately.} \end{aligned} \tag{9}$$

Hence the effect of moving the portion EF of the plane away from AB by an amount ap is to decrease the effective length of the portion CD of the other plane by $0.21 ap$. If it had been moved in the other direction the effective length would have been increased by the same absolute amount. Hence if there is a step of height a_1p in the upper plane and one of height ap in the lower one (the planes of the steps coinciding) the charge on the section CD is very nearly equal to that on a section of one of a pair of infinite parallel planes at a distance apart equal to that of the section CD and of length greater than that of CD by an amount

$$\delta l_1 = \frac{pa}{\pi} \left\{ \log \frac{2+a}{a} - 1 + \frac{3a}{4} \right\} \mp \frac{pa_1}{\pi} \left(\log 2 - \frac{5a_1}{4} \right)$$

The minus sign is to be taken if the relative positions of the steps is as shown in Fig. 18.

The charge on the side of the step, between $t=0$ and $t=a$, is

$$E_s = \frac{V}{4\pi^2} \log (1-a) = -\frac{V}{4\pi^2} \log \frac{q^2}{p^2}$$

To reduce this to a form suitable for computation, when q is nearly equal to p , put $q = p(1+a)$, and we find

$$\begin{aligned} E_s &= -\frac{V}{4\pi^2} \log (1+a)^2 \\ &= -\frac{V}{2\pi^2} \left\{ a - \frac{a^2}{2} + \dots \right\} \end{aligned} \quad (10)$$

By the method of images we find for the case of two equal steps, as shown in Fig. 18, the expression

$$\begin{aligned} E'_s &= -\frac{V}{4\pi^2} \log \frac{q_1}{p_1} \\ \therefore E_s - E'_s &= -\frac{V}{4\pi^2} \log \frac{q^2}{p^2} \cdot \frac{p_1}{q_1} \end{aligned}$$

If we regard p as remaining fixed, then

$$\begin{aligned} p_1 &= p \\ q_1 &= q + ap = p(1+2a) \end{aligned}$$

$$\begin{aligned}\therefore E_s - E'_s &= -\frac{V}{4\pi^2} \log \frac{(1+a)^2}{1+2a} \\ &= -\frac{V}{4\pi^2} (a^2 - 2a^3 + \dots)\end{aligned}$$

Hence the effect of moving EF away from AB by an amount $a\rho$ is to decrease the charge on the side of the step by the amount, $\frac{Va^2}{4\pi^2}$, approximately. Similarly, if we regard q as remaining fixed, we must take

$$\begin{aligned}q_1 &= q = \rho(1+a) \\ \rho_1 &= \rho(1-a) \\ \therefore E_s - E'_s &= -\frac{V}{4\pi^2} \log (1-a^2) \\ &= +\frac{V}{4\pi^2} (a^2 + \frac{a^4}{2} + \dots)\end{aligned}$$

Therefore, the effect of moving GH toward CD by an amount $a\rho$ is to increase the charge on the side of the step by an amount $\frac{Va^2}{4\pi^2}$, approximately. It is readily seen that these effects are exactly equal to half the effect produced by moving the entire upper plane by the same amount. Hence the addition of a step equal to $\beta\rho$ in the upper plane will increase the charge on the side of the step in the lower plane by $\frac{\beta a V}{4\pi^2} = \frac{V}{4\pi} (0.32a\beta)$. Since this effect varies as the product of a and β , it is very small as compared with the effects given by expressions (9) and (10), if a and β are small.

The charge on the semi-infinite plane at the bottom of the step between, $t=0$ and $t=-\tau$, where τ is very large, is

$$\begin{aligned}E_s &= \frac{V}{4\pi^2} \log \left(\frac{1}{1+\tau} \right) \\ &= -\frac{V}{4\pi^2} \log \tau, \text{ approximately.}\end{aligned}$$

But for $t=-\tau$, τ being large, we have

$$x_{-\infty} = -\frac{q}{\pi} \log \tau + \frac{q}{\pi} \log \frac{q^2 - p^2}{4q^2} + \frac{p}{\pi} \log \frac{q+p}{q-p}$$

$$\therefore E_s = -\frac{V}{4\pi q} \left[X - \frac{q}{\pi} \log \frac{4q^2}{q^2 - p^2} + \frac{p}{\pi} \log \frac{q+p}{q-p} \right]$$

where $X = -x_{-\infty}$. That is, X is the distance from B. To reduce this expression to a form more suitable for computation when q is nearly equal to p , put

$$p = q(1 - \beta)$$

Then

$$E_s = -\frac{V}{4\pi q} \left[X - \frac{q}{\pi} \log \frac{4}{\beta(2-\beta)} + \frac{q(1-\beta)}{\pi} \log \frac{2-\beta}{\beta} \right]$$

$$= -\frac{V}{4\pi q} \left[X - \frac{q\beta}{\pi} \left\{ 1 + \log \frac{2-\beta}{\beta} + \frac{\beta}{4} + \dots \right\} \right]$$

Applying the method of images, we find for the case of equal steps, as shown in Fig. 18,

$$E'_s = -\frac{V'}{4\pi q'} \left[X - \frac{q'\beta'}{\pi} \left\{ 1 + \log \frac{2-\beta'}{\beta'} + \frac{\beta'}{4} + \dots \right\} \right]$$

where p' and q' are the half distances between the plates, V' is half the potential difference, and $p' = q'(1 - \beta')$. If we take $2q' = q$, and $p' = p - \beta q$, we find

$$\beta' = 2\beta$$

$$\therefore E'_s = -\frac{V}{4\pi q} \left[X - \frac{q\beta}{\pi} \left\{ 1 + \log \frac{1-\beta}{\beta} + \frac{\beta}{2} + \dots \right\} \right]$$

$$\therefore E_s - E'_s = +\frac{V\beta}{4\pi^2} \left\{ \log \frac{2-\beta}{1-\beta} - \frac{\beta}{4} \right\}$$

$$= +\frac{V\beta}{4\pi^2} \left\{ \log 2 + \frac{\beta}{4} + \frac{3\beta^2}{8} + \dots \right\}$$

Hence the effect of moving the portion GH of the plane nearer to CD by an amount βq is to increase the effective length of the portion AB by an amount

$$\frac{\beta q}{\pi} \left\{ \log 2 + \frac{\beta}{4} + \dots \right\}$$

Which for small values of β becomes, approximately, $0.21 \beta q$. Hence if there is a step of $\beta_1 q$ in the upper plane and one of βq in the lower one—the planes of the steps coinciding—the charge on the section AB is very nearly equal to that of a section of one of a pair of infinite parallel planes at a distance apart equal to that of the section AB and of a length greater than that of AB by an amount

$$\delta l_s = -\frac{q\beta}{\pi} \left\{ 1 + \log \frac{2-\beta}{\beta} + \frac{\beta}{4} \right\} \pm \frac{q\beta_1}{\pi} \left\{ \log 2 + \frac{\beta_1}{4} + \dots \right\}$$

The plus sign is to be taken if the relative position of the steps is as shown in Fig. 18.

If there is no offset in the upper plate, the total charge on the plate AD is

$$E = -\frac{V}{4\pi} \left[\begin{array}{l} x_1 + \frac{a}{\pi} \left\{ \log \frac{2+a}{a} - 1 + \frac{3a}{4} + \dots \right\} \\ + \frac{a}{\pi} \left\{ \log \frac{2-a}{a} - 1 + \frac{3a}{4} + \dots \right\} \\ + \frac{X}{q} - \frac{\beta}{\pi} \left\{ \log \frac{2-\beta}{\beta} + 1 - \frac{\beta}{4} + \dots \right\} \end{array} \right]$$

But in this case

$$1 - \beta = \frac{1}{1+a}$$

$$\therefore \beta = \frac{a}{1+a}$$

$$\begin{aligned} \therefore E &= -\frac{V}{4\pi} \left[\begin{array}{l} x_1 + \frac{a}{\pi} \left\{ \log \frac{2+a}{a} - 1 + \frac{3a}{4} + \dots \right\} \\ + \frac{a}{\pi} \left\{ \log \frac{2-a}{a} - 1 + \frac{3a}{4} + \dots \right\} \\ + \frac{X}{q} - \frac{a}{\pi} \left\{ \frac{1}{1+a} \log \frac{2+a}{a} + 1 - \frac{5a}{4} + \dots \right\} \end{array} \right] \\ &= -\frac{V}{4\pi} \left[\frac{x_1}{\pi} + \frac{X}{q} + \frac{a}{\pi} \left\{ \frac{a}{1+a} \log \frac{2+a}{a} + a + \dots \right\} \right] \end{aligned}$$

Hence the effect of a step in one of a pair of parallel plates is to

increase the magnitude of the charge by

$$\delta E = \frac{V a^2}{4 \pi^2} \left\{ 1 + \frac{1}{1+a} \log \frac{2+a}{a} + \dots \right\}$$

Let us now determine the distance to which the effect of the step makes itself felt. To do this, differentiate the charge with respect to the distance to get the density, and determine the distance we must go from the step before the density becomes sensibly equal to its value at infinity.

Side CD at the top of the step:

$$\begin{aligned} \frac{\partial E}{\partial x} &= \frac{\partial E}{\partial t} \cdot \frac{\partial t}{\partial x} = + \frac{V}{4 \pi^2} \cdot \frac{1}{1-t_1} \cdot \frac{(t_1-1)\pi}{q} \cdot \frac{t_1}{\sqrt{t_1-a}} \\ &= - \frac{V}{4 \pi q} \sqrt{\frac{t_1}{t_1-a}} \equiv \sigma_1 \end{aligned}$$

Put $t_1 = 1 - \epsilon$, then

$$\begin{aligned} \sigma_1 &= - \frac{V}{4 \pi q} \sqrt{\frac{1-\epsilon}{1-a-\epsilon}} \\ &= - \frac{V}{4 \pi q} \cdot \frac{1}{\sqrt{1-a}} \sqrt{1-\epsilon + \frac{\epsilon}{1-a}} \\ &= - \frac{V}{4 \pi q} \cdot \frac{1}{\sqrt{1-a}} \left\{ 1 + \frac{a \epsilon}{2(1-a)} \right\} \\ &= - \frac{V}{4 \pi p} \left\{ 1 + \frac{1}{2} \frac{a \epsilon}{1-a} \right\} \end{aligned}$$

The density at $x = \infty$ is

$$\begin{aligned} \sigma_0 &= - \frac{V}{4 \pi p} \\ \therefore \frac{\sigma_1 - \sigma_0}{\sigma_0} &= \frac{a \epsilon}{2(1-a)} \end{aligned}$$

If this ratio is equal to a very small quantity, say η , then

$$\epsilon = \frac{2(1-a)\eta}{a}$$

or
$$\epsilon = \frac{2 \rho^3 \eta}{q^3 - \rho^3}$$

Hence, the distance at which $\frac{\sigma_1 - \sigma_0}{\sigma_0} = \eta$ must sensibly be

$$\begin{aligned} x_1' &= -\frac{q}{\pi} \log \frac{q+\rho}{q-\rho} + \frac{\rho}{\pi} \log \frac{4\rho^3}{q^3 - \rho^3} - \frac{\rho}{\pi} \log \frac{2\rho^3 \eta}{q^3 - \rho^3} \\ &= -\frac{q}{\pi} \log \frac{q+\rho}{q-\rho} + \frac{\rho}{\pi} \log \frac{2}{\eta} \end{aligned}$$

Example:

If $q = 1.0000$ cm, $q - \rho = 0.006$ cm,
then,

$$x_1 = \frac{0.994}{\pi} \log \frac{1}{\eta} = 1.629$$

For $\eta = 10^{-5}$ this becomes

$$x_1 = 2.01 \text{ cm}$$

Hence, for such a step, the density on the plane at the top of the step and at a distance from it greater than 2 cm does not differ from its value at an infinite distance by more than 1 in 100,000.

Side AB at the bottom of the step:

$$\begin{aligned} \frac{\partial E}{\partial \tau} \cdot \frac{\partial \tau}{\partial x} &= -\frac{V}{4\pi^3} \cdot \frac{1+\tau}{(1+\tau)^3} \cdot \frac{\pi(1+\tau)}{q} \sqrt{\frac{\tau}{a+\tau}} \\ &= -\frac{V}{4\pi q} \sqrt{\frac{\tau}{a+\tau}} \equiv \sigma_s \end{aligned}$$

Here,

$$\begin{aligned} \sigma_0 &= -\frac{V}{4\pi q} \\ \therefore \frac{\sigma_s - \sigma_0}{\sigma_0} &= -1 + \sqrt{\frac{\tau}{a+\tau}} \end{aligned}$$

Since τ is very large we may write

$$\frac{\sigma_s - \sigma_0}{\sigma_0} = -1 + \left(1 + \frac{a}{\tau}\right)^{-\frac{1}{2}} = +\frac{1}{2} \frac{a}{\tau}$$

If this ratio is to be a very small quantity, say η , then

$$\frac{1}{2} \frac{a}{\tau} = \eta$$

$$\text{or, } \tau = \frac{1}{2} \frac{a}{\eta}$$

Hence, the distance at which $\frac{\sigma_s - \sigma_o}{\sigma_o} = \eta$ must sensibly be

$$\begin{aligned} X \equiv -x_{-\infty} &= \frac{q}{\pi} \log \frac{a}{2\eta} - \frac{q}{\pi} \log \frac{q^2 - p^2}{4q^2} - \frac{p}{\pi} \log \frac{q+p}{q-p} \\ &= \frac{q}{\pi} \log \frac{2}{\eta} - \frac{p}{\pi} \log \frac{q+p}{q-p} \end{aligned}$$

Example:

$$\text{If } q = 1.000 \text{ cm; } q - p = 0.006 \text{ cm}$$

$$\text{then, } X = 0.318 \log \frac{1}{\eta} - 1.398$$

For $\eta = 10^{-8}$ this becomes

$$X = 2.26 \text{ cm}$$

Let us now apply these equations to the cases that arise in the work with the cylinders. Denoting by h the distance between the opposed surfaces at the end of the sections; by δ the excess of the radius of the upper section of the inner cylinder above that of the lower one, measured at the adjoining ends, and by δ_1 , the corresponding quantity for the outer cylinders, we obtain from our measurements the following table:

TABLE XI.

Junction of sections	h		δ	δ_1
	Upper	Lower		
1 and 2	0.98793	0.98434	0.00074	0.00433
1 and 3	0.98351	0.98434	0.00016	-0.00067
2 and 3	0.98351	0.98559	-0.00011	-0.00219
2 and 4	0.98099	0.98559	-0.00110	-0.00570
3 and 4	0.98099	0.98604	-0.00107	-0.00612

The greatest correction term will evidently be that which arises from the offsets at the junction of sections 2 and 4. Taking p as the distance between the surfaces at the upper end of section 2, we have:

$$p = 0.98559$$

$$q = 0.98099$$

$$a = 0.00111,$$

$$\beta = 0.00111,$$

$$a_1 = 0.00578,$$

$$\beta_1 = 0.00580.$$

Hence, the charge on the inner cylinder of section No. 2—neglecting the face of the step—is greater than it would be on an equal length of an infinite cylinder of the same cross sectional dimensions by an amount

$$\frac{V}{4\pi p} \Delta l_1 = \frac{V}{4\pi} \left[a \left\{ \log \frac{2+a}{a} - 1 + \frac{3a}{4} \right\} + a_1 \left\{ \log 2 - \frac{5a_1}{4} \right\} \right] = 0.01121 \frac{V}{4\pi}$$

Similarly, the charge on the inner cylinder of section No. 4 is greater than it would be on an equal length of an infinite cylinder of the same cross sectional dimensions by

$$\frac{V}{4\pi q} \Delta l_2 = -\frac{V}{4\pi} \left[\beta \left\{ \log \frac{2-\beta}{\beta} + 1 + \frac{\beta}{4} \right\} + \beta_1 \left\{ \log 2 + \frac{\beta_1}{4} \right\} \right] = -0.01353 \frac{V}{4\pi}$$

The charge on the side of the step is

$$\frac{V}{4\pi} \left[2a \left(1 - \frac{a}{2} \right) + aa_1 \right] = 0.002235 \frac{V}{4\pi}$$

Hence the net effect of these offsets is to increase the total charge on the inner cylinder by

$$-0.00008 \frac{V}{4\pi}$$

which is equivalent to decreasing the length of the condenser by only 0.2μ , a quantity entirely negligible.

Hence the only case in which the presence of the offsets can produce an appreciable effect is when they occur at the gap separating the guard cylinder from the condenser proper, under which condition the charge corresponding to one of the Δl correction terms is not measured.

30. CORRECTION DUE TO IRREGULAR DISTRIBUTION OF CHARGE.

The determination of the various corrections due (1) to the irregular distribution of charge near the end of the inner cylinder, (2) to the charge on the interior of the inner cylinder and (3) to the variation of the charge on the outside of the outer cylinder as the latter is changed in height, are all parts of the same mathematical problem. Being unable to treat directly this problem for the case of circular cylinders, we shall solve two cases for systems of semi-infinite plates, viz, (1) the case of four infinitely thin semi-infinite plates (this will give us an idea of the distribution on the interior of the inner cylinder); (2) the case of a thick semi-infinite plate midway between two infinitely thin semi-infinite plates (this will give us a minimum value for the charge on the end of the inner cylinder when the latter is closed). A more interesting case is that in which the outer plates

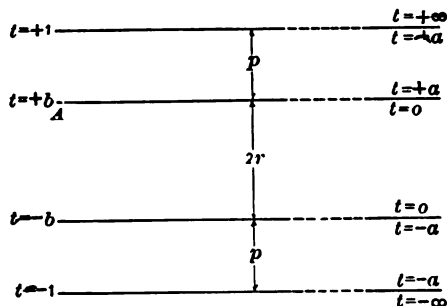


Fig. 19.

are of finite thickness, but the increased mathematical difficulties more than offset the gain.

Case I. *A condenser consisting of two mutually exterior, identical pairs of semi-infinite planes, all bounded by straight lines lying in a plane normal to the four.*—The innermost planes are both at zero potential, while the outermost ones are at the common potential V . Assume that the origin lies at the point A, Fig. 19, the y axis being normal to the planes. Assume $t=+\infty$ at the point $x=\infty$ on the upper side of the top plane; $t=+a$ at the corresponding point on the lower side; $t=+1$ at the bounding edge of the top plane; $t=+b$ at the bounding edge of the next to the top plane; $t=+a$ on the top face of the next to the top plane at $x=\infty$; $t=0$ at the corresponding point on the lower side of this plane. At the

corresponding points of the other pair of planes t will have the same absolute values, but will be of the opposite sign. The values of a and b must be determined from the distances between the planes. Let p be the distance between the planes composing either pair, and $2r$ be the distance between the two innermost planes.

The transformation is

$$\frac{dz}{dt} = C \frac{(t^2 - 1)(t^2 - b^2)}{t(t^2 - a^2)}$$

$$\text{or, } z = -\frac{C}{2a^2} \left[b^2 \log \frac{t^2}{b^2} + (1 - a^2)(a^2 - b^2) \log \frac{t^2 - a^2}{a^2 - b^2} - a^2(t^2 - b^2) - i\pi(1 - a^2)(a^2 - b^2) \right]$$

The conditions that z shall increase by $2ir$ as t increases through zero, and by ip as t increases through a give us the equations

$$C = \frac{2ra^2}{\pi b^2} = \frac{2pa^2}{\pi(1 - a^2)(a^2 - b^2)}$$

These, combined with the condition that $x = 0$ for $t = +1$, give the equation

$$\frac{ra^2(1 - b^2)}{b^2} = 2r \log \frac{1}{b} + p \log \frac{1 - a^2}{a^2 - b^2} \quad (11)$$

From the expression for C , we find

$$b^2 = \frac{a^2 r (1 - a^2)}{p + r - a^2 r} \quad (12)$$

The last two equations allow us to determine the values of a and of b . Replacing C by its value as given above, we find

$$z = -\frac{2r}{\pi} \log \frac{t}{b} - \frac{p}{\pi} \log \frac{t^2 - a^2}{a^2 - b^2} + \frac{ra^2}{\pi b^2} (t^2 - b^2) + ip \quad (13)$$

The diagram in the w plane consists of two infinite parallel lines; the values of t at the extremities of these lines are $+a$ and $-a$. The transformation is

$$\frac{dw}{dt} = B \frac{1}{t^2 - a^2}$$

$$\therefore w = \frac{V}{\pi} \left[\log \frac{t+a}{t-a} + i\pi \right]$$

Hence, the charge per unit length of the strip bounded by $t=t_1$ and $t=t_2$ is

$$E = \frac{V}{4\pi^2} \log \left[\frac{t_2-a}{t_2+a} \cdot \frac{t_1+a}{t_1-a} \right], \quad t_1 < t_2$$

The charge on the top face of the top plate is obtained by putting $t_1=1$; $t_2=t$, where t is very large. Then

$$\begin{aligned} E_1 &= \frac{V}{4\pi^2} \log \left[\frac{t-a}{t+a} \cdot \frac{1+a}{1-a} \right] \\ &= \frac{V}{4\pi^2} \left[\log \frac{1+a}{1-a} - \frac{2a}{t} \right] \end{aligned} \quad (14)$$

But, for t very large

$$t^2 = \frac{b^2}{ra^2} \left[\pi x - 2r \log b - p \log(a^2 - b^2) \right]$$

$$\therefore E_1 = \frac{V}{4\pi^2} \left[\log \frac{1+a}{1-a} - \frac{2a^2}{b} \sqrt{\frac{r}{\pi x - 2r \log b - p \log(a^2 - b^2)}} \right]$$

Hence, at a long distance from the edge, the charge on this face will be sensibly equal to

$$\frac{V}{4\pi^2} \left[\log \frac{1+a}{1-a} - \frac{2a^2}{b} \sqrt{\frac{r}{\pi x}} \right]$$

The charge on the under face of the top plate is obtained by putting $t_2=1$, $t_1=t$, where t is nearly equal to a . Then

$$E_2 = \frac{V}{4\pi^2} \log \left[\frac{2a}{t-a} \cdot \frac{1-a}{1+a} \right]$$

But, for t nearly equal to, but greater than a

$$x = -\frac{2r}{\pi} \log \frac{a}{b} - \frac{p}{\pi} \log \frac{2a(t-a)}{a^2 - b^2} + \frac{ra^2(a^2 - b^2)}{b^2\pi}$$

$$\therefore E_2 = +\frac{V}{4\pi^2} \left[x + \frac{p}{\pi} \log \left(\frac{1-a}{1+a} \cdot \frac{4a^2}{a^2 - b^2} \right) - \frac{r}{\pi} \left\{ \frac{a^2(a^2 - b^2)}{b^2} - \log \frac{a^2}{b^2} \right\} \right]$$

The charge on the top face of the next to the top plate is obtained by putting $t_1 = b$, $t_2 = t$ nearly $= a$. Then

$$E_3 = \frac{V}{4\pi^2} \log \left(\frac{a-t}{2a} \cdot \frac{a+b}{a-b} \right)$$

But for t nearly equal to a

$$x = -\frac{2r}{\pi} \log \frac{a}{b} - \frac{\rho}{\pi} \log \frac{2a(a-t)}{(a^2-b^2)} + \frac{a^2 r (a^2-b^2)}{b^2 \pi}$$

$$\therefore E_3 = -\frac{V}{4\pi^2} \left[x + \frac{\rho}{\pi} \log \frac{4a^2}{(a+b)^2} - \frac{r}{\pi} \left\{ \frac{a^2(a^2-b^2)}{b^2} - \log \frac{a^2}{b^2} \right\} \right]$$

The charge on the under side of the next to the top plane is obtained by putting $t_1 = t_0$, a very small quantity, and $t_2 = b$. Then

$$E_4 = \frac{V}{4\pi^2} \log \left[\frac{a-b}{a+b} \cdot \frac{a+t_0}{a-t_0} \right]$$

$$= \frac{V}{4\pi^2} \left[\log \frac{a-b}{a+b} + \frac{2t_0}{a} \right]$$

But, for very small values of t

$$x = -\frac{2r}{\pi} \log \frac{t_0}{b} - \frac{\rho}{\pi} \log \frac{a^2}{a^2-b^2} - \frac{ra}{\pi}$$

$$\therefore t_0 = b \left(\frac{a^2-b^2}{a^2} \right)^{\frac{\rho}{2r}} e^{-\frac{\pi}{2} \left(\frac{x}{r} + \frac{a^2}{\pi} \right)}$$

$$\therefore E_4 = \frac{V}{4\pi^2} \left[\log \frac{a-b}{a+b} + \frac{2b}{a} \left(\frac{a^2-b^2}{a^2} \right)^{\frac{\rho}{2r}} e^{-\frac{\pi}{2} \left(\frac{x}{r} + \frac{a^2}{\pi} \right)} \right]$$

$$= -\frac{V}{4\pi^2} \left[\log \frac{a+b}{a-b} - \frac{2b}{a} \left(\frac{a^2-b^2}{a^2} \right)^{\frac{\rho}{2r}} e^{-\frac{\pi}{2} \left(\frac{x}{r} + \frac{a^2}{\pi} \right)} \right] \quad (15)$$

From these expressions we find the total charge on the inner cylinder between its end and a long distance, x , from the end is

$$E = -\frac{V}{4\pi^2} \left[x + \frac{\rho}{\pi} \log \frac{4a^2}{a^2-b^2} - \frac{r}{\pi} \left\{ \frac{a^2(a^2-b^2)}{b^2} - \log \frac{a^2}{b^2} \right\} \right]$$

These equations are sufficient to enable us to determine the edge effect and the variations of the density with the distance from the edge, provided this distance is not too small. However, in order to study the variation of density on the upper face of the next to the top plane it is advisable to obtain a general expression for the density. It is

$$\begin{aligned}\sigma_s &= -\frac{1}{4\pi} \frac{\partial \phi}{\partial t} \cdot \frac{\partial t}{\partial x} \\ &= -\frac{V'}{4\pi p} \left[\frac{(1-a^2)(a^2-b^2)}{a} \cdot \frac{t}{(1-t^2)(t^2-b^2)} \right]\end{aligned}$$

For $t=a$, the density is uniform and equal to

$$\sigma_0 = -\frac{V'}{4\pi p}$$

Hence,

$$\frac{\Delta \sigma}{\sigma_0} = \frac{\sigma_s - \sigma_0}{\sigma_0} = \frac{(1-a^2)(a^2-b^2)}{a} \cdot \frac{t}{(1-t^2)(t^2-b^2)} - 1$$

If the density is sensibly equal to σ_0 , this expression must be sensibly zero. If $t^2 = a^2(1-\epsilon)$, we find

$$\begin{aligned}\frac{\sigma_s - \sigma_0}{\sigma_0} &= \frac{(1-a^2)(a^2-b^2)}{a} \cdot \frac{a(1-\epsilon)^{\frac{1}{2}}}{(1-a^2+a^2\epsilon)(a^2-b^2-a^2\epsilon)} - 1 \\ &= \frac{(1-a^2)(a^2-b^2)(1-\epsilon)^{\frac{1}{2}}}{(1-a^2)(a^2-b^2)-a^2\epsilon[1-a^2-(a^2-b^2)]-a^4\epsilon^2} - 1 \\ &= (1-\epsilon)^{\frac{1}{2}} \left\{ 1 + a^2\epsilon \left(\frac{1}{a^2-b^2} - \frac{1}{1-a^2} \right) + \dots \right\} - 1 \\ &= \epsilon \left\{ a^2 \left(\frac{1}{a^2-b^2} - \frac{1}{1-a^2} \right) - \frac{1}{2} \right\}\end{aligned}$$

Hence, in order that

$$\frac{\sigma_s - \sigma_0}{\sigma_0} = +10^{-7}$$

we must have

$$t^2 = a^2 \left[1 - \frac{10^{-\eta}}{a^2 \left(\frac{1}{a^2 - b^2} - \frac{1}{1 - a^2} \right) - 0.5} \right] \quad (16)$$

Knowing a^2 and b^2 , and assuming a suitable value for η , we can find t^2 , and from this can find x .

Case 2. *A condenser consisting of a thick semi-infinite plate midway between, and parallel to two infinitely thin semi-infinite parallel plates; the edges of all the plates lie in the same plane normal to the system of plates.*—The diagram in the z plane is as shown in Fig. 20. Assume

$$\begin{array}{ll} t = +\infty \text{ at A} & t = -b \text{ at E} \\ = +1 \text{ at B} & = -a \text{ at F} \\ = +a \text{ at C} & = -1 \text{ at G} \\ = +b \text{ at D} & = -\infty \text{ at H} \end{array}$$

The transformation is

$$\frac{dz}{dt} = C \frac{t^2 - 1}{t^2 - a^2} \sqrt{t^2 - b^2}$$

$$\begin{aligned} \text{or, } \frac{dz}{dt} &= C \left[t^2 - (1 + b^2 - a^2) - \frac{(1 - a^2)(a^2 - b^2)}{2a} \left\{ t - a - \frac{1}{t + a} \right\} \right] \left(\frac{1}{\sqrt{t^2 - b^2}} \right) \\ \therefore z &= C \left[\begin{aligned} &\frac{t}{2} \sqrt{t^2 - b^2} - \left(1 + \frac{b^2}{2} - a^2 \right) \log (t + \sqrt{t^2 - b^2}) \\ &- \frac{(1 - a^2) \sqrt{a^2 - b^2}}{2a} \left\{ \log \frac{\sqrt{a^2 - b^2} \sqrt{t^2 - b^2} - b^2 - at}{(t + a) \sqrt{a^2 - b^2}} \right. \\ &\quad \left. - \log \frac{\sqrt{a^2 - b^2} \sqrt{t^2 - b^2} - b^2 + at}{(t - a) \sqrt{a^2 - b^2}} \right\} \end{aligned} \right] + C_1 + i D_1 \end{aligned}$$

C_1 and D_1 being constants of integration.

If $z = 0$ at the point $t = 0$, we have

$$0 = C \left[- \left(1 + \frac{b^2}{2} - a^2 \right) \log b - i \left\{ \left(1 + \frac{b^2}{2} - a^2 \right) \frac{\pi}{2} + \frac{\pi(1 - a^2) \sqrt{a^2 - b^2}}{2a} \right\} \right] + C_1 + i D_1$$

$$\therefore C_1 = C \left(1 + \frac{b^2}{2} - a^2 \right) \log b$$

$$D_1 = C \left[\left(1 + \frac{b^2}{2} - a^2 \right) \frac{\pi}{2} + \frac{\pi(1-a^2)\sqrt{a^2-b^2}}{2a} \right]$$

$$\therefore z = C \left[\begin{aligned} & \frac{t}{2} \sqrt{t^2 - b^2} - \left(1 + \frac{b^2}{2} - a^2 \right) \log \frac{t + \sqrt{t^2 - b^2}}{b} \\ & - \frac{(1-a^2)\sqrt{a^2-b^2}}{2a} \left\{ \log \frac{\sqrt{a^2-b^2}\sqrt{t^2-b^2}-b^2-at}{(t+a)\sqrt{a^2-b^2}} \right. \\ & \quad \left. - \log \frac{\sqrt{a^2-b^2}\sqrt{t^2-b^2}-b^2+at}{(t-a)\sqrt{a^2-b^2}} \right\} \\ & + \frac{i\pi}{2} \left\{ \left(1 + \frac{b^2}{2} - a^2 \right) + \frac{(1-a^2)\sqrt{a^2-b^2}}{a} \right\} \end{aligned} \right]$$

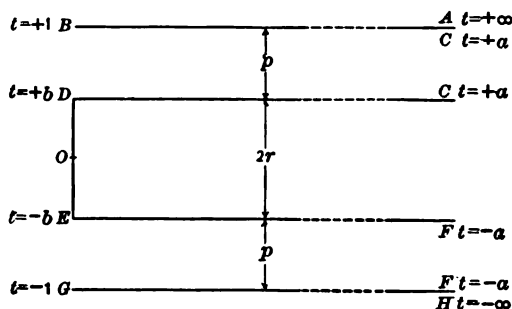


Fig. 20.

For $t=b$, we have $z=ir$

$$\begin{aligned} \therefore ir &= C \left[-\frac{(1-a^2)\sqrt{a^2-b^2}}{2a} \left\{ \log \frac{-b(a+b)}{(a+b)\sqrt{a^2-b^2}} - \log \frac{b(a-b)}{-(a-b)\sqrt{a^2-b^2}} \right\} \right. \\ & \quad \left. + \frac{i\pi}{2} \left\{ 1 + \frac{b^2}{2} - a^2 + \frac{(1-a^2)\sqrt{a^2-b^2}}{a} \right\} \right] \\ &= C \left[1 + \frac{b^2}{2} - a^2 - \frac{(1-a^2)\sqrt{a^2-b^2}}{a} \right] \frac{\pi i}{2} \\ \therefore r &= C \left[1 + \frac{b^2}{2} - a^2 - \frac{(1-a^2)\sqrt{a^2-b^2}}{a} \right] \frac{\pi}{2} \end{aligned} \quad (17)$$

For $t=1$, $z=i(b+r)$

$$\begin{aligned} \therefore i(p+r) &= C \left[\begin{aligned} &\frac{1}{2}\sqrt{1-b^2} - \left(1 + \frac{b^2}{2} - a^2\right) \log \left(\frac{1+\sqrt{1-b^2}}{b}\right) \\ &- \frac{(1-a^2)\sqrt{a^2-b^2}}{2a} \left\{ \log \frac{\sqrt{a^2-b^2}\sqrt{1-b^2}-b^2-a}{(1+a)\sqrt{a^2-b^2}} \right. \\ &\quad \left. - \log \frac{\sqrt{a^2-b^2}\sqrt{1-b^2}-b^2+a}{(1-a)\sqrt{a^2-b^2}} \right\} \\ &+ \frac{i\pi}{2} \left\{ 1 + \frac{b^2}{2} - a^2 + \frac{(1-a^2)\sqrt{a^2-b^2}}{a} \right\} \end{aligned} \right] \\ &= C \left[\begin{aligned} &\frac{1}{2}\sqrt{1-b^2} - \left(1 + \frac{b^2}{2} - a^2\right) \log \left(\frac{1+\sqrt{1-b^2}}{b}\right) \\ &- \frac{(1-a^2)\sqrt{a^2-b^2}}{2a} \left\{ \log \frac{a+b^2-\sqrt{a^2-b^2}\sqrt{1-b^2}}{(1+a)\sqrt{a^2-b^2}} \right. \\ &\quad \left. - \log \frac{a-b^2+\sqrt{a^2-b^2}\sqrt{1-b^2}}{(1-a)\sqrt{a^2-b^2}} \right\} \\ &+ \frac{i\pi}{2} \left\{ 1 + \frac{b^2}{2} - a^2 \right\} \end{aligned} \right] \\ \therefore 0 &= C \left[\begin{aligned} &\frac{1}{2}\sqrt{1-b^2} - \left(1 + \frac{b^2}{2} - a^2\right) \log \frac{1+\sqrt{1-b^2}}{b} \\ &- \frac{(1-a^2)\sqrt{a^2-b^2}}{2a} \log \left(\frac{a+b^2-\sqrt{a^2-b^2}\sqrt{1-b^2}}{a-b^2+\sqrt{a^2-b^2}\sqrt{1-b^2}} \cdot \frac{(1-a)}{(1+a)} \right) \end{aligned} \right] \quad (18) \end{aligned}$$

and
$$C \left[1 + \frac{b^2}{2} - a^2 \right] \frac{\pi}{2} = p+r \quad (19)$$

As t decreases through the value a , z must decrease by ip

$$\therefore C \left[\frac{(1-a^2)\sqrt{a^2-b^2}}{2a} \right] \pi = p \quad (20)$$

Substituting this value of C in (17), we obtain the relation

(19) which becomes

$$\left(1 + \frac{b^2}{2} - a^2 \right) = \frac{p+r}{p} \left[\frac{(1-a^2)\sqrt{a^2-b^2}}{a} \right]$$

Solving this equation for b^2 we find

$$b^2 = \frac{2(1-a^2)}{p^2 a^2} \left\{ \sqrt{[p^2 + (1-a^2)(2pr+r^2)]^2 + a^4 p^2 (2pr+r^2)} - [p^2 + (1-a^2)(2pr+r^2)] \right\} \quad (21)$$

This and (18) serve to determine a and b when p and r are known. Substituting the value of C in the expression for z and rearranging terms, we find

$$z = \frac{p}{\pi} \cdot \frac{a t \sqrt{t^2 - b^2}}{(1-a^2)\sqrt{a^2 - b^2}} - \frac{2(p+r)}{\pi} \log \frac{t + \sqrt{t^2 - b^2}}{b} - \frac{p}{\pi} \log \left(\frac{a t + b^2 - \sqrt{a^2 - b^2} \sqrt{t^2 - b^2} t - a}{a t - b^2 + \sqrt{a^2 - b^2} \sqrt{t^2 - b^2} t + a} \right) + i(p+r)$$

The diagram in the w plane is as before. Hence, the charge per unit length of the strip between the values $t=t_1$, and $t=t_2$; $t_2 > t_1$ is

$$E = \frac{V}{4\pi^2} \log \left[\frac{t_2 - a}{t_2 + a} \cdot \frac{t_1 + a}{t_1 - a} \right]$$

The total charge on unit length of the middle plate between O and C is

$$E_t = \frac{V}{4\pi^2} \log \left[\frac{t-a}{2a} \cdot \frac{a}{-a} \right] = \frac{V}{4\pi^2} \log \frac{a-t}{2a}$$

where t is less than but nearly equal to a . But, for t nearly equal to a , we find

$$x = \frac{p a^2}{\pi(1-a^2)} - \frac{2(p+r)}{\pi} \log \frac{a + \sqrt{a^2 - b^2}}{b} - \frac{p}{\pi} \log \frac{b^2}{2a(a^2 - b^2)} - \frac{p}{\pi} \log(a-t)$$

$$\therefore E_t = -\frac{V}{4\pi p} \left[x + \frac{p}{\pi} \left\{ 2 \log \frac{a + \sqrt{a^2 - b^2}}{\sqrt{a^2 - b^2}} + \frac{2r}{p} \log \frac{a + \sqrt{a^2 - b^2}}{b} - \frac{a^2}{1-a^2} \right\} \right]$$

Hence, the effective length of the thick plate is greater than its actual length by an amount

$$\delta l_t = \frac{p}{\pi} \left[2 \log \frac{a + \sqrt{a^2 - b^2}}{\sqrt{a^2 - b^2}} + \frac{2r}{p} \log \frac{a + \sqrt{a^2 - b^2}}{b} - \frac{a^2}{1-a^2} \right] \quad (22)$$

31. EFFECT OF VARYING THE DISTANCE BETWEEN THE CLOSED ENDS OF CONCENTRIC CYLINDERS.

Lord Rayleigh³ has shown that the force of attraction between two semi-infinite cylinders placed as shown in Fig. 21, and maintained at unit difference of potential is approximately

$$F = \sum \frac{(a+b)^3 J_0^3 \left[\frac{1}{2} k(a+b) \right]}{8b^3 \sinh^3 kl \cdot J_0'^3(kb)}$$

the summation being with reference to k , which is given by the roots of the equation

$$J_0(kb) = 0$$

Hence the values of kb are 2.404, 5.520, 8.657, etc. The upper limit for the force is obtained by putting $a=b$ in the above expression, when it becomes

$$\frac{1}{2} \sum [\sinh kl]^{-2}$$

But, the force is

$$F = \frac{dC}{dl}$$

the potential being constant and equal to unity. Hence,

$$\frac{dC}{dl} < \frac{1}{2} \sinh^{-2} k_1 l \quad (\text{approximately})$$

where $k_1 = \frac{2.404}{b}$. If $b = 7.25$,

$$l = 12, \text{ then } kl = 3.98$$

$$\therefore \frac{dC}{dl} < 0.0007$$

Hence, even though l should vary by 0.1 mm the capacity would be changed by but 7×10^{-6} cm, that is by but one ten-millionth part of the capacity of one section of the condenser.

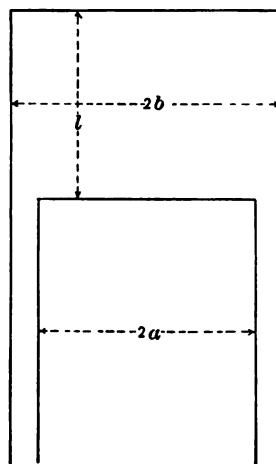


Fig. 21.

32. CORRECTION FOR THE GAP BETWEEN THE GUARD CYLINDER AND THE MIDDLE CYLINDER.

Here we must take into account the fact that the guard cylinder and the middle cylinder are not continuations of one another, but there is an offset where they come together. Proceeding as before, we shall now solve the problem for a condenser of planes, and shall treat the plates as though they were of infinite thickness. The diagram in the z plane is as shown in Fig. 22.

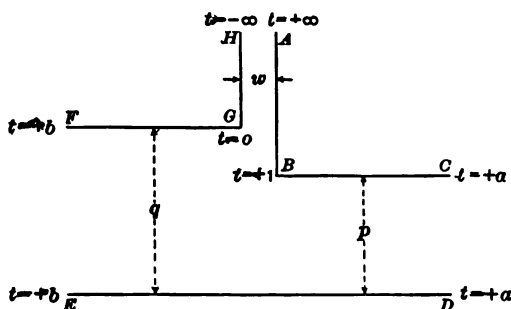


Fig. 22.

Assume $t = -\infty$ at $y = \infty$ on the HG face,

$t = 0$ at G,

$t = +b$ at $x = -\infty$ on the GF or the DE plane,

$t = +a$ at $x = +\infty$ on the BC or the DE plane,

$t = +1$ at B,

$t = +\infty$ at $y = \infty$ on the AB face.

The transformation is given by

$$\frac{dz}{dt} = C \frac{\sqrt{t(t-1)}}{(t-a)(t-b)}$$

$$\text{or, } \frac{dz}{dt} = iC \left[-1 - \frac{1}{a-b} \left\{ \frac{b(1-b)}{t-b} - \frac{a(1-a)}{t-a} \right\} \right]$$

$$\therefore z = -iC \sin^{-1}(2t-1)$$

$$- \frac{iC}{a-b} \left[\sqrt{a(1-a)} \log \frac{2\sqrt{t(1-t)}\sqrt{a(1-a)} + t(1-2a) + a}{2(t-a)\sqrt{a(1-a)}} - \sqrt{b(1-b)} \log \frac{2\sqrt{t(1-t)}\sqrt{b(1-b)} + t(1-2b) + b}{2(t-b)\sqrt{b(1-b)}} \right] + C_1$$

If we take B as origin, we have

$$0 = -iC \left[\frac{\pi}{2} + \frac{1}{a-b} \left\{ \sqrt{a(1-a)} \log \frac{1}{2\sqrt{a(1-a)}} - \sqrt{b(1-b)} \log \frac{1}{2\sqrt{b(1-b)}} \right\} \right] + C_1$$

$$\therefore z = -iC \left\{ \sin^{-1}(2t-1) - \frac{\pi}{2} \right\} - \frac{iC}{a-b} \left\{ \sqrt{a(1-a)} \log \frac{2\sqrt{t(1-t)}\sqrt{a(1-a)} + t(1-2a) + a}{t-a} - \sqrt{b(1-b)} \log \frac{2\sqrt{t(1-t)}\sqrt{b(1-b)} + t(1-2b) + b}{t-b} \right\}$$

As t decreases through the value a , z increases by $-ip$

$$\therefore -iC = \frac{p(a-b)}{\pi\sqrt{a(1-a)}}$$

Similarly, as t decreases through the value b , z increases by $+iq$

$$\therefore -iC = \frac{q(a-b)}{\pi\sqrt{b(1-b)}}$$

At G, $t=0$, $z = -w + i(q-p)$

$$\therefore -w + i(q-p) = i\pi C - \frac{\pi C}{a-b} \left\{ \sqrt{a(1-a)} - \sqrt{b(1-b)} \right\}$$

$$\therefore -iC = \frac{w}{\pi} = \frac{p(a-b)}{\pi\sqrt{a(1-a)}} = \frac{q(a-b)}{\pi\sqrt{b(1-b)}}$$

Solving these equations for a and b , we find

$$\begin{aligned} a &= \frac{1}{2} \left[\frac{\sqrt{[q^2 - p^2 + w^2]^2 + 4p^2 w^2} + [q^2 - p^2 + w^2]}{\sqrt{[q^2 - p^2 + w^2]^2 + 4p^2 w^2}} \right] \\ &= \frac{1}{2} \left[1 + \frac{q^2 - p^2 + w^2}{\sqrt{[q^2 - p^2 + w^2]^2 + 4p^2 w^2}} \right] = \frac{1}{2} \left[1 + \frac{q^2 - p^2 + w^2}{\sqrt{[q^2 + p^2 + w^2]^2 - 4q^2 p^2}} \right] \\ b &= \frac{1}{2} \left[\frac{\sqrt{[q^2 - p^2 - w^2]^2 + 4w^2 q^2} + [q^2 - p^2 - w^2]}{\sqrt{[q^2 - p^2 - w^2]^2 + 4w^2 q^2}} \right] \end{aligned}$$

$$= \frac{1}{2} \left[1 + \frac{q^2 - p^2 - u^2}{\sqrt{[q^2 - p^2 - u^2]^2 + 4u^2 q^2}} \right] = \frac{1}{2} \left[1 + \frac{q^2 - p^2 - u^2}{\sqrt{[q^2 + p^2 + u^2]^2 - 4q^2 p^2}} \right]$$

Substituting the values of iC in the expression for z we find

$$z = \left[\begin{array}{l} \frac{u}{\pi} \left\{ \sin^{-1}(2t-1) - \frac{\pi}{2} \right\} \\ + \frac{p}{\pi} \log \frac{2\sqrt{t(1-t)}\sqrt{a(1-a)} + t(1-2a) + a}{t-a} \\ - \frac{q}{\pi} \log \frac{2\sqrt{t(1-t)}\sqrt{b(1-b)} + t(1-2b) + b}{t-b} \end{array} \right]$$

The transformation from the w plane to the t plane is given by

$$\frac{dw}{dt} = \frac{B}{(t-a)(t-b)}$$

$$\therefore w = \phi + i\psi = \frac{V}{\pi} \left[\log \frac{t-b}{t-a} + i\pi \right]$$

Now the density at any point is

$$\begin{aligned} \sigma &= -\frac{1}{4\pi} \frac{d\psi}{dv} = -\frac{1}{4\pi} \frac{d\phi}{ds} \\ &= -\frac{1}{4\pi} \frac{d\phi}{dt} \cdot \frac{dt}{ds} \end{aligned}$$

$\frac{dt}{ds}$ is positive for all these surfaces.

Hence, the charge between t_1 and t_2 , $t_2 > t_1$, is

$$E = \frac{V}{4\pi^2} \log \left(\frac{t_1-b}{t_1-a} \cdot \frac{t_2-a}{t_2-b} \right)$$

The charge on AB is found by putting $t_1 = 1$, $t_2 = t$, a very large quantity, hence

$$E_1' = \frac{V}{4\pi^2} \left[\log \left(\frac{1-b}{1-a} \right) - \frac{a-b}{t} \right]$$

in the limit this becomes

$$E_1' = \frac{V}{4\pi^2} \log \frac{1-b}{1-a}$$

The charge on BC is found by putting $t_2 = 1$, $t_1 = t$, nearly equal to a , hence

$$E_1'' = \frac{V}{4\pi^2} \log \left[\frac{1-a}{1-b} \cdot \frac{a-b}{t-a} \right]$$

But, for t nearly equal to a , we have

$$z = \left[\frac{w}{\pi} \left\{ \sin^{-1}(2a-1) - \frac{\pi}{2} \right\} + \frac{p}{\pi} \log 4a(1-a) \right. \\ \left. - \frac{q}{\pi} \log \frac{2\sqrt{ab(1-a)(1-b)} + a(1-2b) + b}{a-b} + \frac{p}{\pi} \log \frac{1}{t-a} \right]$$

$$\therefore \log \frac{1}{t-a} = \frac{\pi}{p} \left[x + \frac{w}{\pi} \left\{ \frac{\pi}{2} - \sin^{-1}(2a-1) \right\} - \frac{p}{\pi} \log 4a(1-a) \right. \\ \left. + \frac{q}{\pi} \log \frac{2\sqrt{ab(1-a)(1-b)} + a(1-2b) + b}{a-b} \right]$$

$$\therefore E_1'' = \frac{V}{4\pi p} \left[x + \frac{w}{\pi} \left\{ \frac{\pi}{2} - \sin^{-1}(2a-1) \right\} - \frac{p}{\pi} \log \frac{4a(1-a)}{a-b} \right. \\ \left. + \frac{q}{\pi} \log \frac{2\sqrt{ab(1-a)(1-b)} + a(1-2b) + b}{a-b} \right]$$

Hence, the charge from A to a point on BC far from B is

$$E_1 = \frac{V}{4\pi p} [x + \delta l_1]$$

where,

$$\delta l_1 = \left[\frac{w}{\pi} \left\{ \frac{\pi}{2} - \sin^{-1}(2a-1) \right\} - \frac{p}{\pi} \log \frac{4a(1-a)}{a-b} \right. \\ \left. + \frac{q}{\pi} \log \frac{2\sqrt{ab(1-a)(1-b)} + a(1-2b) + b}{a-b} \right]$$

$$\text{or, } \delta l_1 = \left[\frac{w}{\pi} \left\{ \frac{\pi}{2} - \sin^{-1} \left(\frac{q^2 - p^2 + w^2}{\sqrt{[q^2 + p^2 + w^2]^2 - 4p^2 q^2}} \right) \right\} \right. \\ \left. - \frac{p}{\pi} \log \frac{4p^2}{\sqrt{[q^2 + p^2 + w^2]^2 - 4p^2 q^2}} \right. \\ \left. + \frac{q}{\pi} \log \frac{(p+q)^2 + w^2}{\sqrt{[p^2 + q^2 + w^2]^2 - 4p^2 q^2}} \right] \quad (23)$$

Similarly, the charge on HG is found by putting $t_1 = -\tau$, τ being very large, $t_2 = 0$

then

$$E_s' = \frac{V}{4\pi^2} \left[\log \frac{a}{b} - \frac{a-b}{\tau} \right]$$

In the limit this becomes

$$E_s' = \frac{V}{4\pi^2} \log \frac{a}{b}$$

The charge between G and a point near F is found by putting $t_1=0$, $t_2=t$, t nearly $=b$, then

$$E_s'' = \frac{V}{4\pi^2} \log \frac{b}{a} \cdot \frac{a-b}{b-t}$$

But, for t nearly equal to b , we have

$$x = \left[\begin{aligned} & \frac{w}{\pi} \left\{ \sin^{-1}(2b-1) - \frac{\pi}{2} \right\} \\ & + \frac{p}{\pi} \log \frac{2\sqrt{ab(1-a)(1-b)} + b(1-2a) + a}{a-b} \\ & - \frac{q}{\pi} \log 4b(1-b) - \frac{q}{\pi} \log \frac{1}{b-t} \end{aligned} \right]$$

$$\therefore \log \frac{1}{b-t} = \frac{\pi}{q} \left[\begin{aligned} & -x + \frac{w}{\pi} \left\{ \sin^{-1}(2b-1) - \frac{\pi}{2} \right\} \\ & + \frac{p}{\pi} \log \frac{2\sqrt{ab(1-a)(1-b)} + b(1-2a) + a}{a-b} \\ & - \frac{q}{\pi} \log 4b(1-b) \end{aligned} \right]$$

If X = distance from G to the point $(-x)$, $-x = X + w$

$$\therefore E_s'' = \frac{V}{4\pi q} \left[\begin{aligned} & X + \frac{w}{\pi} \left\{ \frac{\pi}{2} + \sin^{-1}(2b-1) \right\} \\ & + \frac{p}{\pi} \log \frac{2\sqrt{ab(1-a)(1-b)} + b(1-2a) + a}{a-b} \\ & - \frac{q}{\pi} \log \frac{4a(1-b)}{a-b} \end{aligned} \right]$$

Hence, charge from H to a point on FG at a great distance, X , from G is

$$E_s = \frac{V}{4\pi q} [X + \delta_s]$$

where

$$\begin{aligned} \delta l_2 &= \left[\frac{w}{\pi} \left\{ \frac{\pi}{2} + \sin^{-1}(2b-1) \right\} \right. \\ &\quad \left. + \frac{p}{\pi} \log \frac{2\sqrt{ab(1-a)(1-b)} + b(1-2a) + a}{a-b} - \frac{q}{\pi} \log \frac{4b(1-b)}{a-b} \right] \\ &= \left[\frac{w}{\pi} \left\{ \frac{\pi}{2} + \sin^{-1} \left(\frac{q^2 - p^2 - w^2}{\sqrt{[q^2 + p^2 + w^2]^2 - 4p^2q^2}} \right) \right\} \right. \\ &\quad \left. + \frac{p}{\pi} \log \frac{(p+q)^2 + w^2}{\sqrt{[p^2 + q^2 + w^2]^2 - 4p^2q^2}} \right. \\ &\quad \left. - \frac{q}{\pi} \log \frac{4q^2}{\sqrt{[p^2 + q^2 + w^2]^2 - 4p^2q^2}} \right] \quad (24) \end{aligned}$$

If we put $A \equiv \sqrt{[p^2 + q^2 + w^2]^2 - 4p^2q^2}$ and rearrange the terms, equations (23) and (24) can be put in the more simple form

$$\delta l = \left[\frac{w}{2} - \frac{w}{\pi} \sin^{-1} \frac{w^2 + q^2 - p^2}{A} + \frac{p}{\pi} \log \frac{(p+q)^2 + w^2}{4p^2} \right. \\ \left. + \frac{q-p}{\pi} \log \frac{(p+q)^2 + w^2}{A} \right] \quad (25)$$

with the understanding that δl is the correction for the portion of the plate corresponding to p ; nothing is implied as to the relative magnitudes of p and q .

33. NUMERICAL VALUES OF THE CORRECTION TERMS AND ERRORS.

1. **Cylinders without guards.**—Let us now apply these results to the cases that occur in our experimental work. We shall first consider the case in which the plates are of infinitesimal thickness.

(a) *At what distance from the end does the distribution of the charge on the opposed surfaces become uniform?*—Referring to equation (16), we find that, if t^2 is equal to, or greater than, the value defined by the expression

$$t^2 = a^2 \left[1 - \frac{10^{-5}}{a^2 \left(\frac{1}{a^2 - b^2} - \frac{1}{1 - a^2} \right) - 0.5} \right]$$

the density of the charge differs from its value at an infinite distance from the end by only 1 in 100,000.

Taking r as the radius of the inner cylinder = 6.25 cm and p as 1 cm we find

$$\begin{aligned}a^2 &= 0.6926 \\b^2 &= 0.4555 \\t^2 &= a^2 (1 - 5.85 \times 10^{-5}) \\\therefore t^2 &= 0.69256\end{aligned}$$

But from equation (13) we know

$$x = -\frac{r}{\pi} \log \frac{t^2}{b^2} - \frac{p}{\pi} \log \frac{a^2 - t^2}{a^2 - b^2} + \frac{r}{\pi} \frac{a^2(t^2 - b^2)}{b^2}$$

Hence, the distance beyond which the density differs from its value at infinity by an amount which is less than 1 in 100,000 is

$$x = 2.52 \text{ cm.}$$

Hence, the distribution on the opposing surfaces at the central section of a pair of cylinders 12 cm long can not differ by an appreciable amount from its value at the center of a pair of infinitely long cylinders.

(b) *Effect of charge on the interior of the inner cylinder.*—If the cylinders are uncapped, there will be a certain charge on the interior of the inner one, and the magnitude of this charge will increase as the length of the cylinder is increased. It is necessary now to form an estimate of this charge. Throughout the work the lower end of the inner cylinder was always electrically closed. Hence, as the cylinder is reduced in length, the charge on its inner wall will decrease, but the charge on the upper surface of the cap on the lower end will increase. That is, this cap tends to compensate for the removal of the further end of the cylinder, and, if the cylinder is very short, it will overcompensate. Hence, the increase in the total charge on the interior of the inner cylinder as its length is increased from 12 to 32 cm is less than the charge that lies upon a length of 20 cm of the interior, between 12 and 32 cm from the end of a semi-infinite cylindrical condenser. For a series of four thin plates we find, by a slight modification of equation (15) the charge on the interior and between x_1 and x_2 centimeters from the end to be

$$E = \frac{V}{4\pi^2} \cdot \frac{2b}{a} \left(\frac{a^2 - b^2}{a^2} \right)^{\frac{p}{2r}} e^{-\frac{a^2}{2}} \left\{ e^{-\frac{\pi x_1}{2r}} - e^{-\frac{\pi x_2}{2r}} \right\}$$

The charge per unit length upon the opposed surfaces is

$$\sigma_0 = \frac{V}{4\pi p}$$

$$\therefore \frac{E}{\sigma_0} = \frac{2bp}{\pi a} \left(\frac{a^2 - b^2}{a^2} \right)^{\frac{p}{2r}} e^{-\frac{a^2}{2}} \left\{ e^{-\frac{\pi x_1}{2r}} - e^{-\frac{\pi x_2}{2r}} \right\}$$

If $p=1$, $r=6.25$; then $a^2=0.6926$, $b^2=0.4555$; if we also take $x_1=12$, $x_2=32$ we find

$$\frac{E}{\sigma_0} = 0.0165 \text{ cm}$$

Hence, the variation of the charge upon the interior of the inner cylinder with the length of the cylinder, is probably inappreciable.

This conclusion is borne out by the results of experiments on the effect of capping the inner cylinder. These experiments showed that the increase in capacity, produced by capping the condenser by means of a cap fitting inside the inner cylinder and resting upon the stiffening flange near its upper end, when the condenser was 32 cm high, did not differ from that when it was 12 cm high by more than 0.0005 micro-microfarads.

(c) *Effect of the charge on the outside of the outer cylinder.*—In considering the system of four planes, we found that there is an appreciable charge on the outside of the outer planes even at great distances from the end. The same must be true for the cylinders. Hence, the shortening of the cylinders will tend to decrease the capacity of the remaining portion. This decrease will depend upon the magnitude of the charge originally upon the portion which has been removed, and also upon the distribution of the charge; the further the electrical center of this charge is from the end of the cylinder, the less will the removal of this portion of the outer cylinder affect the capacity of the remaining portion. Being unable to give at present a fair estimate of the actual amount by which the shortening of the cylinder by a given amount will decrease the capacity of the remaining portions, we shall limit ourselves to the calcula-

tion of the charge on sections of the outside of the outer plane at

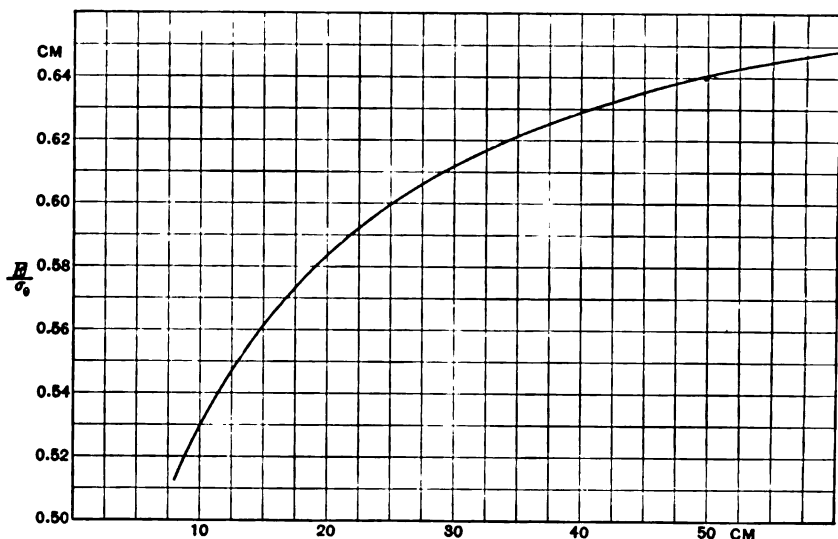


Fig. 23.—Curve Showing Distribution of Charge on Outside of Outer Cylinder.

different distances from the end. For this purpose we shall make use of equations (13) and (14), which, for convenience, we may rewrite as follows:

$$x = \frac{r}{\pi} \left[\frac{a^2(t^2 - b^2)}{b^2} - \log \frac{t^2}{b^2} \right] - \frac{\rho}{\pi} \log \frac{t^2 - a^2}{a^2 - b^2}$$

$$\frac{E}{\sigma_0} = \frac{\rho}{\pi} \left[\log \frac{1+a}{1-a} - \frac{2a}{t_1} - \frac{2}{3} \left(\frac{a}{t_1} \right)^3 \dots \right]$$

where t is greater than a . E is the total charge on a strip of the outside of the outer plane, of unit length normal to the plane of the paper, and extending from the edge of the plane to the point where t has the large value t_1 ; σ_0 is the density on the under surface of this plane and at a long distance from the edge. Taking $\rho = 1$, $r = 6.25$, we find $a^2 = 0.6926$, $b^2 = 0.4555$; and from these values we obtain the curve connecting the ratio $\frac{E}{\sigma_0}$ with x . This curve is given in

Fig. 23, and from it we find that the charge on the section extending from $x = 12$ to $x = 32$ is equivalent to the charge on a length of 0.72 mm of the opposed surfaces; this is equal to 36 parts in 10,000

of the whole charge on this section. Similarly the charge between $x=32$ and $x=52$ is equal to the charge on a length of 0.26 mm of the opposed surfaces; this equals 13 parts in 10,000 of the whole charge on this section. Hence, it is very possible that the charge on the end of the inner cylinder may be increased by 10 parts in 10,000 of the charge on a 20 cm section of the condenser when the height of the cylinder is increased from 12 cm to 32 cm.

(d) *Correction for different diameters.*—There is a further correction due to the fact that the ends of the various sections of the cylinders have slightly different diameters. This will cause the charges on the ends of the various sections to differ. To get an idea as to the magnitude of this variation when the outer cylinder is not capped, it is necessary to study the distribution of charge on the end of a semi-infinite thick plate midway between two equal semi-infinite plates of finite thickness. This problem, however, is more difficult than the one for which the outer planes are of infinitesimal thickness, and its result must lie between that given by the latter and that obtained from the problem of a semi-infinite thick plate midway between two infinite plates. This last problem has been solved,⁹ and applies approximately to the case in which the outer cylinder extends some distance beyond the ends of the inner one, and both are capped.

The expression for the excess over its true length of the effective length of a thick plate between two semi-infinite plates of infinitesimal thickness is given by equation (22). It is

$$\delta l_t = \frac{p}{\pi} \left[\frac{2r}{p} \log \frac{a + \sqrt{a^2 - b^2}}{b} + 2 \log \frac{a + \sqrt{a^2 - b^2}}{\sqrt{a^2 - b^2}} - \frac{a^2}{1 - a^2} \right]$$

where $2r$ is the thickness of the plate, $2(p+r)$ is the distance between the thin plates, and a and b are functions of the ratio of r to p .

With the same notation, the expression given by J. J. Thomson for this quantity, when the outermost plates are of infinite length, is

$$\delta l_r = \frac{p}{\pi} \left[\frac{r}{p} \log \frac{2p+r}{r} + 2 \log \frac{2p+r}{p} \right]$$

If we take $p=1$, $r=6.25$, the quantities a and b have the values $a^2=0.80340$, $b^2=0.68783$; while if $p=1$, $r=6.25$ (1.001), we find

⁹ J. J. Thomson, Recent Researches, § 237.

$a^2 = 0.80355$, $b^2 = 0.68804$. From these we find the values in the following table:

p	r	δl_1	δl_2
1	6.25	1.110	1.896
1	6.25 (1.001)	1.110	1.896

Hence, the quantity in the brackets varies but slowly as the ratio of r to p is changed. But the charge per unit length of the plate is inversely proportional to p , hence the additional charge added depends only upon the quantity in the brackets, and so is almost independent of p . Changing the ratio of p to r by so much as 1 in 1,000 will, in our case, change the effective length of the condenser by less than 10μ , which is only 5 parts in 100,000 of the length of one section.

This concludes the mathematical discussion of the problem of cylinders without guard cylinders; and it is evident that if both cylinders are not capped at both ends, there is a considerable error introduced by the variation in the charges on the ends of the inner cylinder, produced by the change in the distribution of the charge on the outside of the outer cylinder as the length of the condenser is varied. Even were both cylinders capped at both ends it would still be necessary to determine experimentally the effect, if any, of the varying height of the cylinder upon the capacity of the lead wire and commutator.

In this work, the lower end of the inner cylinder was always closed, but the lower end of the outer one was never closed; it was, however, only 17 cm above a large (50 x 70 cm) earthed sheet of metal, and the heavy brass ring carrying the feet of the condenser was separated from the lower end of the outer cylinder by but 2 cm of ebonite (the base of the instrument) and was also earthed. We think this adjustment very completely shielded the lower end of the inner cylinder from variations in charge produced by a change in the height of the outer cylinder. For heights above 24 cm this was tested experimentally as follows:

Sections No. 1 being in place, outer cylinder No. 4 was placed on

outer cylinder No. 1, and was closed with a brass plate. The speed being adjusted and held constant by an auxiliary observer, the bridge was exactly balanced. Then, leaving everything else as before, another outer section was placed on top of the brass plate closing No. 4. On repeated trial this was found to produce an inappreciable increase in capacity, certainly not greater than 1 in 200,000 of the maximum capacity of the condenser.

Unfortunately, similar tests can not be made for the upper end of the cylinder. We have, however, made a few runs in which the upper ends of both cylinders were capped. This should completely eliminate the effect of varying the length of the outer cylinder. This being eliminated the remaining errors which we have estimated that we still have to contend with are

$$\text{Lateral displacement} \dots \frac{\Delta_1 C}{C_0} < + 0.2 \times 10^{-6}$$

$$\text{Inclination of the axis} \dots \frac{\Delta_2 C}{C_0} < + 1.0 \times 10^{-6}$$

$$\text{Errors in figuring (estimated)} \frac{\Delta_3 C}{C_0} < + 20.0 \times 10^{-6}$$

$$\text{Offsets} \dots \frac{\Delta C}{C_0} < + 1.0 \times 10^{-6}$$

$$\begin{array}{l} \text{Variation in distances be-} \\ \text{tween the surfaces at the} \\ \text{ends of the sections} \dots \end{array} \frac{\Delta C}{C_0} < 50.0 \times 10^{-6}$$

$$\text{Variation in position of cap} \dots \frac{\Delta C}{C_0} < 1.0 \times 10^{-6}$$

$$\text{Total} \dots \frac{\Delta C}{C_0} < + 70 \times 10^{-6}$$

Hence, the actual capacity of the cylinders without guard cylinders, but closed at both ends, may exceed the calculated capacity by about 7 in 100,000. To these errors is to be added the uncertainty introduced by variations in the capacity of the commutator.

2. Cylinders with guards.—In this part of the work we are not concerned with the charge on the outside of the outer cylinder or upon the ends of the inner cylinder; and, each guard cylinder being electrically closed, there can be no charge on the inside of the inner cylinder. We have also seen (p. 518) that the irregular distribution of charge on the outer surface of the inner cylinder does not

extend beyond 2.52 cm from the end (to an accuracy of 1 in 100,000 in the density), hence there is no correction to be applied for this irregularity. Also, as we do not have to stop to build up the condenser, we can pass rapidly from a determination of the capacity of the condenser, leads, and commutator to that of the leads and commutator only, and back again. Thus, variations in the capacity of the commutator can produce but a slight effect in the result of even a single determination. The other corrections considered under the discussion of the correction for cylinders without guards apply here, and in addition we must determine the correction for the width of the gap between the guard cylinder and the portion of which the capacity is measured.

In equation (25) we have the expression for the correction for a gap in one electrode of an infinite plate condenser, the distance between the plates being different on the two sides of the gap. Whence we find that the charge on the plate on one side of the gap is

$$E = \frac{V}{4\pi p} (x + \delta l)$$

where

$$\delta l = \frac{w}{\pi} \left[\frac{\pi}{2} - \sin^{-1} \frac{w^2 + q^2 - p^2}{A} \right] + \frac{p}{\pi} \log \frac{(p+q)^2 + w^2}{4p^2} \\ + \frac{q-p}{\pi} \log \frac{(p+q)^2 + w^2}{A}$$

In this expression p is the distance between the plate under consideration and the plate ED (Fig. 22), q is the corresponding distance for the plate on the other side of the gap, w is the width of the gap, and $A \equiv \sqrt{(q^2 + p^2 + w^2)^2 - 4p^2q^2}$

If $q = p(1 + \beta)$, where β is a small quantity, we may write

$$\delta l = \frac{w}{\pi} \left[\frac{\pi}{2} - \sin^{-1} \frac{w^2 + q^2 - p^2}{A} \right] + \frac{p\beta}{\pi} \left[1 + \log \frac{(p+q)^2 + w^2}{A} - \frac{\beta^2}{4} + \frac{w^2}{4p^2} \right]$$

These expressions are obtained on the hypothesis that the face ED is a continuous plane. If, however, there is an offset in the opposite plane, as shown in Figs. 24, 26, or 27, the problem is more

complicated. If HGF represents the portion corresponding to the guard cylinder, then in the cases with which we are concerned the step B' is never in the plane AB. Now, since the gap correction depends almost entirely upon the rearrangement of the tubes of induction that would end upon the portion

GG' (Fig. 24) of the plate if the gap were absent, it is evident that in calculating the correction for the gap we must take as ρ , in our formulæ, the distance between BC and that plane which is cut by the prolongation of AB; q will then be $\rho + a_1$ where a_1 is the excess of the radius of the inner cylinder of the condenser proper over the radius of the guard cylinder. The correction thus obtained is given in terms of the capacity per unit of length of the condenser for which the distance between the plates is

ρ . If A'B' is on the guard cylinder side of AB, ρ will be the distance between the plates of the condenser the capacity of which we measure; if A'B' is on the other side of AB, ρ will be less than this distance by an amount a_2 (where a_2 is the excess of the radius of the outer cylinder of the condenser proper over the radius of the outer cylinder of the guard section).

The calculated correction in this case must be multiplied by $\left(1 + \frac{a_2}{\rho}\right)$

in order to express it in terms of the capacity of unit of length of the section BC.

Now in the actual cases that occur in this work A'B' is always displaced 0.06 cm from the plane of AB (0.06 cm is the width of the lower gap), except when the middle cylinder consists of one section only. Since the total correction due to the step A'B' is small, and this displacement is small, no appreciable error can be introduced by assuming that the actual effect of A'B' is the same as if there were no gap and the two steps were in the same plane. We shall make this assumption. For the upper gap the offsets were as shown in Fig. 24, $a_1 = 0.00107$ cm, $a_2 = 0.00612$ cm; distance between FG and EB' = 0.98099 cm; distance between BC and

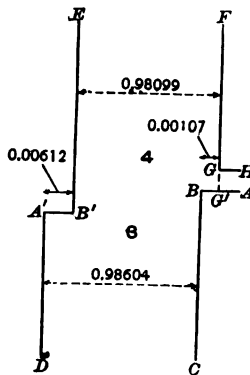


Fig. 24.

$A'D = 0.98604$ cm. Hence for the gap formula we must take

$$p = 0.97992$$

$$q = 0.98099$$

From these we obtain Table XII, in which w = width of gap; δl = increase of capacity produced by the gap, expressed in terms of the capacity of unit of length of a condenser for which the distance between the plates is 0.97992; $\delta l'$ is the product of δl by

$$\left(\frac{0.98371}{0.97992} \right) \frac{1}{40}$$

where 0.98371 is the mean of the mean distances between the plates for the two sections, and 40 is the sum of the lengths of the two sections; hence $\delta l'$ is the correction in terms of the capacity of the total condenser formed of section 2 and section 3; the fourth column contains the differences between the value of $\delta l'$ and the value of $\delta l'$ for $w=0$; the last column contains the product of the values in the preceding column by $\left(1 + \frac{1}{4} \frac{p}{r} \right)$.

TABLE XII.

Variation of Capacity with Width of Guard Gap.

w	δl	$\delta l'$	$\delta l' - \delta l'_0$	$(\delta l' - \delta l'_0) \left(1 + \frac{1}{4} \frac{p}{r} \right)$
0.00	0.00290	7.3×10^{-5}	0.0×10^{-5}	0.0×10^{-5}
.01	.00679	17.0 "	9.7 "	10.1 "
.02	.01150	28.8 "	21.5 "	21.4 "
.05	.02584	64.8 "	57.5 "	59.8 "
.10	.04941	124.0 "	116.7 "	121.3 "
.20	.09414	236.2 "	228.9 "	238.0 "
.30	.13615	341.7 "	334.4 "	347.6 "
.40	.17497	439.1 "	431.8 "	448.8 "
.50	.21072	528.8 "	521.5 "	542.1 "

The factor $\left(1 + \frac{1}{4} \frac{p}{r} \right)$ is the correction introduced by J. J. Thomson¹⁰

¹⁰ Phil. Trans., 181 A, p. 590; 1890.

on account of the curvature of the plates. The justification of this

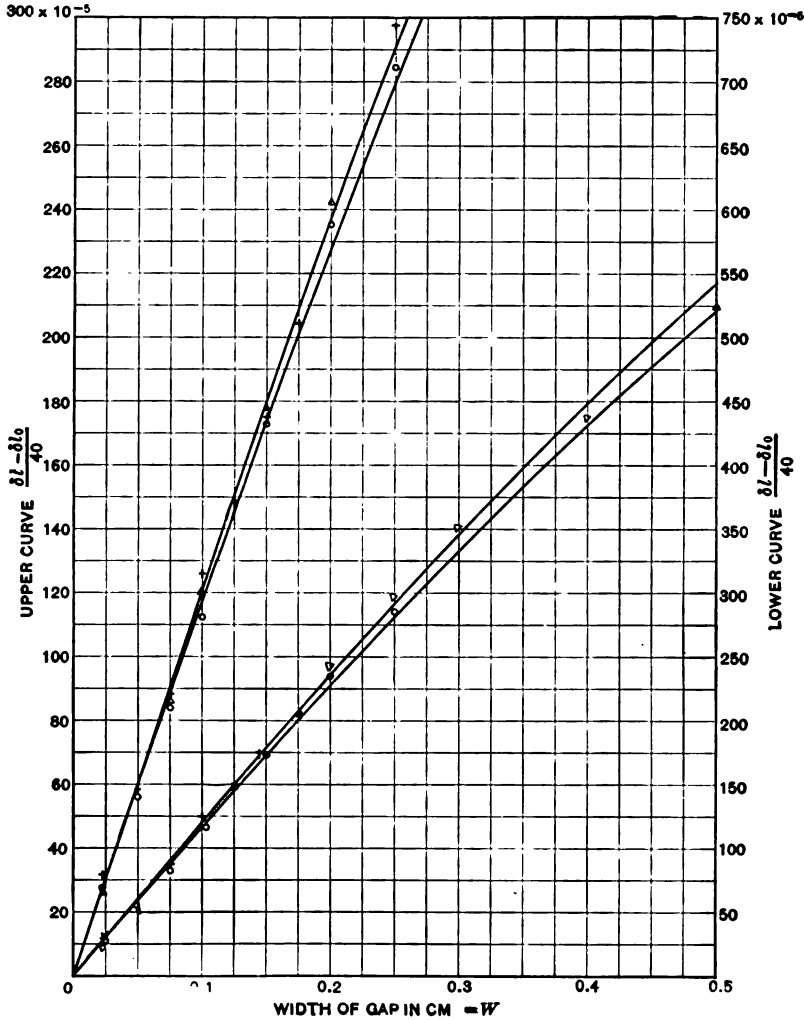


Fig. 25.—Curve Showing Variation of Capacity with Width of Guard Gap.

particular factor is not obvious. In Maxwell's Treatise, § 200, there is a similar factor of curvature deduced for a system of coaxial cylinders, *all of which* are at the same potential and placed end-on at a certain distance from a plane at a different potential ; and its justifi-

cation is based explicitly upon the fact that the electric force in the direction of the radius of curvature is negligible except near the edges of the ends of the cylinders. None of these conditions are fulfilled in the case which we are considering, so this deduction can not be appealed to in justifying the factor.

The variation of the capacity with the width of the gap was measured for gaps of 0.25 mm to 5.0 mm, and the results of three independent series of such observations are plotted in Fig. 25, and the curves determined by the figures in the last two columns of Table XII are drawn on the same plot. From this it is evident that the experimental results lie between the two curves, and, on the whole, rather nearer the lower curve than the one which contains the $\left(1 + \frac{1}{4} \frac{p}{r}\right)$ factor. The measurements are not of sufficient accuracy, however, to enable us to decide as to the correctness of the $\left(1 + \frac{1}{4} \frac{p}{r}\right)$ factor. For gaps of 0.5 mm the difference in the correction as given by the two expressions is only 2×10^{-8} of the total capacity we are measuring, and so is negligible; in this work we shall (for such small gaps) assume that the simpler expression is correct.

Except in one instance, the width of the upper gap was 0.05 cm during the entire course of the work; hence, the total correction, in terms of centimeters of length of section No. 3, for the upper gap, offsets, etc., is as follows:

Correction for 0.05 cm gap, opposite face plane

$$= 0.02584 \times \frac{0.9809}{0.9799} = 0.02587$$

Correction for offset in outer cylinder

$$= \left\{ \begin{array}{l} 0.21 \times 0.00612 \\ -0.32 \times 0.00612 \times 0.0017 \end{array} \right\} = 0.00128$$

Correction for elevation of upper end of inner section

No. 3 by 0.06 cm above top of outer section No. 3

$$= 0.06 \times \frac{0.00612}{0.986} \times 0.981 = 0.00036$$

Total end correction = 0.02751

If the condenser is composed of section No. 2 and section No. 3 its length will be 40 cm and the mean distance between the charged surfaces will be 0.98371 cm; hence, for this case the above correc-

tion will amount to $\frac{0.02751}{40} \times \frac{0.9837}{0.9809} = 69.0$ parts in 100,000 of the total.

If the condenser is composed of section No. 3 only, the correction will be $\frac{0.02751}{20} = 137.6$ parts in 100,000 of the total, if the lower gap is 0.06 cm; if the gap is 0.05 cm the correction will be $\frac{0.2745}{20} = 137.2$ parts in 100,000 of the total.

For the lower gap, also, two cases are to be considered, according as section No. 2 or section No. 3 adjoins section No. 1.

Here the gap width is 0.05 cm in the former case and 0.06 cm in the latter.

With section No. 2 next to section No. 1, the dimensions are as shown in Fig. 26. From this data we find the correction for a gap of 0.06 cm to be equivalent to a length of 0.03028 cm of a condenser for which the plate distance is 0.98793. Since the mean distance for section No. 2 is 0.98651 we find the correction in terms of centimeters of length of section No. 2 to be as follows:

Correction for 0.06 cm gap, opposite face plane

$$= 0.03028 \times \frac{0.9865}{0.9879} = 0.03024$$

Correction for offset in outer cylinder

$$\left\{ \begin{array}{l} 0.21 \times 0.00433 \\ + 0.32 \times 0.00433 \times 0.00074 \end{array} \right\} = 0.00091$$

$$\text{Total correction} \dots \dots \dots = 0.03115$$

If the condenser is composed of both section No 2 and section No. 3 this correction is equal to $\frac{0.03115}{40} \times \frac{0.9837}{0.9865} = 77.6$ parts in 100,000 of the total.

With section No. 3 next to section No. 1, the dimensions are as shown in Fig. 27. From this data we find the correction for a gap of 0.05 cm to be equivalent to a length of 0.02572 cm of a condenser for which the plate distance is 0.98351 cm. Since the mean

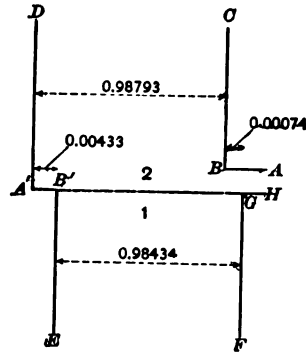


Fig. 26.

distance for section No. 3 is 0.98091 cm, we find the correction in terms of centimeters of length of section No. 3 to be as follows:

Correction for 0.05 cm gap, opposite face plane

$$= 0.02572 \times \frac{0.9809}{0.9835} = 0.02565$$

Correction for offset in outer cylinder

$$\left\{ \begin{array}{l} -0.21 \times 0.00067 \\ -0.32 \times 0.00067 \times 0.00016 \end{array} \right\} = -0.00014$$

$$\text{Total correction} \dots \dots \dots = 0.02551$$

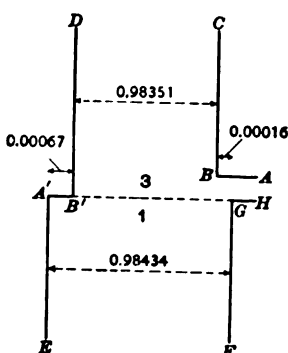


Fig. 27.

For this case the main condenser consists of section No. 3 only; hence, this correction amounts to 127.6 parts in 100,000 of the whole.

There are also offsets where section No. 2 and section No. 3 meet, but as shown on page 501 the effect of such small offsets occurring in the middle of a long condenser has the effect of increasing the effective length of the condenser by a negligible amount. Hence, when the guard cylinder condenser consists of section

section 2 and section 3, plus the guard cylinders, its electrostatic capacity in centimeters is the sum of the following quantities:

Sum of the two sections regarded as portions of infinite cylinders	= 137.004
Correction for upper gap and offset = $137.0 \times 69.0 \times 10^{-5}$	= 0.095
Correction for lower gap and offset = $137.0 \times 77.6 \times 10^{-5}$	= 0.106
Total capacity at 20° C	= 137.205

If the main condenser consists of section No. 3 only, its capacity is

Section No. 3 regarded as portion of an infinite cylinder	= 68.696
Correction for upper gap and offset = $68.70 \times 137.2 \times 10^{-5}$	= 0.094
Correction for lower gap and offset = $68.70 \times 127.6 \times 10^{-5}$	= 0.088
Total capacity at 20° C	= 68.878

VI. CORRECTIONS AND SOURCES OF ERROR—PLATE CONDENSER.

The corrections and sources of error that must be considered in the calculation of the electrostatic capacity of the plate condenser are as follows:

- (1) Stability, scale error, error in setting, temperature changes.
- (2) Inclination of the normal of the movable plate to the direction of motion.
- (3) Lack of parallelism of the plates.
- (4) Finite width of guard ring.
- (5) Correction for the width of the gap between the plate and the guard ring.
- (6) Displacement of the plate from the plane of the guard ring.

34. STABILITY, SCALE ERROR, ERROR IN SETTING, TEMPERATURE CHANGES.

By means of a silk thread passing through a hole in the top of the instrument, a 50 g weight can be lowered upon the collector plate of the instrument or lifted from it. It was found that the lowering of the weight upon the plate produced no appreciable change in the electromagnetic capacity. Also, by means of a Zeiss vertical comparator, the change in the position of the movable plate when a 100 g weight was placed upon it, was measured and found to be less than 0.1μ . Hence the stability of the plates is all that is required.

The position of the lower plate was determined by means of a silver scale attached to the vertical column carrying the lower plate, and read by a fixed microscope. This scale was graduated to 0.1 mm divisions, and was calibrated by the Division of Weights and Measures of this Bureau, and the corrections, so determined, have in every case been applied.

Every setting of the plate was so made that a millimeter division of the scale coincided with the zero setting of the microscope to within two microns. Then, five settings upon the line were made with the micrometer of the microscope, and the mean taken. The microscope was by Zeiss, and each division of the divided head corresponded to one micron. Except in rare instances the extreme range in any group of settings did not exceed 0.5μ , and was usually less. The microscope mounting was very rigid. A load of 100 g

on the eye-piece end of the microscope changed the reading by only 1.2μ ; on the removal of the load the reading returned to its original value to within less than 0.1μ . Hence, we think the settings are known to within 0.2μ at least.

The method of procedure adopted in the work with the plate condenser is based upon the assumption that the distance between the plates, for a given scale reading, is the same throughout each entire set of observations. For an instrument composed of brass, steel, and ebonite this necessitates very constant temperature conditions; and at first these conditions were not fulfilled. This rendered valueless the earlier observations. For the later work the condenser was inclosed in a glass case in which, by electrical heating, the temperature could be raised to one or two degrees above the room temperature. By frequent measurement of the resistances of two groups each of three coils of copper wire, distributed about the condenser, it was possible to detect slight changes in the temperature, and, by suitable hand regulation of the heating current, to maintain the temperature constant to within one or two tenths of a degree. This procedure sufficiently eliminates this source of error.

35. INCLINATION OF THE NORMAL OF THE MOVABLE PLATE TO THE DIRECTION OF MOTION.

The only error introduced by the inclination of the normal to the movable plate to the direction of motion is that due to the fact that the measured motion of the column carrying the plate is greater than the change in the distance between the plates in the ratio of one to the cosine of the angle of inclination. By the method of adjustment described on page 454 it is easy to determine this angle and to reduce it sufficiently to make this error negligible. If this angle is not more than one-fourth of one degree the error introduced will be only 1 in 100,000.

36. LACK OF PARALLELISM OF THE PLATES.

As we saw on page 487 (equation 7) the capacity of a rectangular section of length l , normal to the axis of inclination, and of breadth b of a pair of slightly inclined plates is

$$C = \frac{bl}{4\pi d_1} \left(1 + \frac{1}{3} \frac{\delta^2}{d_1^2} + \frac{1}{5} \frac{\delta^4}{d_1^4} + \dots \right)$$

where d_1 is the mean distance between the plates, and δ is half of the difference between the greatest and the least distances between the plates in this section. We may write this expression in the form

$$C = \frac{bl}{4\pi d_1'}$$

where
$$d_1' = d_1 \left[1 - \frac{1}{3} \frac{\delta^2}{d_1^2} - \frac{4}{45} \frac{\delta^4}{d_1^4} - \dots \right]$$

We shall call d_1' the effective distance between the plates.

If we now increase the distance between the plates until d_1 becomes d_2 , the effective distance will become

$$d_2' = d_2 \left[1 - \frac{1}{3} \frac{\delta^2}{d_2^2} - \frac{4}{45} \frac{\delta^4}{d_2^4} - \dots \right]$$

$$\therefore d_2' - d_1' = d_2 - d_1 + \frac{d_2 - d_1}{3 d_2 d_1} \delta^2 + \frac{4}{45} \frac{d_2^3 - d_1^3}{d_2^3 d_1^3} \delta^4 + \dots$$

$$= d_2 - d_1 + e$$

Where e is the quantity defined by the equation

$$e = \frac{(d_2 - d_1) \delta^2}{3 d_2 d_1} \left[1 + \frac{4}{15} \frac{d_2^3 + d_2 d_1 + d_1^3}{d_1^2 d_2^2} \delta^2 + \dots \right] \quad (26)$$

Hence, the effective distance that the plate has been moved is greater than the actual distance by the amount e .

By the method of procedure adopted in this work we measure d_1' and $d_2 - d_1$, and assume that $d_2' = d_1' + d_2 - d_1$. Hence the relative excess of the true value of the electrostatic capacity for the distance d_2 over its assumed value is

$$\frac{A_1 C}{C_0} = -\frac{e}{d_1'} = -\frac{d_2 - d_1}{3 d_2^2 d_1} \delta^2, \text{ approximately.} \quad (27)$$

It is evident, other things being equal, that this error is greater the smaller the initial distance, d_1 ; and for a given value of d_1 it is greater the smaller the value of d_2 , provided d_2 is not too nearly equal to d_1 .

In Table XIII we give the values of $\frac{A_1 C}{C_0} \cdot \frac{10^6}{\delta_\mu^2}$. To get $\frac{A_1 C}{C_0}$ for any value of δ , therefore, we multiply the tabular value of $\frac{A_1 C}{C_0} \cdot \frac{10^6}{\delta_\mu^2}$ by $\delta^2 10^{-6}$, expressing δ in microns. Thus if d_1 is 0.05 and $d_2 = 0.15$, then $\frac{A_1 C}{C_0}$ will amount to 0.0001 if $\delta = 18.4\mu$, which corresponds to an inclination of but $38''$. If δ were so much as 100μ the inclination would be but $3' 26''$, while the assumed capacity would be in error by nearly 3 parts in 1,000.

TABLE XIII.

Error Produced by Inclination of Plates.

$d_1 = 0.05$		$d_1 = 0.06$	
d_2	$\frac{\Delta_1 C}{C_0} \times \frac{10^6}{\delta^2 \mu}$	d_2	$\frac{\Delta_1 C}{C_0} \times \frac{10^6}{\delta^2 \mu}$
0.15	—0.296	0.16	—0.217
0.25	.213	0.26	.164
0.55	.110	0.56	.088
1.05	.060	1.06	.049
2.05	.032	2.06	.026
2.55	.026	2.56	.021

The importance of this adjustment was not recognized until the conclusion of the work, and the lack of proper care in this respect explains one of the most annoying characteristics of all the work on the plates, viz, the values of ∇ obtained for the shorter distances were almost invariably higher than the values found from the other condensers. If δ were 50μ , which might easily have been the case, the calculated electrostatic capacity for $d_1 = 0.15$ would be too high by 7.4 in 10,000 and the value found for ∇ would be too high by 3.7 in 10,000, which is of the same order of magnitude as the variations actually observed.

Furthermore, while the upper plate was ground until it was very truly plane, the movable plate had a surface which was appreciably cylindrical. This error in figuring would of itself introduce an error of the kind which we have just considered.

37. CORRECTION FOR THE FINITE WIDTH OF THE GUARD RING.

J. J. Thomson¹¹ has shown that the charge on a strip of unit length normal to the paper and extending from the edge C (Fig. 28) of one of a pair of semi-infinite parallel plates to a point P at a distance x , is (if we consider the opposed surfaces only)

$$E = -\frac{1}{2} \frac{V}{4\pi^2} \log(t+1)$$

¹¹ Recent Researches, § 235.

where the distance (x) from the edge is given by the expression

$$x = \frac{p}{2\pi} [t - \log (1+t)]$$

where t lies between -1 and 0 , and p is the distance between the plates. Putting in the last expression

$$t = -1(1 - 10^{-n})$$

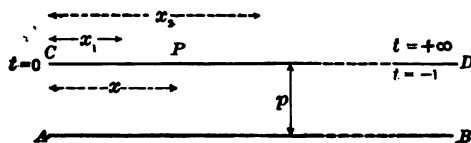


Fig. 28.

we find

$$\frac{2\pi x}{p} = 2.302 n + 10^{-n} - 1$$

Hence, if n is not less than 2, that is, if $\frac{2\pi x}{p}$ is not less than 3.614, the value of $\frac{2\pi x}{p}$ is given with an accuracy of at least 1 in 300 by the approximate formula

$$\frac{2\pi x}{p} = \log \left(\frac{1}{1+t} \right) - 1$$

$$\text{or, } t = -1 + e^{-\left(\frac{2\pi x}{p} + 1\right)}$$

In the actual case with which we are at present concerned, x is not less than 5 (the width of the guard ring) and p is not greater than 2.5, hence $\frac{2\pi x}{p}$ is not less than 12.56. Hence, as a second approximation we have

$$\log \frac{1}{1+t} = \frac{2\pi x}{p} + 1 - e^{-\left(\frac{2\pi x}{p} + 1\right)}$$

$$\therefore E = \frac{V}{4\pi p} \left[x + \frac{p}{2\pi} - \frac{p}{2\pi} e^{-\left(\frac{2\pi x}{p} + 1\right)} \right]$$

Hence, the charge on the strip included between $x=x_1$ and $x=x_2$ is

$$E = \frac{V}{4\pi p} \left[x_2 - x_1 + \frac{p}{2\pi} \left\{ e^{-\left(\frac{2\pi x_1}{p} + 1\right)} - e^{-\left(\frac{2\pi x_2}{p} + 1\right)} \right\} \right]$$

That is, the effective width of the strip is greater than its true width by an amount

$$\delta l = \frac{p}{2\pi} \left\{ e^{-\left(\frac{2\pi x_1}{p} + 1\right)} - e^{-\left(\frac{2\pi x_2}{p} + 1\right)} \right\}$$

The actual increase in the capacity of our circular plate will be less than that obtained by assuming that this entire excess of charge is situated on the extreme edge of the collector plate. Hence, the effect of the finite width of the guard ring is to increase the capacity of the plate by an amount which is less than

$$A_1 C = \frac{R \delta l}{2p} \left(1 + \frac{1}{4} \frac{p}{R} \right)$$

where the factor $\left(1 + \frac{1}{4} \frac{p}{R} \right)$ is J. J. Thomson's correction for curvature. Hence the relative increase in capacity will be less than

$$\frac{A_1 C}{C_0} = \frac{2 \delta l}{R} \left(1 + \frac{1}{4} \frac{p}{R} \right)$$

In the present case the width of the guard ring is 5 cm, R is 10 cm, hence

$$\begin{aligned} \frac{A_1 C}{C_0} &= \frac{p}{10\pi} \left\{ e^{-\left(\frac{10\pi}{p} + 1\right)} - e^{-\left(\frac{30\pi}{p} + 1\right)} \right\} \left(1 + \frac{p}{40} \right) \\ &= 0.0118 p \left\{ e^{-\frac{31.4}{p}} - e^{-\frac{94.2}{p}} \right\} \left(1 + 0.025 \frac{p}{10} \right) \end{aligned}$$

Giving p the greatest value used in this work (2.5 cm), we find

$$\frac{A_1 C}{C_0} = 1.4 \times 10^{-7}$$

Hence the width of the guard ring is more than sufficient for the highest possible accuracy.

38. CORRECTION FOR GUARD RING GAP AND FOR DISPLACEMENT OF THE PLATE FROM THE PLANE OF THE GUARD RING.

The capacity of a circular plate condenser is¹³

$$C = \frac{\left[R + \delta l \left(1 + \frac{1}{4} \frac{p}{R} \right) \right]^2}{4p}$$

Where R is the radius of the plate, p is the distance between the plates and δl is the edge correction when R is infinite.

On page 517 (equation 25) we found the following expression for δl for the case in which the distance between the plates is different on the two sides of the gap:

$$\begin{aligned} \delta l &= \frac{w}{2} + \frac{1}{\pi} \left[p \log \frac{(p+q)^2 + w^2}{4p^2} + (q-p) \log \frac{(p+q)^2 + w^2}{A} \right. \\ &\quad \left. - w \sin^{-1} \frac{w^2 + q^2 - p^2}{A} \right] \\ &= \frac{w}{2} + \Delta l \end{aligned}$$

where p is the distance between the plates on the collector plate side of the gap, q is the distance between the plates on the guard ring side of the gap, w is the width of the gap, and

$$A \equiv \sqrt{[p^2 + q^2 + w^2]^2 - 4p^2q^2}$$

For abbreviation we write

$$\Delta l \equiv \frac{1}{\pi} \left[p \log \frac{(p+q)^2 + w^2}{4p^2} + (q-p) \log \frac{(p+q)^2 + w^2}{A} - w \sin^{-1} \frac{w^2 + q^2 - p^2}{A} \right] \quad (28)$$

Hence,

$$C = \frac{\left(R + \frac{w}{2} \right)^2}{4p} \left\{ 1 + \frac{wp}{8R^2} + \frac{\Delta l}{R} \left(1 + \frac{1}{4} \frac{p}{R} \right) \right\}^2$$

¹³Maxwell, Electricity and Magnetism, Vol. I, § 201.

which may be written

$$C = \frac{\left(R + \frac{w}{2}\right)^2}{4d}$$

We shall call the quantity d the effective distance between the plates. If p_1 is the actual distance between the plates, the effective distance is

$$d_1 = p_1 \left\{ 1 - \frac{wp_1}{4R^2} - \frac{2(\Delta l)_1}{R} \left(1 + \frac{1}{4} \frac{p_1}{R} \right) \right\} \text{ approximately.}$$

Increasing the distance between the plates until it is equal to p_2

$$\begin{aligned} d_2 &= p_2 \left\{ 1 - \frac{wp_2}{4R^2} - \frac{2(\Delta l)_2}{R} \left(1 + \frac{1}{4} \frac{p_2}{R} \right) \right\} \\ \therefore d_2 - d_1 &= p_2 - p_1 + \left[\frac{2p_1(\Delta l)_1}{R} \left(1 + \frac{1}{4} \frac{p_1}{R} \right) + \frac{wp_1^2}{4R^2} \right. \\ &\quad \left. - \frac{2p_2(\Delta l)_2}{R} \left(1 + \frac{1}{4} \frac{p_2}{R} \right) - \frac{wp_2^2}{4R^2} \right] \end{aligned}$$

By the method of procedure adopted in this work we calculate a_1 , measure $p_2 - p_1$, and assume $d_2 = d_1 + p_2 - p_1$. Hence, the relative amount which must be added to the assumed capacity at the distance p_2 to make it equal to the true capacity is

$$\frac{\Delta_2 C}{C_0} = \left\{ \frac{2(\Delta l)_2}{R} \left(1 + \frac{1}{4} \frac{p_2}{R} \right) + \frac{wp_2}{4R^2} \right\} - \frac{p_1}{p_2} \left\{ \frac{2(\Delta l)_1}{R} \left(1 + \frac{1}{4} \frac{p_1}{R} \right) + \frac{wp_1}{4R^2} \right\} \quad (29)$$

We may write this in the form

$$\frac{\Delta_2 C}{C_0} = F(p_2) - \frac{p_1}{p_2} F(p_1)$$

where

$$F(p) \equiv \frac{2(\Delta l)}{R} \left(1 + \frac{1}{4} \frac{p}{R} \right) + \frac{wp}{4R^2}$$

If $(q-p)$ is not too great, we can expand $F(p)$ in ascending powers of $(q-p)$. Omitting all after the term of the first power, we have

$$F(p) = F_0(p) + (q-p) F'_0(p) + \dots$$

$$= \frac{2}{R\pi} \left[p \log \frac{4p^2 + w^2}{4p^2} - w \sin^{-1} \frac{w}{\sqrt{4p^2 + w^2}} \right] \left(1 + \frac{1}{4} \frac{p}{R} \right) + \frac{wp}{4R^2}$$

$$+ \frac{q-p}{R\pi} \left[\log \frac{4p^2 + w^2}{w^2} \right] \left(1 + \frac{1}{4} \frac{p}{R} \right)$$

and

$$\frac{A_2 C}{C_0} = F_0(p_1) - \frac{p_1}{p_2} F_0(p_1) + (q-p) \left[F'_0(p_1) - \frac{p_1}{p_2} F'_0(p_1) \right] \quad (30)$$

The values of $F_0(p)$ and of $F'_0(p)$ for several values of p , and for $w=0.0179$, and $R=10$ are given in Table XIV. These values apply directly to the results obtained from plate B.

TABLE XIV.

Correction for Displacement of Guard Ring.

p	$F_0(p)$	$F'_0(p) \times 10^{-4}$
0.05	-99×10^{-6}	11.1×10^{-6}
0.15	$-28 \quad "$	18.0 $"$
0.25	$-9 \quad "$	21.4 $"$
0.55	$+15 \quad "$	26.6 $"$
1.05	$+42 \quad "$	31.1 $"$
2.05	$+89 \quad "$	36.4 $"$
2.55	$+112 \quad "$	38.3 $"$

If $(q-p)$ is expressed in microns, the numbers in the third column are the values of $F'_0(p)$ to be substituted directly in equation 30. Hence, even if $(q-p)$ were so small as 5μ the assumed capacity for p_2 equal to 0.25 would be too small by 11 parts in 100,000, if p_1 were equal to 0.05; for p_2 equal to 2.55 it would be too small by 30 in 100,000.

An accuracy of adjustment approaching this was attained only in the last few determinations. For all the former work the accuracy of the adjustment is unknown and can not be satisfactorily determined; the relative displacement of the collector plate and the ring was probably large.

[The electromagnetic measurements and the concluding portion of this article will be given in the next number of this Bulletin].

WASHINGTON, May 20, 1907.

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(Continued from second page of cover.)

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47. On the Geometrical Mean Distances of Rectangular Areas and the Calculation of Self-Inductance. *E. B. Ross*
48. The Compensated Two-Circuit Electrodynamometer. *E. B. Ross*
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50. A Comparison of the Unit of Luminous Intensity of the United States with those of Germany, England, and France. *E. P. Hyde*
51. Geometrical Theory of Radiating Surfaces with Discussion of Light Tubes. *E. P. Hyde*
52. The Influence of Basic Lead Acetate on the Optical Rotation of Saccharose in Water Solution. *F. J. Bates and J. C. Smith*
53. On the Colorimetric Determination of Iron with Special Reference to Chemical Reagents. *H. N. Stokes and J. R. Cain*
54. On Sulphocyanic Acid. *H. N. Stokes and J. R. Cain*
55. Radiation from and Melting Points of Palladium and Platinum. *E. W. Wagner and G. K. Burgess*
56. The Mutual Inductance of a Circle and a Coaxial Single-Layer Coil—The Lorenz Apparatus and the Ayton-Jones Absolute Electrodynamometer. *E. B. Ross*
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59. The Mutual Inductance of Coaxial Solenoids. *E. B. Ross and L. Cohen*
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62. Melting Points of the Iron Group Elements by a New Radiation Method. *E. K. Buehler*
63. On the Determination of the Mean Horizontal Intensity of Incandescent Lamps. *Edmund P. Hyde and F. B. Cary*
64. Simultaneous Measurement of the Capacity and Power Factor of Condensers. *E. W. Genies*
65. A New Determination of the Ratio of the Electromagnetic to the Electrostatic Unit of Electricity. *E. B. Ross and N. B. Doner*

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- No. 13. Standard Specifications for the Purchase of Carbon-Flameless Incandescent Lamps.

MISCELLANEOUS

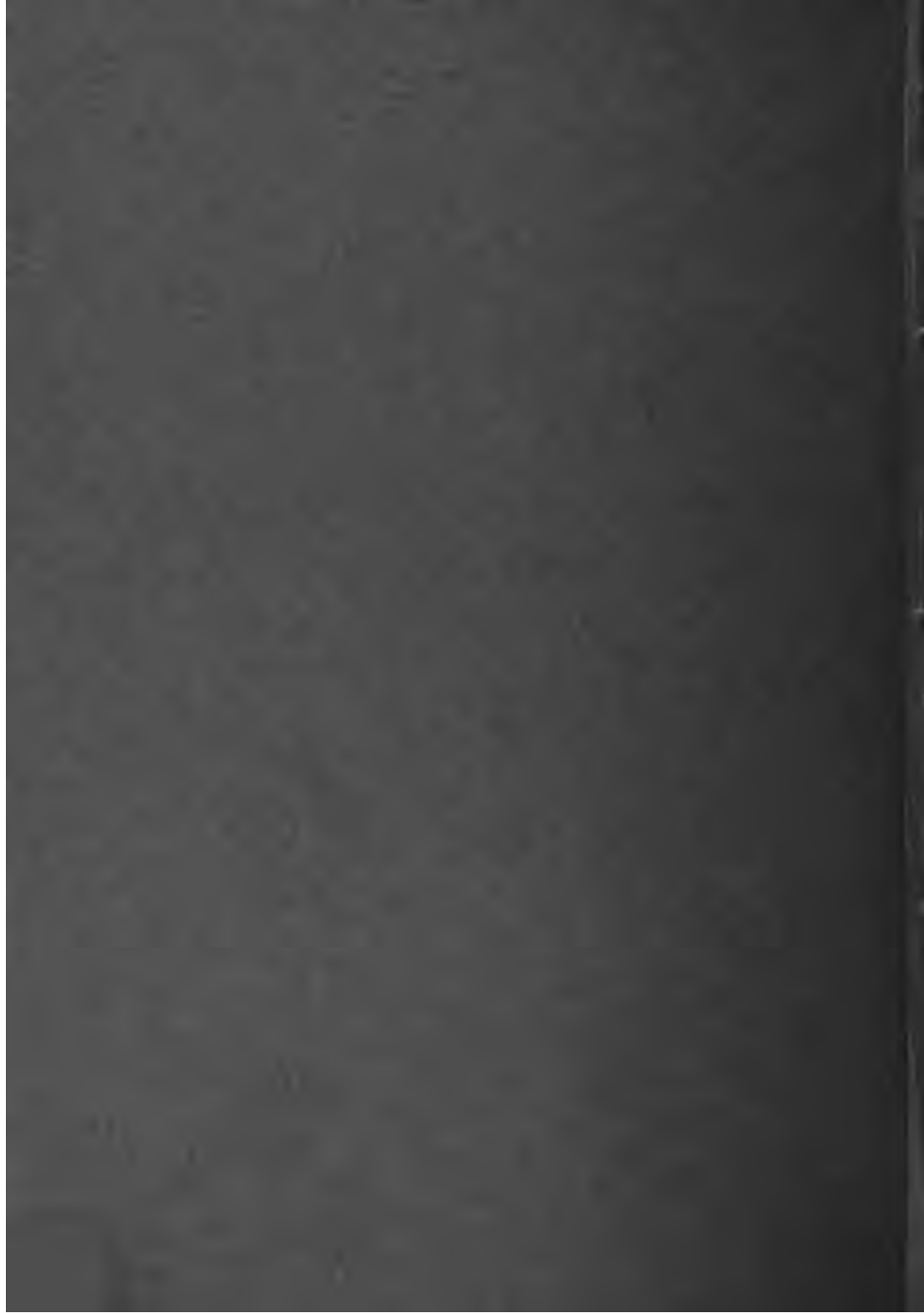
International Metric System (chart).

Table of the Equivalents of Customary and Metric Weights and Measures.

The International Metric System of Weights and Measures. (Pamphlet.)

First Conference on the Weights and Measures of the United States.

Second Annual Conference on the Weights and Measures of the United States.



DEPARTMENT OF COMMERCE AND LABOR

Vol. 3

BULLETIN

No. 4

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

OCTOBER, 1907



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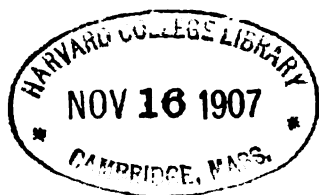
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The Bureau.

A NEW DETERMINATION OF THE RATIO OF THE ELECTROMAGNETIC TO THE ELECTROSTATIC UNIT OF ELECTRICITY.

[Continued.] *Vol. P 433*

By E. B. Rosa and N. E. Dorsey.

VII. DETERMINATION OF CAPACITIES IN ELECTROMAGNETIC MEASURE.¹³

In the electromagnetic measurement of the capacity two distinct methods were employed, viz: (1) The method described by Maxwell;¹⁴ (2) the differential galvanometer method. For the latter method connections were so made that we could readily measure either the charging or the discharging current.

39. MAXWELL'S BRIDGE METHOD.

As is well known this method consists in replacing one of the resistances in a Wheatstone bridge by a condenser and a suitable device for rapidly and regularly charging and discharging the condenser. Maxwell originally suggested a periodic reversal of the terminals of the condenser, but this is not so good, mechanically, as the periodic charging and discharging by touching one terminal of the condenser *P* (Figs. 29, 30) alternately to *R* and *S*. Then, by a suitable adjustment of the frequency of the charging, and of the magnitude of the resistances of the net, a balance of the galvanometer can be obtained. When this is done the relation between the electromagnetic capacity and the resistance will be ¹⁵

¹³ We are under obligations to Dr. F. W. Grover and Mr. H. D. Babcock for assistance in some of the experimental work, and to Mr. G. W. Vinal and Miss G. C. U. McDermut for assistance in preparing the results for publication.

¹⁴ Electricity and Magnetism, Vol. II, § 776.

¹⁵ J. J. Thomson, Phil. Trans., 174; 1884. Proc. Roy. Soc. 35, p. 346; 1883. Rosa and Grover, this Bulletin, 1, p. 153; 1905.

$$C = \frac{a \left[1 - \frac{a^2}{(a+c+g)(a+b+d)} \right]}{ncd \left[1 + \frac{ab}{c(a+b+d)} \right] \left[1 + \frac{ag}{d(a+c+g)} \right]} = \frac{aF}{ncd}$$

where

$$F \equiv \frac{1 - \frac{a^2}{(a+c+g)(a+b+d)}}{\left[1 + \frac{ab}{c(a+b+d)} \right] \left[1 + \frac{ag}{d(a+c+g)} \right]}$$

The letters stand for the values of the resistances (in international ohms) of the branches against which they are placed in Figs. 29, 30; n is the number of charges of the condenser per second. F is a correction factor differing but little from unity, in our work only a few parts in 100,000.

In the case of guard ring condensers it is necessary that the guard ring be charged to the same potential as the main condenser. This

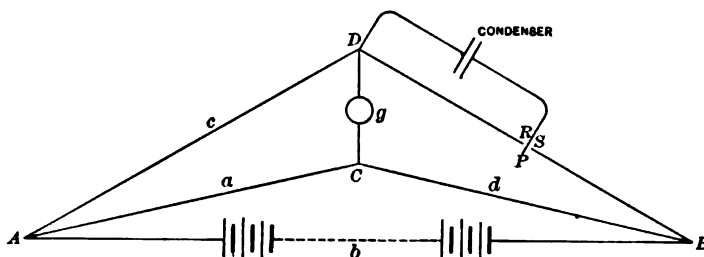


Fig. 29.—Maxwell Bridge for Measuring the Capacity of a Condenser.

The points P, R, S are joined to B_3 , B_2 , B_1 , respectively, of the commutator, Figs. 32 and 32a.

condition is readily obtained by constructing, as shown in Fig. 30, a second Wheatstone net exactly duplicating the main net and containing the same battery; for, if the two nets contain the same resistances branch for branch, the maximum potential of D' must be the same as the maximum potential of D , and this maximum potential is the potential to which the condenser and ring are charged, provided neither of the charging points P and P' breaks contact with the corresponding charging (or discharging) points R , R' (or S , S') before *both* condenser and ring are fully charged (or discharged). This condition is easily fulfilled. It is also necessary,

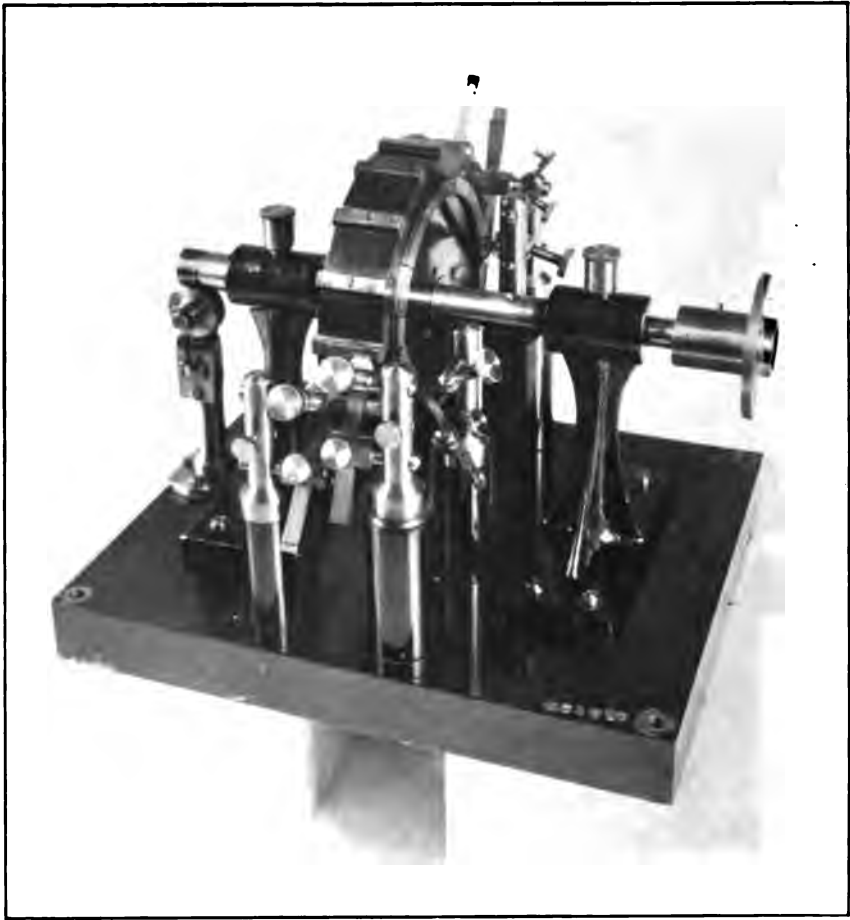


Fig. 31.—*Rotating Commutator.*

for exact work, for the continuous plate of the condenser to be connected to the point B instead of D.

The charging and discharging of the condenser is effected by means of a rotating commutator similar to the commutator described in this Bulletin, vol. 1, p. 172, except that there are sixteen contact pieces, and the ring of brass to which the contact pieces are connected is divided into sixteen segments T_1, T_2 , etc., insulated from one another so that but one segment and contact piece is charged at a time. This materially lessens the capacity of the commutator, and allows the guard ring of the condensers to be charged by the

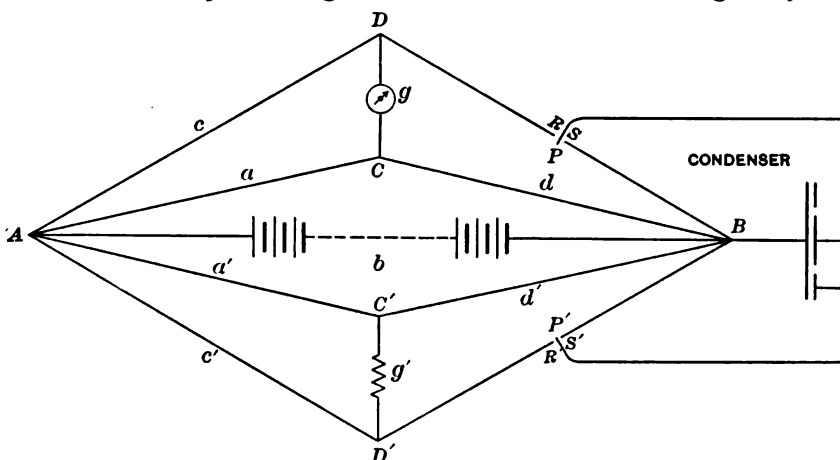


Fig. 30.—Double Maxwell Bridge, for Measuring the Capacity of a Guard Ring Condenser.

The corresponding arms $a, a'; c, c'$ etc., have identical resistances, and the potentials of D, D' , and C, C' are therefore the same. P', R', S' are joined to the second set of brushes of the commutator B_6, B_5, B_4 of Fig. 32a.

same commutator as that employed for the main condenser, the charging of the ring being effected by a second set of three brushes, B_4, B_5, B_6 , placed diametrically opposite those which serve for the main condenser. The insulation across the commutator was frequently tested and was always found to be as nearly perfect as could be desired; its resistance was of the order of a million megohms. The contact pieces S_1, S_2 , etc., of the commutator are of phosphor-bronze, and the brushes B_1, B_2 , etc., are of thin brush copper clamped between springs reaching to within about 4 mm of their tips; steadying screws with dampers reduced the vibration. As shown in Figs. 31, 32, and 32a, the phosphor-bronze segments project

beyond the surface of the ebonite disk of the commutator so that the tips of the brushes B_1 , B_2 , etc., are *air insulated*, except when in contact with the metal segments. The commutator was ordinarily driven at the rate of 1,200 to 1,700 revolutions per minute. A counter C made contact and recorded every fiftieth revolution on the chronograph.

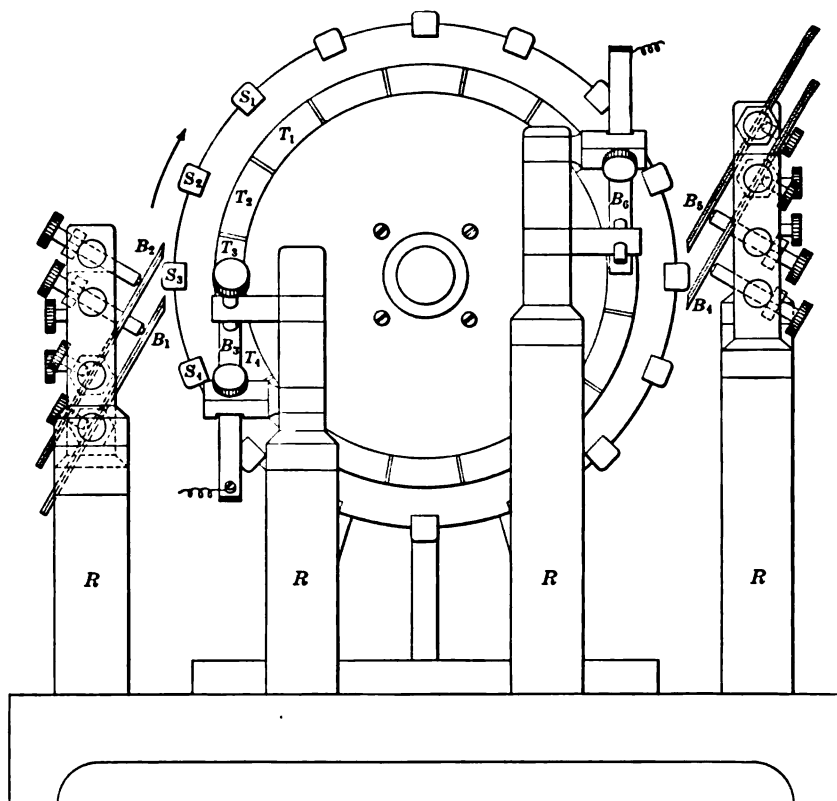


Fig. 32.—Rotating Commutator for Charging and Discharging the Condensers.

The six metal brushes are mounted on six ebonite columns, three on each side. The condenser is joined to B_3 , and then to T_3 and the segment S_3 when the commutator is in the position shown in the figure. The condenser is charged when S_3 touches the brush B_1 and discharged when it touches B_2 . It is charged again an instant later when S_4 touches B_1 , etc. The guard ring of the condenser is joined to B_6 , and is similarly charged and discharged by the brushes B_4 and B_5 simultaneously with the charging and discharging of the main condenser. The brushes are carefully adjusted so that the interval between charging and discharging shall coincide as nearly as possible for the condenser and its guard ring.

40. DIFFERENTIAL METHOD.

In this method the charging or the discharging current is passed through one coil of a differential galvanometer, while through the other coil is passed a steady current shunted off from a portion of a high resistance which permanently connects the terminals of the

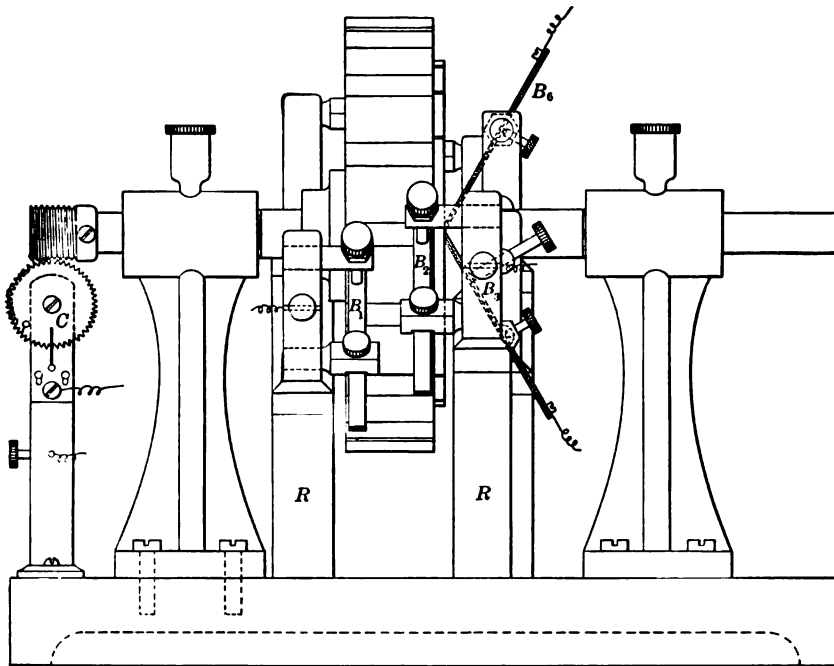


Fig. 32a.—Rotating Commutator; Side View.

battery employed to charge the condenser. The continuous lines in Fig. 33 show the connections in case we wish to send the discharge current through the galvanometer. If we wish to send the charging current through it, we disconnect BE and connect $B'E'$. If the condenser has a guard ring the connections for measuring the discharge current are shown by the continuous lines in Fig. 34. To measure the charging current we disconnect JK and connect $J'K'$; and interchange the connections at the posts S and R also at the

posts S' and R' . If as before we denote the resistances of the various branches by the small letters written near them, and call E the emf.

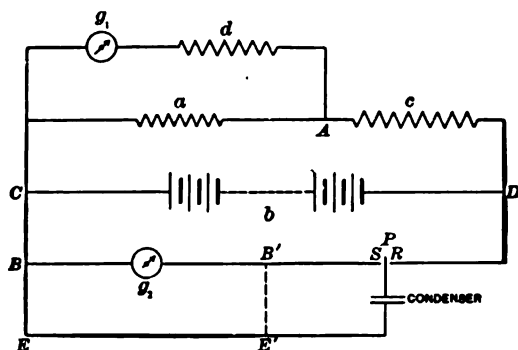


Fig. 33.—Arrangement of Resistances, Galvanometer Coils, etc., for Measuring the Capacity of a Condenser by the Differential Method.

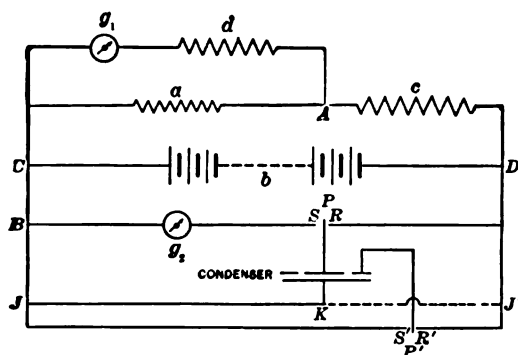


Fig. 34.—Arrangement of Resistances, Galvanometer Coils, etc., for Measuring the Capacity of a Guard Ring Condenser by the Differential Method.

Same as Fig. 33, except for connections of guard ring to second set of commutator brushes, etc.

of the battery, we find the following equations for the differential method:

$$\text{Current through AD} = i = \frac{E}{c + \frac{a(d+g_1)}{a+d+g_1}} = \frac{E(a+d+g_1)}{c(a+d+g_1) + a(d+g_1)}$$

$$\begin{aligned}\text{Current through } g_1 = i_1 &= \frac{ai}{a+d+g_1} = \frac{Ea}{cd \left[1 + \frac{a+g_1}{d} + \frac{a}{cd} (d+g_1) \right]} \\ &= \frac{Ea}{cd \left[1 + \frac{g_1}{d} + a \left(\frac{1}{d} + \frac{1}{c} + \frac{g_1}{cd} \right) \right]} \\ &= \frac{Ea}{cd [1 + A_1]}\end{aligned}$$

where

$$A_1 = \frac{g_1}{d} + a \left[\frac{1}{d} + \frac{1}{c} + \frac{g_1}{cd} \right]$$

The current through $g_2 = i_2 = nCE$, n being the number of charges per second. Hence for balance when $i_1 = i_2$

$$nC = \frac{a}{cd(1 + A_1)}$$

or

$$C = \frac{a}{ncd(1 + A_1)}$$

This formula is strictly analogous to the one obtained for the bridge method. However, the quantity A_1 is here not small, but may amount to several per cent of the whole. The galvanometer coils were shunted so as to make the system aperiodic and at the same time to make the coils magnetically equivalent, so that, when a considerably larger current than any used in measuring capacities is sent through the two coils in opposite directions, the deflection produced is too small to be observed. This equality test was made frequently. The resistances g_1 and g_2 are the effective resistances of the coils of the galvanometer with their respective shunts. The resistances g_1 and g_2 were measured whenever the galvanometer was used.

A great advantage of the differential method is that it gives a direct check against the insulation of the armatures of the condenser. If there is any appreciable leakage between the plates of the condenser, either through their supports or by way of the commutator and conductors leading to it, then the capacity of the condenser as measured by the use of the charging current must of necessity be greater

than the capacity as measured by means of the discharging current. No certain difference of this kind was ever noticed. This method, however, does not give a check against leakage from the brush S to

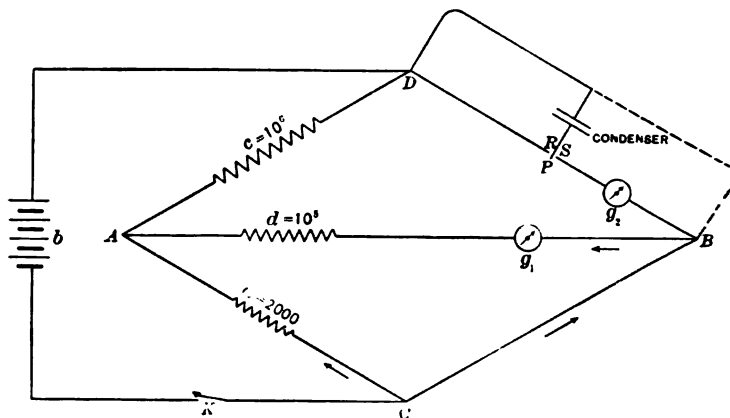


Fig. 35.—Arrangement of Resistances as in Fig. 33 for the Differential Method, Showing the Resemblance to the Maxwell Bridge.

The arm BC has zero resistance.

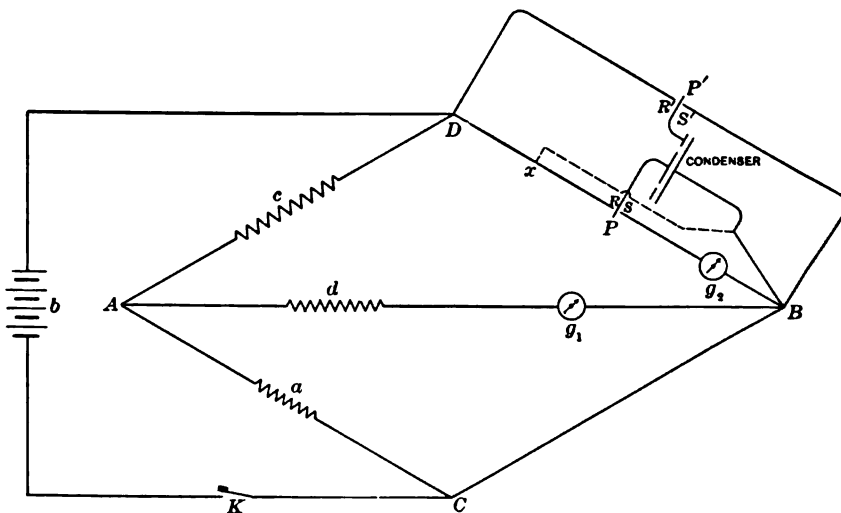


Fig. 36.—For Measuring Guard Ring Condenser by Differential Method.

Same arrangement of resistances as shown in Fig. 34.

the brush R. The resistance between these points was, however, frequently measured and was always of the order of a million megohms.

Another advantage of this method is that the resistance in the charge or discharge circuit is always rather small, being merely that of the galvanometer coil g_1 , which is about 80 ohms. It is the same whatever the capacity, so that the duration of contact of the brushes in this method might be shorter, if desired, than in the bridge method. However, they were always adjusted by the same criterion as described on page 552.

The connections for the differential method are also represented in Figs. 35 and 36, which are exactly equivalent to Figs. 33 and 34.

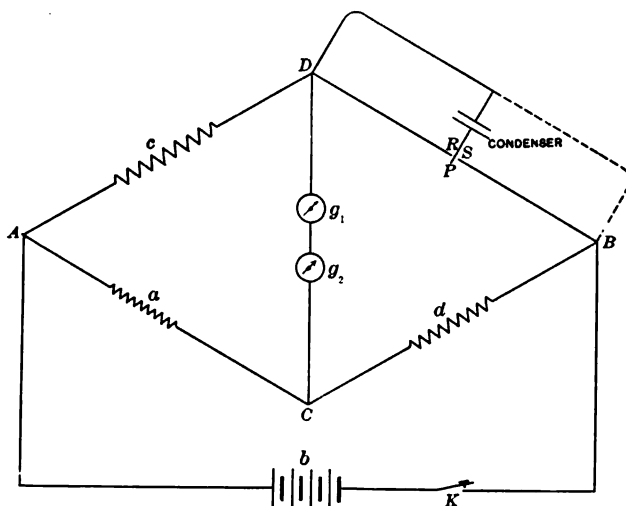


Fig. 37.—Showing how the Maxwell Bridge Arrangement is Obtained by Rearranging the Resistances and Galvanometer Coils of the Differential Method, Fig. 35.

To change the connections shown in Fig. 35 into the proper connection for the bridge method we have only to (1) interchange the battery and galvanometer (putting both halves of the galvanometer in series) and (2) put the resistance d in the arm CB (see Fig. 37). We now have a Maxwell bridge consisting of the same resistances, and it will be balanced when a small change is made in a , corresponding to the difference in the correction factors, F in the one case and $1 + A$, in the other. This change is so quickly made that it is very convenient to compare measurements by the two methods, especially in the case of the spherical condensers where, owing to the absence of a guard ring, a single bridge is used.

41. COMPLETENESS OF CHARGE.

If the duration of contact of the commutator brushes is not sufficient, the condenser will not be completely charged or completely discharged, hence, the measured capacity will be less than the true capacity in the ratio of $(1 - e^{-\frac{t_1}{CR_2}})$ to 1. Where t_1 is the duration of contact of the brushes, C is the capacity, and R_2 is the resistance of the network of conductors joining the points B and D of the bridge.¹⁶ If, however, d is very large (in our case it is 10^5 ohms),

then R_2 may be replaced by its approximate value¹⁷ $R'_2 = b + \frac{c(a+g)}{c+a+g}$

For the large sphere and the high resistance galvanometer $C = 5.5 \times 10^{-11}$ farads, $a < 3,500$ ohms, $b = 100$ ohms $c = 10^5$ ohms, $g = 1,000$ ohms, $\therefore R'_2 < 2.5 \times 10^{-7}$ seconds.

The highest speed ever reached during the work was 1,960 revolutions per minute, the circumference of the commutator over contact pieces is 40 cm, the width of each contact piece is 0.5 cm; hence, the time required for a contact piece to pass any given point is $t = \frac{0.5}{40} \times \frac{60}{1,960} = 3.8 \times 10^{-4}$ seconds. If the brush makes contact over

$\frac{1}{n}$ of the total width of a contact piece $t_1 = \frac{t}{n} = \frac{3.8}{n} \times 10^{-4}$

$$\text{Hence, } e^{-\frac{t_1}{CR_2}} = e^{-\frac{3.8 \times 10^{-4}}{2.5 \times 10^{-7}}} = e^{-\frac{1.5 \times 10^3}{n}}$$

If we allow an error of so much as 1 in 10,000 we can determine the maximum allowable value of n from the equation

$$e^{-\frac{1.5 \times 10^3}{n}} = 10^{-4}$$

$$\therefore n = 160$$

Hence, for this case the error due to this cause will amount to only 1 in 10,000 even when the brushes never make contact over more than $\frac{1}{160}$ of the total width of a commutator segment. By means of the tests with the milammeter, as described on page 552, it was estimated that at no time, when the action of the commutator was

¹⁶ See Maxwell, Elec. and Mag., Vol. II, § 777.

¹⁷ This Bulletin 1, p. 169; 1905.

regarded as good enough to give reliable results, did the brushes make contact over less than 0.75 of the width of the segment, and usually (owing to the finite length of the portion of the brush making contact) the contact extended over the entire width of the segment.

For the largest capacity practically obtainable with the plate condensers used, we have $C = 119 \times 10^{-11}$ farads, $a = 14,000$, $g = 1,000$, $c = 0.250 \times 10^6$, $b = 100$; $\therefore CR_s = 1.7 \times 10^{-5}$. Hence for an accuracy of 1 in 10,000 we must have n no greater than is given by the equation

$$e^{-\frac{3.8 \times 10^{-4}}{1.7n \times 10^{-5}}} = 10^{-4}$$

$$\therefore n = 2.44$$

Hence, even in this extreme case, which was never reached except in the preliminary work, the margin of safety would appear to have been sufficient.

All of this applies directly to the case in which the brushes are working smoothly and possess but little free vibration. This condition can always be obtained by properly adjusting the springs between which the copper brushes are held and the angles at which the segments meet the brushes. Hence for condensers without guard rings this correction can be made vanishingly small. Unfortunately, however, in the case of guard-ring condensers it is often impracticable to get *both* sets of brushes running smoothly simultaneously and still have them in proper phase relation. Under these conditions the brushes are set in rather violent free vibration, which renders the duration of contact uncertain and gradually bends the brushes so that the contact becomes worse and worse. This condition can be improved, but not completely eliminated, by placing soft rubber dampers on the backs of the brushes and extending nearly to their tips. These rubbers are gilded so as to eliminate any irregular creeping of electricity over them. Having got the brushes adjusted as well as possible, they are screwed down against the commutator until they seem to make as long a contact as is possible without harmful vibration. The bridge is then balanced, after which an extra resistance of 10,000 or 20,000 ohms is introduced seriatim in the branches DR, SB, BS', R'D'. If the balance

is not affected by this process we are justified in concluding that the brushes make sufficiently good contact. If the balance is affected by this process the brush or brushes at fault are readjusted. Having thus got the brushes into what is known to be a good adjustment, it is possible to obtain an empirical and more expeditious method of testing the functioning of the brushes. This consists in placing a milammeter and storage cell in series with P and R, D and B being disconnected from R and S, respectively. The reading of the milammeter gives a measure of the average duration of contact, and the constancy of its reading is an index to the regularity of the contact. Similarly the other three contacts are tested. After a little experience in this way under conditions which are known to be good, or bad, as the case may be, one learns what reading of the ammeter to expect under different conditions, after which it serves as a very ready and satisfactory method for determining the behavior of the brushes.

42. RESISTANCES.

The resistances employed were as follows: In the arm *a*, at various times were used, (1) a box by Otto Wolff, No. 2470; resistances range from 0.1 ohm to 50,000 ohms, the coils are of manganin, are wound noninductively upon metal spools, and were oil immersed. (2) An exactly similar box, No. 2471. The coils of these two boxes are shortcircuited, when desired, by conical plugs in the usual way. (3) A dial box, also by Wolff, No. 3087, having a range from 0.1 ohm to 10,000 ohms. These coils are similar to the coils of the other boxes, but were not oil immersed.

Arm *c* always consisted of one or other of two similar boxes by Siemens and Halske, No. 15657 and No. 16228. Each consists of a series of ten manganin coils, nominally of 100,000 ohms each. The coils are embedded in paraffin into which holes are drilled for the insertion of thermometers. The total resistance of No. 15657 is approximately 1,000,880 international ohms at 20° C, and that of No. 16228 is 1,001,180 international ohms at 20° C.

For the larger capacities the megohm in *c* has its two halves connected in parallel, making the resistance nominally 250,000 ohms.

Arm *d* consisted of one of the following: (1) A box by Otto Wolff, No. 2620, consisting of a series of ten manganin coils of approxi-

mately 10,000 ohms each. The coils are always oil immersed. (2)
An exactly similar box, No. 2621, also oil immersed.

Thus, the resistance of d is always nominally 100,000 ohms, and the resistance of c is always nominally 1,000,000 or 250,000 ohms.

The resistance of a is varied so as to balance the bridge.

These coils above 500 ohms were intercompared almost daily when in use, and comparisons were made at frequent intervals with the standards of this Bureau. The intercomparison was performed as follows: The 10,000 ohm combination, obtained by connecting the coils of the megohm in parallel, was compared with the 10,000 ohm combination obtained by connecting the first nine coils of the 100,000 ohm box in square array, three in series and three such groups in parallel; then the coils in the 100,000 ohm box were connected in parallel and the 1,000 ohms thus obtained was compared with the 1,000 ohm coil of the box in arm a . The other coils of a were compared with the 1,000 ohm coil in the usual manner. Thus, we get all of the resistances in terms of the 1,000 ohm coil in the box in arm a . This and other coils are frequently compared with the standards. This method of comparison is justified by the fact that all the coils in any one of the decade boxes considered have nearly the same value; the error thus introduced in the case of these boxes does not amount to more than 2 parts in 1,000,000.

43. STANDARDS OF RESISTANCE.

The standard resistance employed in this work is the mean value of the resistances of 11 one-ohm and 7 tenth-ohm coils belonging to this Bureau. Groups of these coils have recently and at various times in the past been compared with the standards of the Physikalisch-Technische Reichsanstalt, so that we feel confident that the value of this mean is known in terms of the international ohm with an accuracy in excess of that required for the present work.

44. SEASONAL CHANGES OF THE RESISTANCES.

By the process of intercomparison of the various coils and the frequent comparison of certain of them with the standards as just described, we obtain data from which, after proper temperature corrections have been applied, we are able to construct curves showing the relation between the corrections to be applied to the different

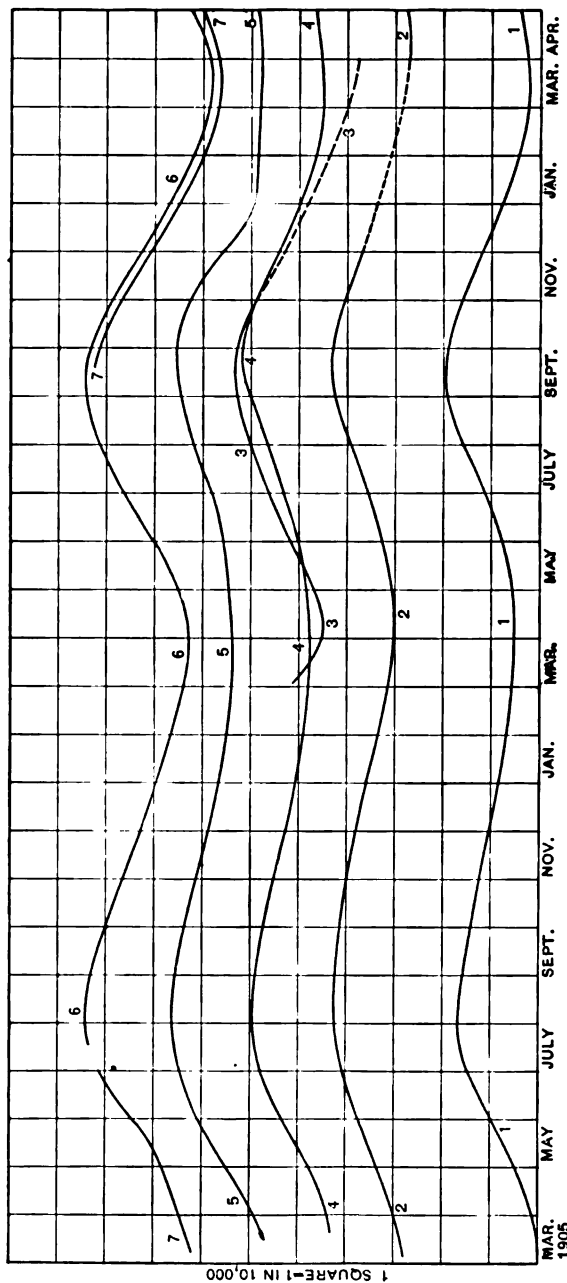


Fig. 38.—Showing the Seasonal Changes of the Resistances Employed in the Measurement of the Capacities of the Condensers.

The resistances were of manganin and were kept submerged in oil. All measurements after being reduced to a common temperature show maximum values in the summer and minimum in the winter. These changes have recently been found to be due to atmospheric moisture.

coils at some constant temperature (say 20° C) and the time. Doing this we obtain the very surprising and interesting group of curves shown in Fig. 38. The ordinates represent the correction in parts per 10,000 to be added to the nominal value of the coil in order to obtain its true resistance in terms of our standard ohm, the origin being arbitrary. The abscissæ represent the time, each division corresponding to one month. It will be noticed that every curve has a minimum between February and April, then rises gradually to a maximum in July and August, then decreases to a minimum again the following February and April. This unanimity is so much the more surprising when we consider the great difference in the coils under study. The behavior of the

1,000-ohm coil of No. 2470 is shown in curve No. 1;

1,000-ohm coil of No. 2471 is shown in curve No. 2;

1,000-ohm coil of No. 3087 is shown in curve No. 3;

Combination of (500+200+200+100) ohm coils of No. 2470 is shown in curve No. 4;

Combination of (500+200+200+100) ohm coils of No. 2471 is shown in curve No. 5;

Ten 10,000-ohm coils of No. 2620, connected in parallel, is shown in curve No. 6;

Ten 10,000-ohm coils of No. 2621, connected in parallel, is shown in curve No. 7.

Boxes No. 2470 and No. 2471 are probably of the same age and have been subjected to the same treatment, except that No. 2471 has been at constant humidity since December, 1906. The coils of these boxes have been in oil since February, 1905. Box No. 3087 is a new box of a different type from No. 2470. It was purchased in the winter of 1905-6; its coils are not oil immersed. Boxes No. 2620 and No. 2621 are probably of the same age and have undergone similar treatment; the coils are oil immersed. Thus four of the seven curves are entirely unrelated to one another. These are

I. Curves 1 and 2, behavior of certain 1,000-ohm coils in oil;

II. Curve 3, behavior of a different 1,000-ohm coil in air;

III. Curves 4 and 5, behavior of coils under 1,000 ohms in oil;

IV. Curves 6 and 7, behavior of coils of 10,000 ohms in oil.

Throughout this time the boxes have all been kept at laboratory temperatures which have fluctuated from about 15° to 30° C, though

the temperature was rarely under 19° or over 27° C; they have not been heavily loaded, and in no sense can their treatment, other than by the weather, be said to have been of a periodic character. The variations can not be due to leakage produced by moisture condensed upon portions of the box, for the resistances of the coils are highest during July and August, when the humidity in this region is the greatest; and it is much too great to be explained by an uncertainty in the temperature coefficients.

It is probably due to strains produced by the expansion of the shellac, in which the coils are embedded, as moisture is absorbed, the oil not protecting the coils completely, even when they are completely submerged. This question has been under investigation in this laboratory for some time and will form the subject of another paper. These curves emphasize the necessity of very frequent comparisons of resistances with the standards when an accuracy of 1 in 10,000, or better, is desired.

The temperature coefficients of the other coils employed were not determined, but they were almost daily compared at laboratory temperatures with the coils of which we know the coefficients. This enables us to correct our results for any variations in the resistances used.

45. GALVANOMETERS.

Two galvanometers, both of the D'Arsonval type, were used. One was a 1,000-ohm galvanometer by Sullivan. It had a sensibility of 1,000 mm, at 1 meter scale distance, per microampere. Its period is about nine seconds, damping on open circuit is very slight, and its zero is very constant. It was used at a scale distance of 2.5 meters, and was shunted by a noninductive resistance of 10,000 ohms. This was just sufficient to make the instrument aperiodic. Its inductance, including the leads, was found to be 66 millihenrys; hence its time constant is negligible as compared with the duration of contact of the commutator brushes.

The other galvanometer was a fine differential D'Arsonval made by the Weston Electrical Instrument Company. When the coils are connected in series, it has a sensibility equal to that of the Sullivan. Each coil has a resistance of about 82 ohms, its period is about five seconds, its damping is slight, and its zero is extraordinarily con-

stant. When each coil is shunted with 2,000 ohms the instrument is nearly aperiodic. The two coils were very nearly equal in their effects, and the two circuits were made equal by a proper adjustment of the shunts. The inductance of the coils is small, but was not measured.

The only remaining question to be considered is that of the division of the current between the galvanometer and its shunt. In deducing our formulæ we assume that the intermittent current from the condenser divides between the galvanometer and its shunt in the same proportion as a constant current does; that is, in inverse proportion to their resistances. We have seen that the time constants of the galvanometers are small as compared with the duration of contact of the commutator brushes, hence the intermittent current starts from zero, rises to a maximum, and falls to zero again. Hence the momentum of the field is the same at the end of the process as it is at the start; hence the integral over this period of the applied emf. must be equal to the integral over the same time of the total dissipative force in this branch. Now the only appreciable dissipative force is $i r$, where i is the current and r is the resistance. Hence

$$\begin{aligned}\int_0^T V dt &= \int_0^T i_1 r_1 dt = \int_0^T i_2 r_2 dt \\ &= Q_1 r_1 = Q_2 r_2\end{aligned}$$

Hence, the quantity of electricity carried is divided between the two circuits in the same proportion as in the case of constant currents.

Here it may be mentioned that at one time both galvanometers, each separately shunted, were connected in series and used. The high-resistance galvanometer had an inductance probably twenty times that of the low-resistance galvanometer. It was found that they were simultaneously balanced, which could not be the case if the division of the current between the galvanometer and the shunt depended on the inductance of the former.

46. REGULATION AND MEASUREMENT OF SPEED.

The commutator was driven by a small motor-generator set (0.5-kw motor and a 0.3-kw alternator), which was driven by the cur-

rent from a large storage battery. On each end of the motor-generator shaft was mounted a heavy fly wheel, and the alternator carried a load composed of two 16-candlepower lamps to steady the speed. In the armature circuit of the motor was a carbon rheostat by which the speed could be very delicately controlled. At other times the speed was regulated by gentle friction applied to the rim of one of the fly wheels on the shaft of the motor. The method of procedure has already been described in this Bulletin¹⁸. The resistance in the bridge having been suitably adjusted, the speed of the motor was varied until a balance was obtained, as indicated by the galvanometer; then a chronographic record of the speed was obtained while the rheostat was manipulated so as to keep the galvanometer continuously as near zero as possible. Under favorable conditions it is possible to hold the speed so nearly constant that the values found for it from successive three-minute intervals of the record will differ from one another by only a few parts in 100,000.

The time was given by one or other of two chronometers which were occasionally rated by comparison with the standard Riefler clock of this Bureau. The latter was daily compared with mean solar time, as determined at the Naval Observatory. The rates of the chronometers were found to be subject to a very slight linear change, an allowance for which is made in the reductions.

In the last part of the work another method of procedure was adopted, which is more expeditious than that just outlined. As stated above, the commutator was attached to one extremity of the axis of a small motor-generator set. To the other extremity of this axis was attached a second commutator making four contacts on each brush per revolution. This commutator, which necessarily has always the same speed as the other, together with a mica condenser (0.2 mf) formed one arm of an auxiliary Wheatstone bridge. By a proper adjustment of resistances the two bridges were balanced simultaneously for a certain speed of the machine. If the capacity of the air condenser is slightly changed (the resistance in the main bridge remaining unchanged), it will be necessary to change the speed, and consequently the resistance in one of the arms of the auxiliary bridge, in order to restore the simultaneous balance of the two bridges. Except for sign, the percentage change in speed,

¹⁸ Vol. I, p. 176.

which is equal to the percentage change in the capacity of the air condenser, is equal to the percentage change in the resistance in the auxiliary bridge, provided the changes are not too great. Hence, we may proceed as follows: With the machine running at a convenient speed, the auxiliary bridge is balanced as nearly as possible. Then the observer at the auxiliary bridge changes the speed of the motor slightly, so as to give an exact balance, and maintains the balance until a sufficiently long chronograph run has been made. This determines the speed which corresponds to the existing values of the resistances in the auxiliary bridge. While this is being done the observer at the main bridge determines the values of the resistances a_1 and a_2 in the arm a of his bridge, which will give the nearest possible balance (1) for the condenser, leads, and commutator, and (2) for leads and commutator without the condenser. Since one-tenth of an ohm is the smallest coil in arm a it is usually impossible to get an exact balance at this particular speed. Having obtained the chronograph run, the resistance of a is made a_1 and the observer at the main bridge slowly alters the resistance in the auxiliary bridge, thus causing the auxiliary observer to slowly change the speed of the machine so as to maintain a balance in his bridge; the main observer continues this process while continually watching his galvanometer, until the main bridge is also balanced. Thus a simultaneous balance is obtained, and the amount by which the resistance in the auxiliary bridge has been altered, together with the former chronograph run, gives us directly the speed at which the main bridge is balanced when a has the value of a_1 . Connections are now changed so as to measure the capacity of the leads and commutator only, a is made equal to a_2 , and the process is repeated. By this process a balance can be obtained in forty to sixty seconds, while two and a half or three minutes is necessary to obtain a chronograph record of suitable length; as a consequence we can pass much more rapidly from the measurement of the capacity of the condenser, commutator, and leads to the measurement of the capacity of the commutator and leads only, and back again. This reduces the chance of the commutator capacity differing appreciable in the two cases, and enables us to make in rapid succession more determinations of the capacity in each case, both of which are conducive to greater accuracy.

This method presupposes, of course, that the capacity in the auxiliary bridge remains constant, that all variations in the resistances in this bridge are known, and that the arm in which the resistance is varied has a resistance sufficiently great to give the required accuracy of setting.

If the condenser is a good mica condenser, and if its temperature is maintained constant, which can readily be done, its capacity will remain constant to the highest degree of accuracy attainable in the measurement of capacities. Added to this, however, we have the capacity of the leads and of the commutator. If the capacity of the condenser is sufficiently large, as is always the case, the variations of these small capacities will be absolutely negligible as compared with the total capacity, unless the commutator is acting very badly. This last condition is, however, easily detected and corrected.

The only thing that can cause unknown variations in the resistances during the course of a day's work is the heating by the current, which is made rather heavy in order to obtain the desired sensibility. This effect, however, is reduced by a sufficient subdivision of the higher resistances and by closing the battery circuit a little while before beginning observations. The remainder of this effect, as well as all other slight variations, may be eliminated by taking a suitable number of chronograph runs at intervals throughout the day's work.

The total resistance in the arm of the auxiliary bridge which was varied was 50,000 ohms, and usually any one setting could be determined to within 0.5 ohm when the air condenser was in the main bridge, and to 1 ohm when the commutator and leads only were being measured. The mean of from three to six independent settings was always taken for each determination of the capacity.

In order to avoid all personal bias, the resistance which was varied was divided into two parts; one was placed near the auxiliary observer and the other near the main observer. After each setting the auxiliary observer changed his portion of the resistance by an amount unknown to the main observer, who then proceeded to change his portion of the resistance so as to restore the balance. In this way every setting was independent of the preceding one.

As it is seldom that both galvanometers are simultaneously at rest, it is often desirable for the auxiliary observer to call out,

at intervals of about one second, whether the speed is fast, slow, or right, according to the indications of his galvanometer. This serves as a guide to the main observer in estimating the correctness of the balance of his bridge. If the galvanometer in the main bridge has a period that is considerably longer than that of the one in the auxiliary bridge it will integrate the current more perfectly and the process of balancing will be rendered easier.

47. THE NEW DIRECT READING CHRONOGRAPH.

In the earlier part of our work we used a chronograph¹⁹ driven electrically and making one revolution of the drum per minute. By means of an electric contact on the shaft of the commutator every fiftieth revolution of the commutator was recorded on the chronograph, and a simultaneous record of the second-beats of a chronometer gave a means of computing with great precision the mean speed of the motor and commutator for any three-minute interval. The great labor of reading the chronograph record and computing the speed, together with the impossibility of detecting small variations in the speed until after the record had been worked up, led us to design and build a new form of chronograph which has been of great value in this and other work where the speed of a machine or the frequency of a current must be known with high precision.

The chronograph is shown in Fig. 39, and the gears and connections to the driving motor are shown in Fig. 40. The chronograph is driven through the worm and bevel gears at a speed 250 times less than the motor, which also drives the commutator and an alternating generator which provides testing current for other purposes, the frequency of which current has sometimes to be determined. The usual speed of the machine was 1,500 revolutions per minute, and the drum of the chronograph therefore made 6 revolutions per minute, or one in ten seconds. This speed could, however, be varied through wide limits, in practice from 500 to 2,000 or more per minute.

A printing magnet²⁰ was used instead of a pen, and once every second it printed (through a typewriter ribbon supported on posts P_1 and P_2) a dot on the paper on the drum D of the chronograph.

¹⁹ By the Societ  Genevoise.

²⁰ Such as used on the Rosa curve tracer.

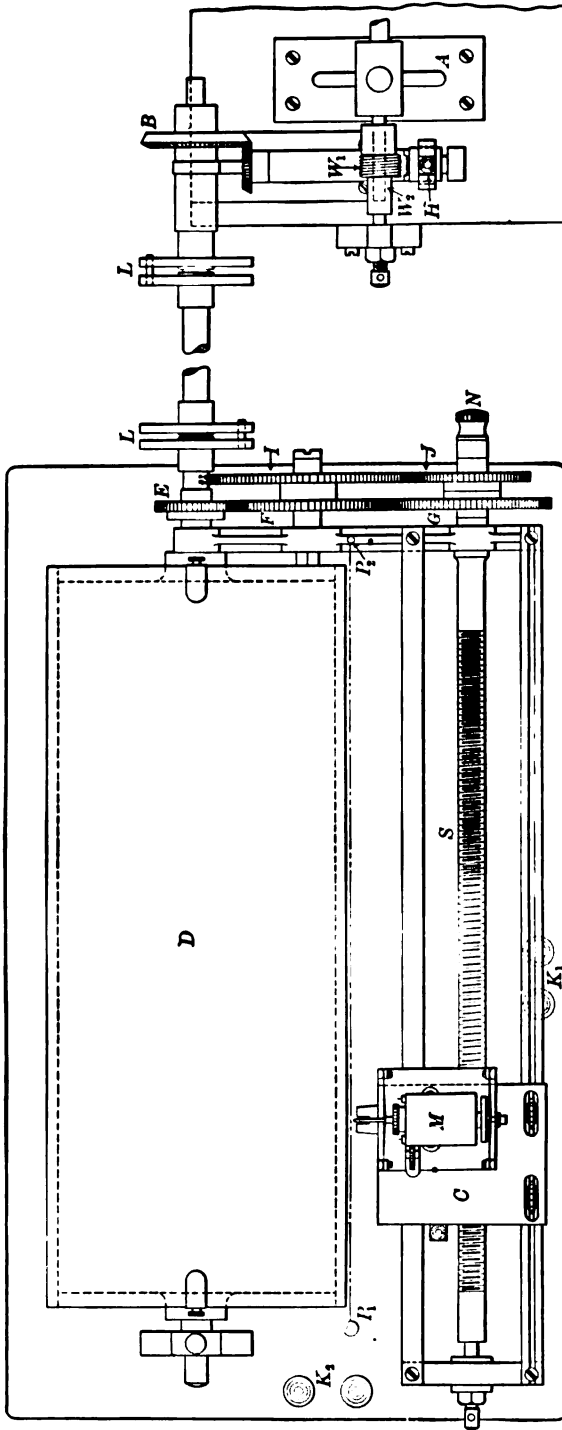


Fig. 40.—Direct Reading Chronograph for Giving the Speed of a Generator, Commutator, or Other Machine.

The chronograph is driven by the machine, the speed of which is to be measured, by means of the worm and gear W_1 and W_2 and the flexible coupling L . The bevel gears B are used merely to keep the direction of the axis of the chronograph parallel to that of the driving machine. The clutch N permits the speed of the driving screws to be changed. The printing magnet M (mounted on the carriage C) prints through the type ribbon R upon the paper wrapped over the drum D . The binding posts K_1 are joined by flexible wires to the printing magnet, and the binding posts K_2 (connected to K_1 in the base) are connected to the battery and chronometer. A dot is printed once a second on the revolving drum.

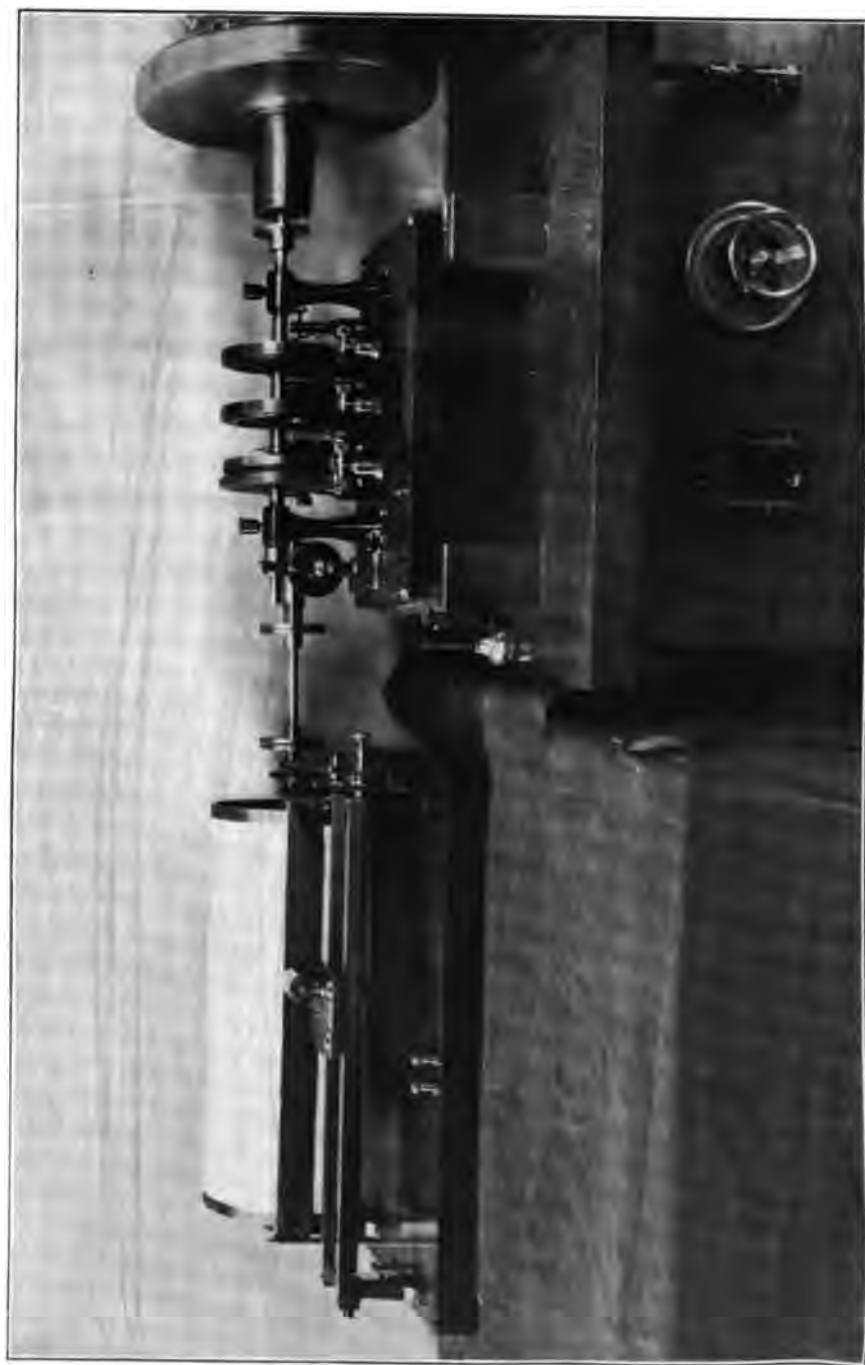


Fig. 39.—*Direct Reading Chronograph.*

made in n seconds, the surface speed is $50/n$ cm per second. The distance $d = BC$ therefore represents a lag of the cylinder of $\Delta t = \frac{dn}{50}$ sec. Let t be the time in which the given number of revolutions have been made. As the carriage advances 1 mm with every revolution of the drum

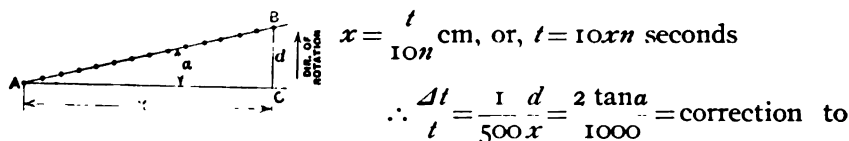
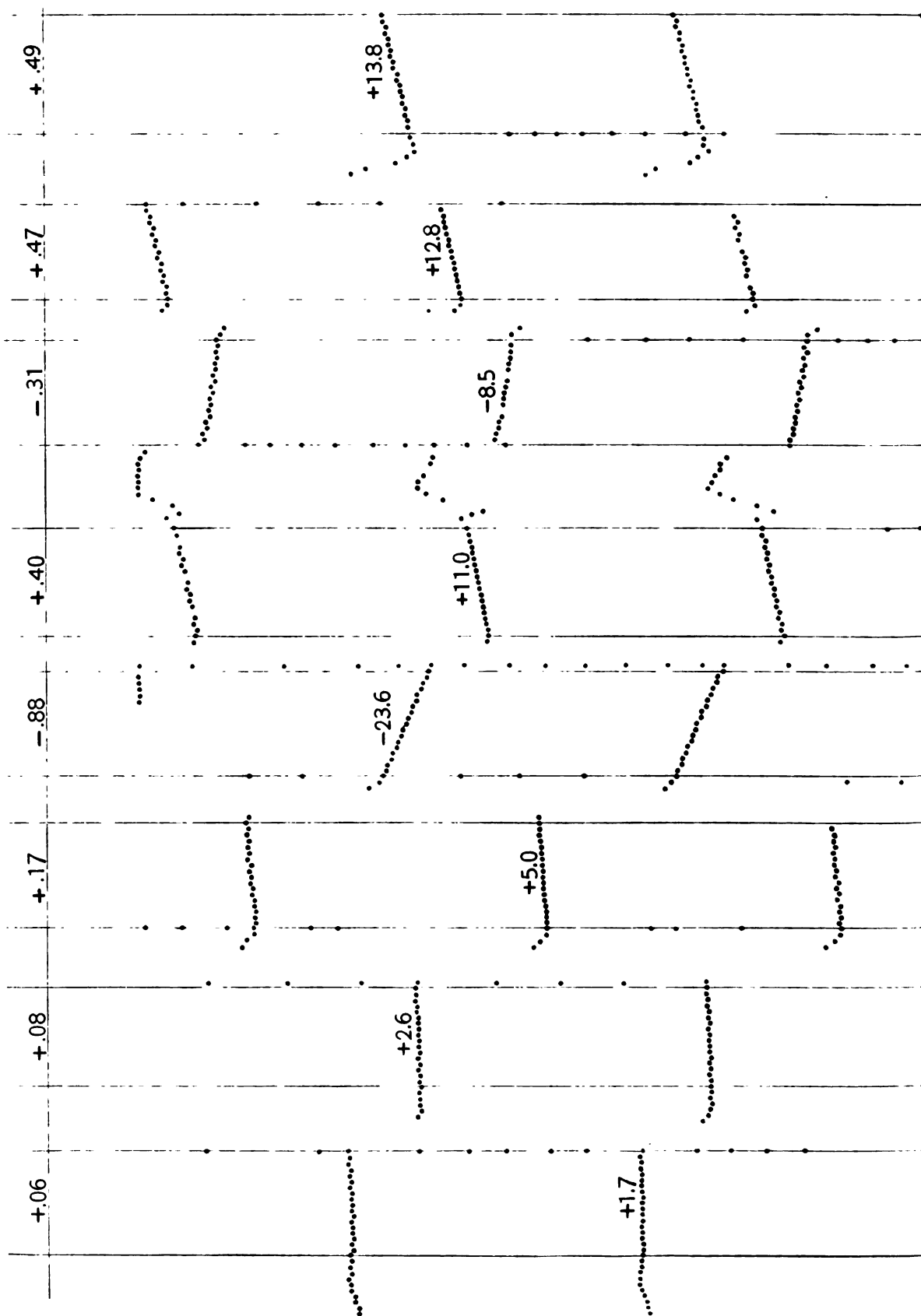


Fig. 41.

be applied to the nominal speed to obtain the actual speed for the given interval. That is, *the correction in thousandths is twice the tangent of a* . For $a = 1^\circ$, $2 \tan a = 0.035$, and the correction is 3.5 in 100,000. As several independent rows of dots are available for the determination of a , the error in a need not be as much as one-fourth degree when the speed is steady for a run of two or three minutes. Hence the uncertainty in the speed is less than 1 in 100,000. For longer runs the uncertainty may be still less.

Fig. 42 is a full-size reproduction of a portion of a chronograph record taken October 9, 1906. It shows 8 runs of about three minutes each, between which are records of somewhat variable speeds. At the top of the sheet is given for each run the mean amount, in parts per thousand, by which the actual speed of the machine falls short of 1,500 revolutions per minute. As explained above, these corrections are equal to the doubled tangents of the angles of slope. For example, the first run on the sheet has a mean slope of 1.8° and the correction amounts to 6 parts in 100,000. The next figure (Fig. 43) is a reduced representation of an entire chronograph sheet, which was made in comparing some condensers by alternating current, the frequency being 100 at a speed of 1,500 of the machine. The corrected frequencies for the different runs are given at the top of the sheet. In this reproduction the tails, frequently produced when the duration of contact at the relay is excessive, have been unduly magnified, and the location of the points have not been faithfully reproduced. Further than this, it gives a fair idea of the general appearance of the sheets as taken.



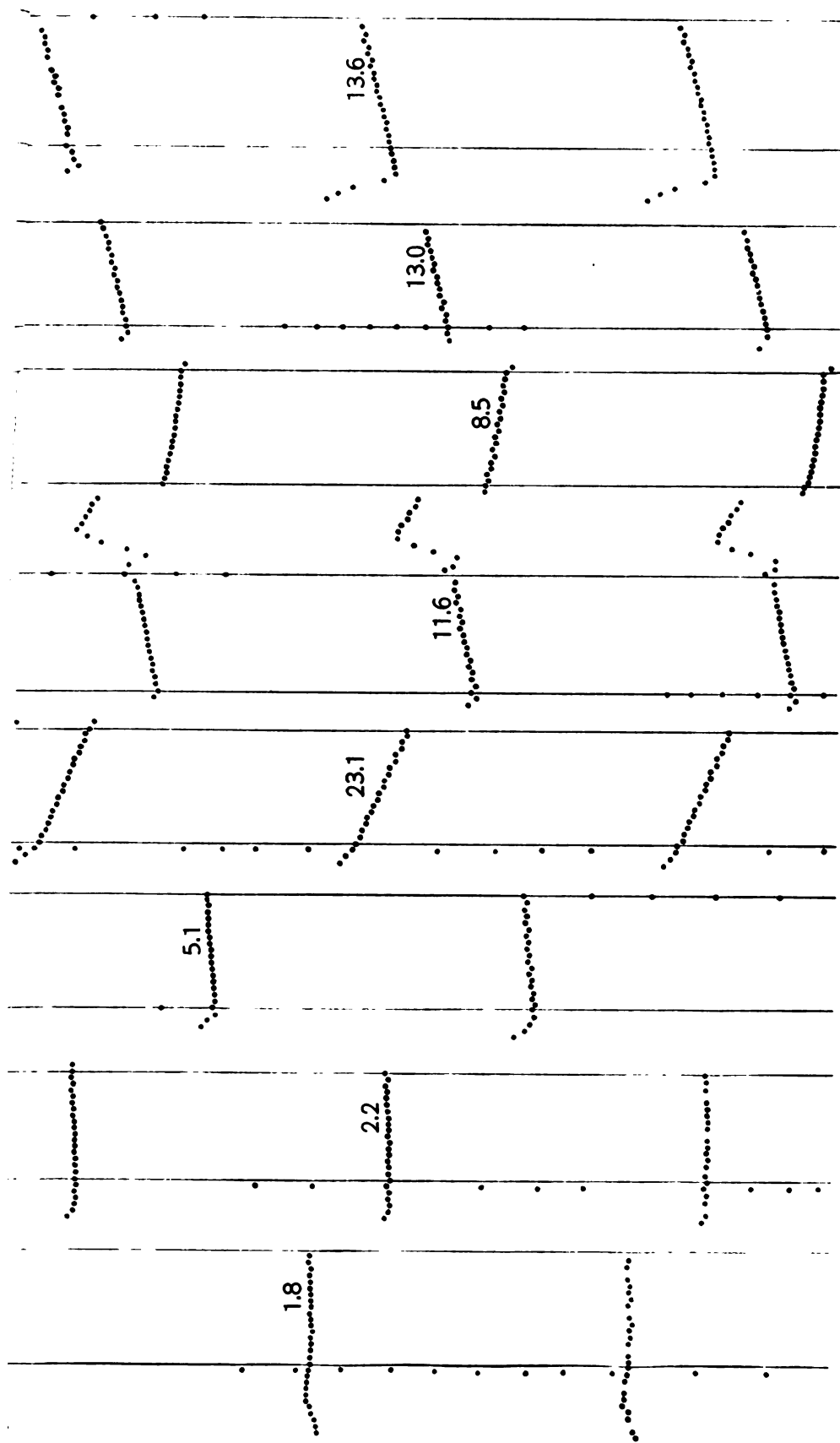


Fig. 42.—*Portion of a Chronograph Record of October 9, 1906. Full size.*

In the last run the upward slope indicates that the speed was slightly slower than the nominal speed, and the capacity being measured therefore a little greater than it would have been if the bridge had been balanced at the nominal speed. The corrections to the capacity are therefore positive when the speed is slower, and negative when faster than the nominal. The numbers on the rows of dots are the angular slopes in degrees as measured by a protractor, the mean corrections to the capacities are given at the top. The correction for the last run is 49 parts in 100,000.

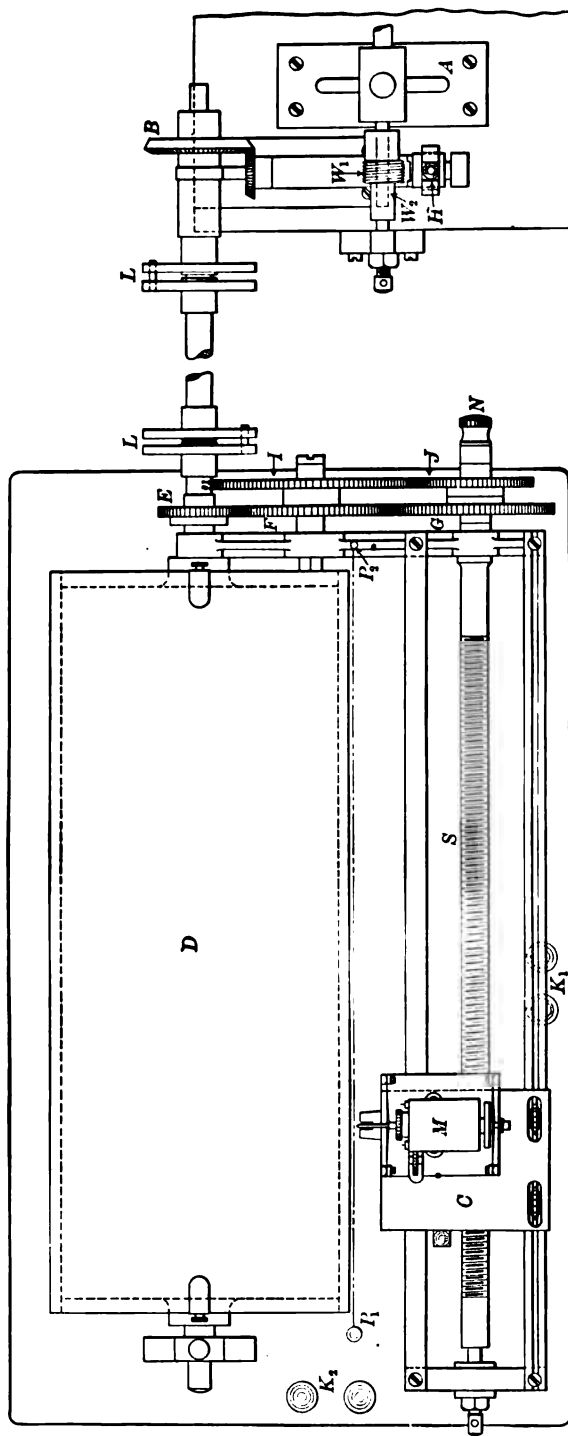


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The chronograph is driven by the machine, the speed of which is to be measured, by means of the worm and gear W_1 and W_2 and the flexible coupling L . The bevel gears B are used merely to keep the direction of the axis of the chronograph parallel to that of the driving machine. The clutch N permits the speed of the driving screws to be changed. The printing magnet M (mounted on the carriage C) prints through the type ribbon R upon the paper wrapped over the drum D . The binding posts K_1 are joined by flexible wires to the printing magnet, and the binding posts K_2 (connected to K_1 in the base) are connected to the battery and chronometer. A dot is printed once a second on the revolving drum.



Fig. 39.—Direct Reading Chronograph.

At a speed of exactly 1,500 revolutions per minute there would be 10 dots in each revolution of the drum. The printing magnet was advanced by a carriage C and screw S like the pen of an ordinary chronograph, and hence there are 10 rows of dots printed, at the rate of 6 dots per minute in each row. If the speed of the motor is 1,250 instead of 1,500 there will be 12 rows of dots, printed at the rate of 5 per minute in each row, etc. The following are some of the different speeds that may be used to obtain horizontal rows of dots:

TABLE XV.

Speed	Number of rows	Number of dots per minute in each row
2,500	6	10
1,875	8	7½
1,666½	9	6⅔
1,500	10	6
1,250	12	5
1,000	15	4
750	20	3
500	30	2

If the speed is a little greater or less than 1,500, for example, the rows of dots will slope one way or the other, the angular deviation from the horizontal being a measure of the variation of the speed from the nominal speed, the direction of the slope showing whether the speed is too fast or too slow. If the speed is variable the line of dots will be wavy or irregular. One can therefore see at a glance whether the speed is regular, and soon learns to estimate the speed without stopping the chronograph (if it is steady and near the nominal value) to within a few parts in 10,000. The interval between successive rows of dots at a speed of 1,500 is 5 cm, and this corresponds to one second. Hence the record can be read much closer than the usual chronograph record.

If AB, Fig. 41, is a row of dots making an angle α with the horizontal line AC, the speed was slower than the nominal speed in that case by an amount proportional to $\tan \alpha$. The direction of rotation of the drum is indicated by the arrow.

The drum is 50 cm in circumference; hence if one revolution is

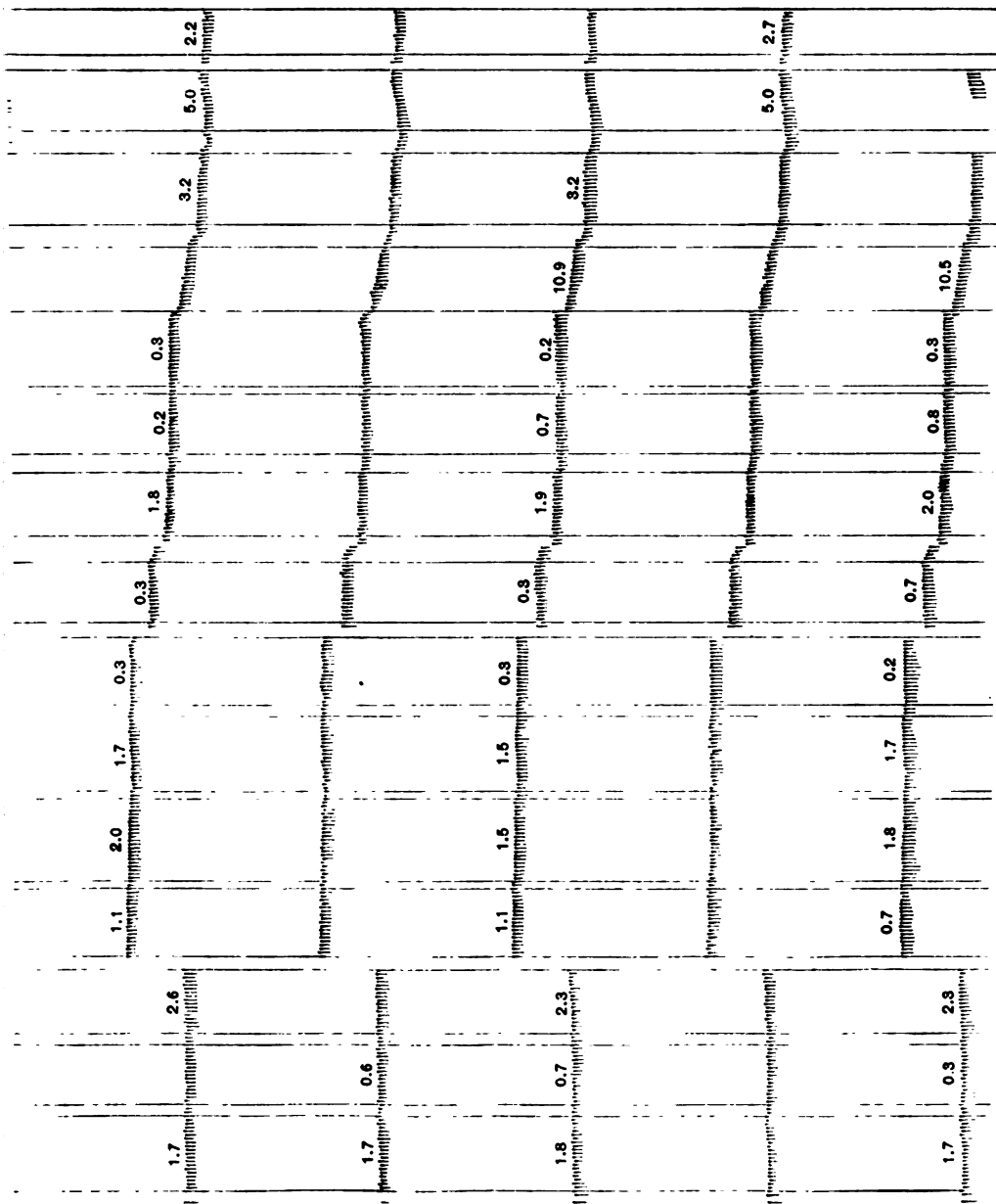


Fig. 43. - *Chronograph Record.*

This record was taken in comparing the capacity of a mica condenser with that of an air condenser, the actual frequency of the charging being given in the third line at the top. In this case the angle of an upward slope is marked negative, as it indicates a slower speed. The uncertainty of the measurement of these angles is indicated by the separate measurements on the different records of the same run. The variation of the separate measurements from the mean is usually less than 1 in 100,000.

the deflection can usually be reduced to a small value. A capacity determination was never attempted if the deflection exceeded 5 millimeters, and for most of the work it was under 2 millimeters. This deflection corresponds to the transfer of a definite quantity of electricity and is independent of the actual capacity being measured. Hence, since the capacity desired is the difference of two measured capacities, this effect will not affect the determination of the capacity desired unless the charges on the commutator are different for the two measurements. Therefore if this effect remains constant, or varies uniformly, it will be eliminated along with the capacity of the leads and of the commutator. However, we can not count on it remaining constant, or varying uniformly for any great length of time, so it is advisable to eliminate it as much as possible from every measurement. Two methods have been used to effect this. The first method employed consisted in measuring a capacity, then reversing the battery and the galvanometer and immediately measuring the same capacity again; if the mean of this effect was the same for both measurements, the mean of the two will be the value of the capacity freed from this effect. In this method the zero of the galvanometer was the zero on open galvanometer circuit. The second method consisted in taking for galvanometer zero its zero on closed circuit, the battery circuit being open. This zero is a spurious one, depending, as we have seen, upon the effect under consideration; hence measurements thus made will be free from this effect provided it remained constant during the observations. When the effect was very variable, the runs were made longer and the zero of the galvanometer was reset at intervals during the run. The variation in this effect was the greatest and most annoying source of error met with in the electromagnetic measurements.

It is evident that the errors mentioned under the second and the third groups also produce a deflection of the galvanometer when the battery is disconnected, and are eliminated by the methods just outlined.

(5) The fifth source of error is of the same nature as the fourth except that the inducing charges are upon the metal portions of the brushes and are maintained by the battery. Though always present when a capacity is measured by means of such a commutator, it can never be independently observed and allowed for. It takes

its rise in the fact that the commutator segment has a somewhat different capacity when it breaks contact with one brush from what it has when it breaks contact with the other. The difference in capacity is, however, independent of the capacity connected with the commutator; hence when the capacity is determined by the method of differences, as is always done, the final result is unaffected by this source of error.

49. EFFECT OF VARIATIONS OF SPEED.

When considering the question of galvanometers, we arrived at the conclusion that even at the highest speed we ever used the duration of contact of the commutator brushes is ample to ensure completeness of the charge and of the discharge. Nevertheless it appeared desirable to test this point experimentally. For this purpose the difference between the capacity of the cylindrical condenser built up with three sections and no guard cylinder and the capacity of the same condenser when built up with but two sections was measured at different speeds, with the following results:

TABLE XVI.

Effect of speed.

Revolutions per minute	Charges per second	Capacity
970	259	76.570×10^{-6} mf
1,130	301	76.558 "
1,175	313	76.556 "
1,280	341	76.555 "
1,320	352	76.556 "
1,370	365	76.556 "
1,450	387	76.556 "
1,530	408	76.575 "
1,660	443	76.560 "
1,680	448	76.559 "
1,700	453	76.560 "
1,850	493	76.555 "
1,870	499	76.579 "
Mean		76.561 "

These results show that within this range the measured capacity is independent of the speed. When we consider that the capacity given above is the difference between two larger capacities, the greater of which was 232×10^{-6} mf the agreement is all that can be expected. The average variation from the mean is only 0.006×10^{-6} mf; that is, about one in 40,000 of the larger capacity, and there is no evidence that the value changes with the speed. The inner cylinder used in this work was not capped.

A few measurements of the capacity of ball B at different frequencies were made. These also indicate that the capacity as measured is independent of the frequency. The values found are given in Table XVII.

TABLE XVII.

Variations with speed. Ball B.

Revolutions per minute	Charges per second	Capacity
1,070	288	32.979×10^{-6} mf
1,250	333	32.981 "
1,350	360	32.986 "
1,600	427	32.977 "
1,800	480	32.983 "
Mean		32.981 "

Here the average variation from the mean is only 0.0026×10^{-6} mf; that is, about one in 13,000 of the mean. When we remember that 0.0026×10^{-6} mf is the capacity of a sphere having a radius of but 23μ , this variation is not astounding.

50. EFFECT OF VARIATION OF VOLTAGE.

The battery, usually of about 200 volts, for charging the condenser consisted of small storage cells. This high voltage was chosen in order to reduce the relative importance of the electrostatic effect of the commutator. To test the effect of different voltages, a key was so connected as to allow the voltage to be suddenly changed between certain limits. Even when these limits were so different as 32 and 214 volts, the change produced no observable change in the

TABLE XVIII.
Spherical Condenser, Ball B.

[illegible]

23	179.517	179.509	2322	51.956		1.676		3.591	
24	179.501								
25	179.365	179.354	2249	50.280					36.580
26	179.343								
27	179.450	179.421	613			13.700			
28	179.392								
29	179.489	179.476	2249	50.314					36.593
30	179.463								
31	179.720	179.692	613			13.721			
32	179.664								
33	179.532	179.512	2162	48.379			34.661		
34	179.493								
35	179.616	179.599	613			13.714			
36	179.582								
37	179.513	179.503	2162	48.376			34.664		
38	179.494								
39	179.560	179.555	613			13.710			
40	179.550								
41	179.490	179.488	2162	48.372			34.664		
42	179.486								
43	179.552	179.513	613			13.707			
44	179.474								
					Mean	1.673	34.663	3.592	36.584
					Capacity of charging wire.		1.673		3.592
					Capacity of Ball B.	32.990×10^{-6} mf	32.990×10^{-6} mf		32.992×10^{-6} mf
					Correction for resistance and time	-0.010,	" "	" "	" "
					Capacity of Ball B at 20° C.	32.979,	" "	" "	32.981,

balance of the bridge. Also for the plate condenser a few complete runs were made at different voltages. These likewise failed to indicate that the measured capacity depended in the least upon the voltage employed. Thus, the means of two runs at different voltages gave for the capacity of the condenser, leads, and commutator the following values:

At 32 volts, $C = 137.481 \times 10^{-6}$ mf,

At 214 volts, $C = 137.486 \times 10^{-6}$ mf.

A difference of only 5 parts in 137,000; the sensibility, and hence the accuracy with which the speed can be controlled, is of course much less at the lower voltage than at the higher one.

51. METHODS FOR REDUCTION OF OBSERVATIONS.

Four different methods for reducing the observations were employed, depending upon the chronograph and the method employed in obtaining them.

In Table XVIII is shown the form of reduction employed in the earlier stages of the work. At this time we were using an electrically driven chronograph of a common form; the electrostatic effect of the commutator was eliminated by balancing to the galvanometer zero obtained from closed galvanometer but open battery circuit (see p. 567); after making a three-minute chronograph run the battery and galvanometer were both reversed and another run was made. The slight difference in the speed in the two cases is due to slight errors in the elimination of the electrostatic effect. The chronograph time for 80,000 charges is given in the second column of the table, and the mean of these times for each pair of runs taken as just described is given in the third column. This mean corrected to mean solar time (in this case a sidereal time chronometer was used) and combined with the assumed values of the resistances, give the capacities recorded in columns 5-8. The values assumed for the resistances are those based upon the most recent preceding comparison of the coils with the standards. As stated in section 44, these assumed values were later corrected for the seasonal changes in the resistances. This correction is combined with the correction for the temperature of the condenser and applied to the final result as given at the foot of the table. In the fourth column of the table

is given the nominal value of the resistance in the arm a of the bridge (see Fig. 29); the resistances in the other arms remained unchanged throughout the day. In the fifth column is given the electromagnetic capacity, K , of the commutator, lead wires, and ball with both charging wires in position. Removing the top charging wire, w_1 (see Fig. 2) the electromagnetic capacity of the remaining portion is denoted by K_1 and is given in column 6. Replacing w_1 and removing the side charging wire, w_2 , the capacity becomes K_2 as given in the seventh column. Finally, removing both charging wires we obtain the capacity of the commutator and leads as given under k in column 8. The effective capacity of the upper charging wire, c_1 , is the difference between K and K_1 , that of the side wire, c_2 , is $K - K_2$. The capacity of the ball increased by the effective capacity of the upper charging wire is equal to $K_1 - k$; the capacity of the ball increased by the capacity of the side charging wire is equal to $K_2 - k$ (see p. 444). These differences are given in the last four columns of the table. The unit in which all these capacities are expressed is the one-millionth of a microfarad.

In Table XIX is shown the method employed in reducing the results obtained by means of the differential galvanometer and the new form of chronograph.

Having decided upon a speed of 1,500 revolutions per minute (400 charges per second) as suitable for the work, a table was constructed giving the values of the capacity corresponding to given values of the resistance a . In this table (not given in this paper) the values of a (in the region with which we are concerned) differed by steps of one ohm each, so that it was easy to interpolate to a hundredth of an ohm in a . These capacities are of course based upon assumed values for the resistances of c , d , and the galvanometer, and so are but approximately correct. From this table we get the values given in the third column of Table XIX. In the fourth column are given the corrections k_1 (in parts per 1,000), which must be added to the approximate capacity on account of the speed of the machine being slightly different from that corresponding to exactly 1,500 revolutions per minute. As explained on page 564, k_1 is merely twice the tangent of the angle of deviation of the rows of second dots on the chronograph sheet from the generating lines of the drum. In the fifth column is the sum of k_1 and k_2 , where k_2 is the correction

TABLE XIX.
Spherical Condenser, Ball B.—Differential Method.

Mean air temperature 27°C . Mean temperature of resistances 20°C for all. Hygrometer 90% .

November 24, 1905.

Run	a corrected	Approx. capacity	k_1	$k_1 + k_2$	Total correct.	k	K_1	K	$K - K_1$	$K_2 - k$	Mean	Remarks
1	2119.61	51.691	+0.38	+0.33	+17			51.708				Speed poor.
2	2051.61	50.069	+0.25	+0.20	+10		50.079					
3	2120.61	51.714	-0.56	-0.61	-32			51.682	1.604		1.610	
4	2052.61	50.093	-0.28	-0.33	-16		50.077		1.608			
5	2119.61	51.691	-0.01	-0.06	-3			51.688	1.612			
6	2052.61	50.093	-0.34	-0.39	-19		50.074		1.614			
7	2119.61	51.691	-0.01	-0.06	-3			51.688				After 15, stopped for lunch. Interchanged coils. Interchanged coils.
8	546.51	13.557	+1.98	+1.93	+26	13.583						
9	547.51	13.581	-0.17	-0.22	-3	13.578						
10	1971.16	48.147	+0.79	+0.74	+36		48.183			34.606		
11	547.51	13.581	-0.37	-0.42	-6	13.575						
12	1972.16	48.171	-0.19	-0.24	-12		48.159			34.584		
13	547.51	13.581	-0.39	-0.44	-6	13.575						
14	1972.16	48.171	-0.13	-0.18	-9		48.162			34.586		
15	547.51	13.581	-0.22	-0.27	-4	13.577						
16	547.51	13.581	-0.34	-0.39	-5	13.576					34.590	
17	1972.16	48.171	-0.19	-0.24	-12		48.159					
18	1972.16	48.171	-0.26	-0.27	-13		48.158				34.586	
19	547.51	13.581	-0.80	-0.81	-11	13.570						

	Charge	Discharge
Capacity of ball and wire.....	34.590×10^{-6} mf	34.588×10^{-6} mf
“ “ wire.....	1.610	“ “ 1.610
“ “ Ball B.....	32.980	“ “ 32.978
Corrections for temperature and errors in the assumed values of the resistances.....	—0.002 ₃	—0.002 ₃
Capacity of Ball B at 20°0 C.....	32.977×10^{-6} mf	32.975×10^{-6} mf

Mean	1.6128	34.5821
Capacity of wire	1.6128	32.9653
Capacity ball B		-0.0023
Correction for temp. and resist.		32.9670
Capacity of ball B at 20.0 C		

(in parts per 1,000) to be added to the capacity on account of the differences between the true values of the resistances of c , d , and the galvanometer, and the values assumed in the construction of the table from which we obtain the approximate capacity as given in column 3. In the sixth column is given the correction $k_1 + k_2$ expressed in thousandths of a micro-microfarad. The symbols at the heads of the following columns have the same meaning as in the former case.

As before, the values of the resistances assumed to be correct when the observations are reduced are slightly in error owing to the seasonal variations in the resistances. The correction for this error and the correction for the temperature of the condenser differing from 20° C are combined and applied to the final result, as shown at the bottom of the table.

The charging current was measured in the first 24 runs, and the discharging current was measured in the rest. These have been separately averaged and reduced, and give results differing by but 1 part in 16,000.

In order to test the equality of the coils of the galvanometer under working conditions four pairs of runs (17-18, 19-20, 21-22, 27-28) were taken, the second in each pair being taken immediately after the first, but with the galvanometer coils interchanged. No certain difference is observable.

As before, the electrostatic effect was eliminated by balancing to the displaced zero of the galvanometer.

In Table XX is shown the form used in the reduction of observations obtained by the method (page 558) in which the speed is controlled by means of an auxiliary bridge containing a large capacity. During this series four chronograph runs were taken, as shown in the table; the temperature of the large condenser was not kept perfectly constant so that the resistance c' , required to balance the auxiliary bridge when the speed of the machine was exactly 1,500 revolutions per minute gradually increased.

The first part of the table is taken up with the determination of the electromagnetic capacity of the upper charging wire (w_2). "Wire in" denotes that this wire is in place so as to touch the ball and to dip into the mercury cup that serves to connect it with the commutator; "Wire out" denotes that it has been entirely removed. Throughout this portion of the work the side charging wire w_1 was continuously in position so as to connect the ball to the commutator.

In column 2 is given the nominal value of the resistance in arm a of the main bridge; in the next four columns are given the values of the arm c' (decreased by 49,500 ohms) of the auxiliary bridge which correspond to a simultaneous balance of both bridges. The order in which these observations were taken is best illustrated by an example. Let us take the first set given in the table. The wire w_1 , being in place, a is made equal to 2,137.5 ohms, and a simultaneous balance is found for $c' = 49,599.0$ ohms; the auxiliary observer now changes the portion of the c' arm that is near him and the main observer (not knowing the amount of this change) proceeds to obtain another simultaneous balance of the bridges; this again occurs for $c' = 49,599.0$ ohms. The wire is now taken out, a is made equal to 2,073.2 ohms, and two simultaneous balances are obtained, as before, for $c' = 49,588.0$ and $49,589.0$ ohms. Then the wire is replaced, a changed to its former value, and balances are obtained for $c' = 49,599.0$ and $49,597.0$ ohms, etc. The mean of these settings for each case is given in column 7. In the eighth column are given the values of $c'_0 - 49,500$ that correspond to the balance of the auxiliary bridge for a speed of exactly 1,500 revolutions per minute; these are obtained by interpolating between the four chronograph runs taken during the day. The following column contains the difference between c' and c'_0 , which (divided by c'_0) gives the correction s (column 10) in parts per 1,000 to be added to a in order to obtain a balance for the even speed of 1,500 revolutions per minute. Column 11 contains the further correction to be added to a in order to compensate for the fact that cd has a value different from that assumed in the calculation of the capacity for a given value of a . The next column (headed A_p) contains the correction that must be applied to the value of the capacity as calculated by the approximate formula $\left(C = \frac{a}{ncd}\right)$ for the bridge in order to obtain the true capacity; the third and the fourth numbers from the end of this column are the corrections that must be applied on account of the galvanometer resistance being different from that assumed in the construction of the table giving the capacity corresponding to a given value of a when the differential method is used. The sum of these three corrections is given in column 13; the product of these sums by

one-thousandth of the corresponding a is the correction in ohms to be applied to a ; this is given in column 14. In column 15 is given the correction to be added to a on account of the errors in the nominal values of the coils of which it is composed. The value of a as given in column 2, increased by the corrections in columns 14 and 15, is exactly 40 times the true electromagnetic capacity, expressed in micro-microfarads, if the bridge method is used. If the differential method is used, the true capacity is that given by our table for this corrected value of a . These capacities are given in column 16, and their differences (corresponding to the capacity of the wire or of the ball and wire) are given in columns 17 and 18.

The runs marked "Bridge" in the first column were obtained by the bridge method by the use of the Weston (low resistance) galvanometer; that marked "Charge" was by the differential method, the charging current being measured; and the last, marked "Bridge S," is by the bridge method, the Sullivan (high resistance) galvanometer being used.

The three methods substantially agree, and give 32.967×10^{-8} mf as the capacity of ball B at 20° C.

52. RESULTS BY BALL A.

Many preliminary determinations of the electromagnetic capacity of ball A were made, but all of those made before March 7, 1905, are unreliable on account of errors in the determination of the capacity of the charging wire, heating of the resistance in arm d , leakage, or other cause. On March 6, 1905, the 100,000 ohms resistance composing arm d was placed in oil. By this time the method for determining the capacity of the charging wire had been perfected, and the leakage had been reduced so as to be usually negligible. This date is therefore taken as the beginning of the actual measurements. However, there would occasionally be enough leakage over the ebonite bushing, which served as a guide for the side charging wire, to render the determination of the capacity of the charging wires and of the ball, when measured by means of the side charging wire, uncertain. This condition of affairs continued until April 4, 1905, when the bushing was replaced by one extending further beyond the surfaces of the shell and having the hole enlarged at each end so as to increase the leakage path. After this no further trouble from this cause was experienced.

Owing to this trouble some of the observations taken between March 7 and April 4 had to be discarded. *All* the determinations of the capacity of ball A, made after April 4, 1905 (except 6 single determinations out of a total of 65), are given in Table XXI. The observations omitted compose two groups of consecutive observations and every indication pointed to the fact that during these observations the commutator brushes were too loose.

TABLE XXI.

Results by Ball A.

Date	Wire	C	δ_1	A	δ_2	N	W	v	Mean
1905									
Mar.	7	1.516	13.928	3	55.920	3	3	$\frac{1}{2}$	2.9965
	7	3.339			55.915	8	2	$\frac{1}{2}$	2.9967
Apr.	10	1.454	14.026	3	55.922	2	3	3	2.9965
	10	3.448			55.919		1	1	2.9966
	17	1.454	13.883	2	55.929	1	2	2	2.9963
	26	1.495	14.106	26	55.928	9	3	2	2.9963
	26	3.484			55.932		1	1	2.9962
									2.99614
May	1	1.482	14.031	2	55.944	3	4	3	2.9959
	1	3.433			55.944	4	4	3	2.9959
	2	1.475	14.014	15	55.939	5	5	4	2.9960
	2	3.429			55.939	6	3	2	2.9960
									2.9960
July	14	1.541	13.787	9	55.923	1	3	3	2.9964
	15	1.504	13.741	15	55.962	2	2		
	15	3.476			55.929	4	2	1	2.9963
	18	1.512	13.761	20	55.950	2	3	2	2.9959
	18	3.471			55.934	12	2	1	2.9963
	19	1.505	13.794	2	55.942	2	2	1	2.9961
	19	3.472			55.944	2	2	1	2.9961
	21	1.574	13.749	3	55.928	1	2	2	2.9963
	21	3.466			55.928	2	2	2	2.9963
	22	1.572	13.766	1	55.938	4	2	1	2.9960
	22	3.472			55.934	6	2	1	2.9962
	24	1.575	13.758	5	55.932	1	2	2	2.9962
	24	3.474			55.932	2	2	2	2.9962
25		13.692	3	55.926	4	4	4	4	2.9964
25				55.920		1	1		2.9965

The only other result with this ball that does not appear in the final mean is the first given under date of July 15, 1905, in the table. Though everything appeared to be going nicely when these observations were taken, except that the humidity was high (70%), the results when reduced were found to be very erratic.

All the determinations with ball A were made by the bridge method.

The second column of the table contains the capacity of the charging wire; the smaller capacity (1.5) is that of the wire which passed through the hole in the top of the shell, the other (3.5) is that of the one which passed through the hole at the equator of the shell. In the third column under C is given the capacity of the commutator and leads; in the next column is given δ_1 , which is the average variation of the capacity of the commutator and leads from the mean value given under C. In the fifth column under A is given the capacity of ball A at 20° C (all corrections applied), and in the following column is given the average variation δ_2 of its measured capacity from this mean. N is the number of distinct determinations included in the mean value of the capacity given under A, and W is the weight assigned to this mean. In the ninth column under $\sqrt{v}$ is the square root of the ratio of the electrostatic capacity of ball A (corrected for the holes in the shell, ebonite bushings, etc.) to the value given in column A.

The unit in which all the electromagnetic capacities are expressed is the one-millionth of a microfarad. The average variations from the mean (δ_1 and δ_2) are expressed in terms of units in the last place—that is, in terms of one-thousandth of a micro-microfarad.

It will be noticed that δ_1 is as a rule considerably greater than δ_2 . The cause of this is obvious. Every change in the adjustment of the brushes changes the value of the commutator capacity, and the brushes were usually readjusted several times in the course of a day. This sometimes caused the commutator capacity to vary over wide limits, and so made δ_1 large. But the value of the capacity of the ball is the difference between the measured capacity of the ball, commutator, and leads, and the mean of two measurements of the capacity of the commutator and leads, one being taken before and the other after the measurement with the ball. The commutator brushes are never touched during such a set of three measurements,

so this difference is independent of the large changes in the commutator capacity and is almost independent of the small ones.

The results naturally fall into four groups, which have been separately averaged. After each group the condenser was entirely dismantled and was reassembled again just before the beginning of the results recorded in the following group.

The mean value of δ_1 is but 0.0037 micro-microfarads, which shows that on the average a single measurement of the capacity of ball A departed from the mean of all measurements of the same day by no more than the capacity of a sphere of radius equal to 33 microns, or 7 parts in 100,000.

The weighted mean of these 64 independent determinations is

$$\bar{v} = 2.99621 \times 10^{10}$$

The average variations of the results in the ninth column of the table from this mean is 0.00018×10^{10} , or 6 parts in 100,000.

53. RESULTS BY BALL B.

The preliminary work having been completed with ball A, ball B was adjusted in the shell on March 13, 1905, and the first determination of its electromagnetic capacity was made on the following day. As stated in the preceding section, we were occasionally troubled with leakage until April 4, 1905, so that some of the results in March had to be rejected. All the determinations of the capacity of ball B made after the latter date (except 3 single determinations out of a total of 202) are given in Table XXII. These omitted determinations were the last observations taken on November 21, 1905, when the commutator brushes were badly worn, and the commutator was working badly. There were in addition a few preliminary runs taken with the differential galvanometer. These were taken for the purpose of studying the instrument and the method and were not used for the purpose of obtaining a measurement of the capacity of the condenser.

Table XXII is analogous to Table XXI; the only change is that the column containing the capacity of the ball is headed B, and a column is inserted between this column and the column containing the mean deviations (δ_1). This column, which is marked M, con-

tains the designations of the methods employed; b, signifies that the Maxwell bridge method has been used; d, that the *discharge* has been measured by the differential galvanometer method; and c, that the *charge* has been measured by the same method. The results obtained on September 1 and 6, 1905, at different speeds have been averaged and recorded separately; the results for each day are arranged in the order of increasing speed. The measurements in October and November of 1906 were made by the method in which

TABLE XXII.
Results by Ball B.

Date	Wire	C	δ_1	B	M	δ_2	N	W	v	Mean
1905										
Mar.	14	1.601	13.935	2	32.984	b	0	3	3	2.9962
	14	3.464			32.980	b	2	3	3	2.9964
	16	1.603	13.932	4	32.984	b		1	1	2.9962
	18	1.609	13.959	1	32.984	b		1	1	2.9962
	21	1.616	13.996		32.977	b		1	1	2.9965
	22	1.627	13.965	3	32.980	b	2	5	5	2.9964
	22	3.449			32.977	b	4	3	3	2.9965
	24	1.621	13.975	4	32.984	b	2	3	3	2.9962
	24	3.440			32.983	b	2	3	3	2.9962
	25	1.632	13.962	2	32.981	b	2	3	3	2.9963
	25	1.632	13.984	1	32.980	b	1	3	3	2.9964
July	26	1.656	13.675	1	32.978	b	1	2	2	2.9965
	26	3.584			32.979	b	2	2	2	2.9964
	27	1.662	13.683	5	32.982	b	8	2	2	2.9963
	27	3.582			32.985	b	2	2	2	2.9962
	28	1.666	13.691	7	32.976	b	1	4	3	2.9965
	28	3.587			32.981	b	4	4	4	2.9963
	31	1.673	13.716	9	32.980	b	1	3	3	2.9964
	31	3.592			32.982	b	5	5	5	2.9963
Sept.	1	1.672	13.814	11	32.977	b	3	5	5	2.9965
	1		13.797	1	32.979	b	3	2	2	2.9964
	1		13.756	1	32.974	b	3	2	2	2.9967
	6	1.666	13.786	1	32.984	b	2	2	2	2.9962
	6		13.764	8	32.976	b	4	4	4	2.9965
	6		13.753	1	32.981	b	1	2	2	2.9962

TABLE XXII—Continued.
Results by Ball B—Continued.

Date	Wire	C	δ_1	B	M	δ_2	N	W	v	Mean
1905.										
Oct.	30	1.623	13.507	1	32.985	d	3	2	2	2.9962
	31	1.645	13.532	0	32.975	b	3	2	2	2.9966
	31		13.541	1	32.980	d	1	3	3	2.9964
	31		13.557	4	32.979	c	1	3	3	2.9964
Nov.	2		13.460	8	32.982	b	2	4	4	2.9963
	3	1.625	13.480	8	32.983	b	5	3	2	2.9962
	3		13.474	7	32.980	c	1	3	3	2.9964
	3		13.528	4	32.980	d	4	3	3	2.9964
	6	1.635	13.549	3	32.974	d	3	4	4	2.9966
	6		13.548	2	32.979	c	6	4	4	2.9964
	6		13.543	3	32.975	b	3	4	4	2.9966
	9	1.632	13.452		32.980	d		1	1	2.9963
	10	1.625	13.429	4	32.976	d	1	2	1	2.9965
	18	1.615	13.721		32.982	d		1	1	2.9963
	20	1.618	13.569	4	32.982	d	1	3	3	2.9963
	20		13.564	3	32.982	c	3	3	3	2.9963
	23	1.611	13.544	3	32.986	b	5	4	4	2.9961
	23		13.550	3	32.998	d	6	2	2	2.9956
	23		13.545	4	32.980	c	6	8	7	2.9964
	24	1.610	13.576	2	32.978	c	6	4	4	2.9964
	24		13.568	2	32.976	d	2	3	3	2.9965
	25	1.607	13.600	1	32.979	d	1	2	2	2.9964
	25		13.632	22	32.979	c	4	4	4	2.9964
	29	1.619	13.702	11	32.977	b	2	5	5	2.9965
1906.										
Oct.	30	1.605	15.393	1	32.975	c	0	2	2	2.9966
	30		15.410	2	32.977	c	1	8	8	2.9965
	30		15.399	1	32.978	c	0	3	3	2.9965
	30		15.415	1	32.974	c	0	2	2	2.9966
Nov.	1	1.604	15.411	2	32.971	c	2	2	2	2.9967
	1		15.407	2	32.974	c	1	10	10	2.9966
	12	1.613	15.337	1	32.967	b	0	5	5	2.9969
	12		15.335	0	32.966	b	1	4	4	2.9970
	12		15.333	2	32.970	c	2	8	8	2.9968
	12		15.328	0	32.966	b	1	4	4	2.9969
	15	1.609	15.154	0	32.981	c	2	6	6	2.9963
	16	1.607	15.381	1	32.977	b	1	8	8	2.9965
	16		15.381	1	32.980	c	1	6	6	2.9964
	16		15.385	1	32.977	c	2	6	6	2.9965
	16		15.390	2	32.980	c	1	7	7	2.9964

the auxiliary bridge is employed (see p. 558). It will be noticed that δ_1 is here very much smaller than it is in the first part of the table; this is a consequence of the greater rapidity with which determinations can be made by this method; the commutator brushes are not touched at all during the determinations composing any one series.

These results fall into five groups, which have been separately averaged. As with ball A the condenser was entirely dismantled after each group and was reassembled again just before the beginning of the preceding group.

The mean value of δ_1 for the observations of 1905 is but 0.0030 micro-microfarads, or the capacity of a sphere of a radius of 27 microns. The mean value of ∇ for this year is 2.99637. In October and November, 1906, when the other method was employed, the mean value of δ_1 was but 0.0011 micro-microfarads, or the capacity of a sphere of a radius of 10 microns. The mean value of ∇ for this period is 2.99661. At the time these measurements were made we were aware that they departed from the mean, and every endeavor was made to locate the cause of the discrepancy. The resistances were carefully compared with the standards and also intercompared, all leakage paths were carefully tested, and on November 12 three methods were employed in determining the capacity. The first two results given under that date were determined by the bridge method, using the low-resistance galvanometer; the fourth was by the same method, but with the high resistance galvanometer, and the third was by the differential method, the charge being measured. The results by these three methods agree almost perfectly. The next day the guard cylinder condenser was assembled and one determination of the capacity was made. This value gave a remarkably low value for ∇ . The next day several results were obtained that differed but little from the mean. The following days the results obtained by the cylinders gave rather low values for ∇ , while those by the ball were rather high, but much lower than the values found on November 12. The cause of these high values has not yet been found. It may possibly be due to some abnormality in the dielectric constant of air during this period.

The weighted mean of the 147 independent determinations made in 1905 is

$$\nabla = 2.99637 \times 10^{10}$$

The average deviation of the results in the tenth column of the table from this value is

$$0.00011 \times 10^{10}, \text{ or four parts in } 100,000.$$

The weighted mean of the 81 independent determinations made in 1906 is

$$\bar{v} = 2.99659 \times 10^{10}$$

The average deviation determined as before is

$$0.00016 \times 10^{10}, \text{ or five parts in } 100,000.$$

From these results we can get a good comparison of the results obtained by the three methods designated as b, c, and d; collecting the results obtained by different methods on the same day we obtain the data presented in Table XXIII.

TABLE XXIII.

Comparison of Different Methods.

Method.	b	c	d
1905.			
Oct. 31.....	2.9966	2.9964	2.9964
Nov. 3.....	2.9962	2.9964	2.9964
6.....	2.9966	2.9964	2.9966
23.....	* 2.9961	* 2.9964	* 2.9956
Mean.....	2.99647	2.99640	2.99647
Nov. 20.....		2.9963	2.9963
24.....		2.9965	2.9964
25.....		2.9964	2.9964
Mean.....		2.99640	2.99637
1906.			
Nov. 12.....	2.9969	2.9968	
16.....	2.9965	2.9964	
Mean.....	2.99670	2.99660	

* Omitted from mean, balance hard to maintain, brushes bad.

54. RESULTS BY CYLINDERS WITHOUT GUARDS.

A series of measurements were made with the upper ends of both cylinders, and with the lower end of the outer one open. These are assembled in Table XXIV. The numbers at the tops of the columns indicate the two capacities of which the difference is given in that column; for example, 1. 2. 3-1. 2 at the head of the second column indicates that each number contained in this column is the

TABLE XXIV.

Difference in the Capacity produced by Changing the Height.

Date.	1.2.3-1.2	W	1.2.3-1.3	W	1.3.2-1.3	W	1.3.2-1.2	W	1.2-1	W	1.3-1	W
1905												
Mar. 24	76.542	1							76.152	1		
Apr 18									76.245	2		
19			75.683(?)	1								
20			76.137	1	76.120	1			76.182	1	76.630	1
22					76.134	1	76.546	2	76.147	2	76.565	4
May 2			76.172	1					76.204	2		
3	76.589	2			76.143	2	76.606	2	76.195	2		
8	76.582	1			76.121	1			76.185	2	76.605	1
9	76.558	2	76.078	2			76.590	2	17.178	3		
Aug. 1	76.564	6			76.132	4			76.174	5	76.618	4
2	76.553	4	76.133	4	76.129	4			76.175	4	76.627	4
3	76.553	3										
	76.577	3										
	76.558	2										
23	76.565	2										
	76.555	2										
24	76.580	3										
	76.578	2										
	76.573	1										
25	76.564	2										
	76.523	1										
	76.519	1										
31	76.554	4										
	76.551	4										
	76.554	2										
Sept. 7	76.563	4										
	76.562	3										
	76.546	3										
12	76.552	3										
13	76.558	3										
Mean	76.560		76.125		76.131		76.581		76.183		76.605	
Correction	-0.005		-0.005		-0.005		-0.005		-0.005		-0.005	
	76.555		76.120		76.126		76.576		76.178		76.600	

difference between the capacity of the condenser as composed of sections 1, 2, and 3 and its capacity as composed of sections 1 and 2. As before, the unit of capacity employed is one micro-microfarad.

While the individual values are rather variable, the means are probably fairly accurate and show that the first section added invariably produces a greater increase in the capacity of the system than the same cylinder produces when it is the second section added. This is exactly what we have found should be observed on account of the distribution of charge on the outside of the outer cylinder. It was not due to the change in position of the upper end with respect to objects above it, for the upper end never came closer than 80 cm to any of these objects, while by actual trial it was found that a metal plate could be brought down from above to within about 10 cm of the upper end of the condenser before it appreciably affected the balance of the bridge.

Taking the weighted means from Table XXIV we find the following values for the increase in the capacity produced by the addition of one section:

TABLE XXV.

Section.	2	3
Section added first	76.178	76.600
" " second	76.124	76.557
Difference	0.054	0.043

Assuming 2.9963×10^{10} as the true value of ν , the electromagnetic capacities of these sections are 76.085 and 76.518 micro-microfarads, respectively. Hence the measured capacity is too great by 124×10^{-6} and 107×10^{-6} of the total for the section first added, and 51×10^{-6} of the total for the section next added. Hence the section first added has a capacity higher than that calculated by 12 parts in 10,000, and the next section is higher by 5 parts in 10,000. Hence in order to use this method the shortest section of the cylinder should be at least 70 or 80 cm in length. This, however, will in turn necessitate a great increase in the accuracy of the electromagnetic measurements, for the determination will then depend upon the dif-

ference of two large quantities. For this reason it is clearly not desirable to use this method with open cylinders.

Capped cylinders.—Recently a few measurements were made on two days with both cylinders capped at the top. As stated on page 522, it seems that with the arrangement adopted no error is introduced by the fact that the lower end of the outer cylinder is not capped. The results are as follows:

TABLE XXVI.

Section.	2	3
First section added	76.109	76.531
Second " "	76.085	76.510
Difference	0.024	0.021

Here we still have a difference in the same direction as before, but a little less than half as large. The values of the capacities are much nearer their true values than those obtained when the ends were open, but for the first section added they are still too large. The capacities for the sections added second give us $\tau = 2.9963 \times 10^{10}$ for section 2, and 2.9964×10^{10} for section 3.

The only explanation of the high values found for the section added first seems to be that the capacity of the lower end of the cylinder, or of the leads and commutator, is appreciably affected by changing the height of the cylinder in spite of the tests described on page 522.

The method of procedure followed in this part of the work necessitates numerous buildings up of the condenser, and the results obtained serve as an indication of the reproducibility of this process. From the table we see that the average variation from the mean for any section is only 0.010 micro-microfarads. This is very good when we consider that this includes the variations in the commutator capacity, and these may well be large, as it requires some time to add and adjust (or to remove) a section.

55. RESULTS BY CYLINDERS WITH GUARDS.

Having found by experience that it is impossible to obtain consistent results with uncapped cylinders without guard cylinders,

the guard cylinders were constructed and after a few preliminary experiments the determination of ∇ by the use of the guard cylinder condenser was begun on March 8, 1906. The use of the second set of commutator brushes (for charging the guard cylinders) necessitated the setting of all the brushes at rather unfavorable angles so that the vibration of these brushes was increased. This in turn made the brushes work loose more rapidly, and rendered the capacity of the commutator more variable. Because of this some of the measurements between March 8 and March 16 had to be discarded, but after the latter date the adjustment of the brushes was maintained by frequently testing with a milammeter, as already described. As a consequence, all observations after March 16, 1906, are included in Table XXVII, except 7 single determinations (out of a total of 457) when there was a slight leakage across the mica insulation separating the segments of the ring on which the side brush bears, causing the capacity of the commutator to be uncertain. On January 3 and 4, 1907, the balance of the bridge could not be maintained, so that the observations made were discarded; they indicated a variable leakage, probably between the segments of the commutator.

The construction of Table XXVII is similar to that of Table XXI, except that in this case there is no charging wire of which the capacity has to be determined. All these determinations were made by the differential method, except those of October 8 and 9, 1906, which were by the bridge method. All the results after October 9 were obtained by the method involving the use of the auxiliary bridge (page 558).

The condenser was dismantled and reassembled again after each group that has been separately averaged.

The average value of δ_s for the observations taken before November 14, 1906, is 0.0150 micro-microfarads; this is only 1 in 10,000 of the capacity of the condenser, but is absolutely about 5 times as large as in the case of the spheres. This increase is due to the additional difficulties introduced by the second set of commutator brushes. The weighted mean of these 174 independent determinations is

$$\nabla = 2.99626 \times 10^4$$

TABLE XXVII—Continued.

Date	C	δ_1	Cylinder	δ_2	N	W	v	Mean
1906								
Nov. 14	18.168	1	152.849	4	9	4	2.9961	2.99616
15	18.181	2	152.864	4	8	4	2.9959	
15	18.186	2	152.866	3	8	4	2.9959	
15	18.182	1	152.873	3	8	4	2.9959	
15	19.300	2	152.922	3	6	3	2.9954	
15	18.199	3	152.884	3	6	3	2.9957	
15	18.188	1	152.871	3	4	2	2.9959	
16	18.216	1	152.869	5	6	3	2.9959	
16	18.215	1	152.873	5	6	3	2.9959	
17	18.208	2	152.847	5	6	3	2.9961	
17	18.205	1	152.857	4	7	3	2.9960	
17	18.205	2	152.869	5	8	4	2.9959	
17	18.207	1	152.855	6	6	3	2.9960	
17	18.197	2	152.840	9	8	3	2.9962	
17	18.195	4	152.854	5	4	2	2.9960	
Dec. 4	16.799	2	152.793	2	6	3	2.9966	
4	16.797	2	152.792	5	6	3	2.9966	
4	16.806	8	152.806	3	4	2	2.9965	
4	16.855	4	152.806	10	8	3	2.9965	
5	18.117	2	152.814	2	4	2	2.9964	
6	18.563	3	152.792	2	6	3	2.9966	
6	18.565	2	152.792	6	6	3	2.9966	
6	18.562	1	152.797	4	8	4	2.9966	
6	18.559	2	152.818	8	6	3	2.9964	
7	18.534	10	152.833	19	6	1	2.9962	
7	18.528	10	152.853	9	6	2	2.9961	
7	18.494	1	152.836	2	4	2	2.9962	
1907								
Jan. 5	17.799	1	152.790	4	8	4	2.9967	2.99570
5	17.768	1	152.790	4	6	3	2.9967	
5	17.770	2	152.788	4	6	3	2.9967	
Feb. 4	20.619	0	152.878	6	7	3	2.9958	
4	20.619	0	152.890	3	6	3	2.9957	
4	20.617	2	152.884	7	8	4	2.9957	
4	20.612	1	*76.745	2	6	3	2.9957	
4	20.604	1	*76.752	2	6	3	2.9957	
4	20.608	1	*76.758	2	6	3	2.9957	

TABLE XXVII—Continued.

Date	C	δ_1	Cylinder	δ_2	N	W	ν	Mean
1907								
Feb. 4	20.615	1	152.879	3	6	3	2.9958	2.99596
5	20.627	1	152.835	2	6	3	2.9962	
5	20.627	2	152.854	4	6	3	2.9960	
5	20.629	1	152.876	4	6	3	2.9958	
5	20.632	1	152.885	8	6	2	2.9957	
5	20.639	2	152.843	2	6	3	2.9962	
5	20.631	2	152.851	4	6	3	2.9961	
7	20.409	3	152.854	4	8	4	2.9960	
7	20.409	1	152.864	3	6	3	2.9959	
7	20.411	1	152.867	3	6	3	2.9959	
7	20.419	1	152.864	3	6	3	2.9959	
7	20.421	1	152.864	3	7	3	2.9959	

* The middle section of the condenser consisted of section three only.

The average deviation of the results in the eighth column of the table from this mean is

$$0.00030 \times 10^{10}, \text{ or } 10 \text{ parts in } 100,000$$

The average value of δ_2 for the remainder of the observations in this table is 0.0046 micro-microfarads. The weighted mean of these 329 independent determinations is

$$\nu = 2.99609 \times 10^{10}$$

and the average deviation from this mean is

$$0.00027 \times 10^{10}, \text{ or } 9 \text{ parts in } 100,000$$

The mean of all the values as they stand is $\nu = 2.99614 \times 10^{10}$; this, however, gives undue weight to the results obtained on those days on which several series of observations were taken. These series are all independent so far as the electromagnetic measurements go, but are mere duplicates so far as the electrostatic capacity is concerned. Hence it seems more fair to give to the mean result of any one day the mean weight attached to the separate series; thus from the daily means we find

$$\nu = 2.99624 \times 10^{10}$$

The average variation of the daily means from this is

$$0.00027 \times 10^{10}, \text{ or } 9 \text{ parts in } 100,000$$

56. RESULTS BY THE PLATE CONDENSERS.

As already stated the importance of the correct adjustment of the plates to parallelism, as well as the importance of the lower plate being truly plane, was not recognized until the conclusion of the work, hence all results obtained are in error from these causes. Also, until near the end of the work (October, 1906), the importance of the plate being strictly in the plane of the guard ring was not recognized. As a consequence the adjustment was but approximate and the error due to this cause was large. Later the adjustment was made with more care, but it was not possible to make the error from this cause negligible.

Consequently, all results obtained by use of the plates have to be corrected for these two sources of error.

By dividing the results into a series of groups such that each group covers a period over which the adjustment of the plates was not knowingly altered, and by assuming that the adjustment (so far as the displacement of the plate from the plane of the guard ring, and the relative inclination of the plates are concerned) remained unchanged throughout each group, we can determine the errors in adjustment which must be assumed in order to bring into accord with an assumed value of v and with one another the results obtained at different distances between the plates.

Proceeding in this manner we have found that all the results obtained can be made concordant with one another and with the value of v given by the other condensers by the assumption of very probable errors in these two adjustments. Until April 17, 1906, there was no attempt made to maintain the temperature of the condenser constant. As a consequence, the results before this date are more variable than the others, and it seems scarcely worth while to present them in detail since the essential features of the uncorrected results and of the corrections are sufficiently shown by the later and more reliable work. We may, however, state that the 66 determinations made before this date can be made to give a mean of 2.99627 by assuming displacements of the plate not exceeding $68\ \mu$, and inclinations not exceeding $4' 42''$. The capacity was varied in the ratio of 1 to 15, and the average deviation from the mean given above is but 6 in 100,000.

The other results are set forth in Tables XXVIII and XXIX.

TABLE XXVIII.

Results by Plate Condenser. Plate A.

(Temperature controlled.)

Date	Distance between the plates in mm							
	1.64	2.64	3.64	4.64	5.64	6.64	7.64	9.64
1906								
Apr. 17.....	2.9971	2.9972						
25.....		2.9966						
26.....		2.9969						
28.....		2.9967						
28.....	2.9976	2.9965						
30.....		2.9971						
May 1.....		2.9969						
1.....		2.9963						
5.....	2.9972	2.9967	2.9966					
9.....			2.9954	2.9951	* 2.9939			
9.....			2.9964		2.9946	2.9952	2.9943	2.9928
9.....					2.9957		2.9948	
11.....		2.9967	2.9967		2.9957		2.9947	2.9937
11.....			2.9960		2.9951		2.9946	2.9946
Mean.....	2.99730	2.99676	2.99622	2.9951	2.99528	2.9952	2.9946	2.9937
Correction.....	-0.00143	-0.00041	+0.00028	+0.00077	+0.00115	+0.0014	+0.0017	+0.0020
$\bar{v}$	2.99587	2.99635	2.99650	2.99587	2.99643	2.9966	2.9963	2.9957
Mean of all $\bar{v}=2.99626 \times 10^{10}$								
$p_1=0.64$ mm; $q-p=66.5$ μ ; $\delta=94.5$ μ ; $\theta=3' 15''$								

* Omitted from mean.

In Table XXVIII are contained the results obtained by the use of Plate A, when the temperature of the condenser was maintained constant. These fall into a single group and but one initial distance, $p_1=0.64$ mm, was employed. Hence the uncorrected values have been tabulated and averaged and the corrections are applied to the means. Though the uncorrected means range from 2.9973 to 2.9937, it is necessary to assume a displacement of but 66.5 μ and an inclination of but $3' 15''$ in order to make them range but from 2.9966 to 2.9957, and now the high and low values alternate while the uncorrected values steadily decreased as the capacity was reduced. The mean of all these results is 2.99626×10^{10} .

TABLE XXIX.
Results by Plate Condenser.—Plate B.

Date	P ₁ mm	P-q μ	δ μ	θ	1			2			3			4			5			10			15			20			25		
					O	C	V	O	C	V	O	C	V	O	C	V	O	C	V	O	C	V	O	C	V	O	C	V	O	C	V
1906.																															
May 11	0.68	170	161.5'	32"				58+2	60	45+18	63																				
25								56+2	58	46+18	64																				
25								69+2	71																						
29								47+2	*49	78+18	*96																				
29								65+2	*67	72+18	*100																				
Oct. 3	0.66	2.7	58.2'	0"	71-8	63				58+18	*76			69-3	66																
8					72-8	64								64-3	61																
1907.																															
Jan. 5	0.59	5.3	37.51'	17"	67-4	63				56-1	55						63+0	63	69+2	71			53+3	56			31+4	*35	45+4	49	
8								65-2	63								61+0	61	63+2	65			75+3	78			72+4	76	70+4	74	
10																															
10																															
18																															
24	0.58	10.6	73.02'	31"	85-17	68											66-2	64	63+2	65			60+4	64			48+5	53	41+6	47	
24																															
25						74-17	57										65-2	63	66+2	68											
28	0.60	13.3	47.41'	38"	68-5	63											59+2	61	50+5	55											
Feb. 1						78-5	73										62+2	64	56+5	61			41+7	48			52+8	60			
1						57-5	52																								
1																															
Mean corrected value					2.99628		2.99630		2.99607		2.99635		2.99626		2.99641		2.99615		2.99570		2.99558										
Mean omitting three smallest capacities is $v=2.99630 \times 10^{10}$.																															

Mean omitting three smallest capacities is $v=2.99630 \times 10^{10}$.

* Omitted from mean.

The results obtained by the use of Plate B are contained in Table XXIX. The temperature throughout each day's work was maintained constant. In the first column is given the date, then follows p_1 , in mm, the initial distance between the plates as calculated from the measured electromagnetic capacity and the assumed value $\mathbf{v} = 2.9963 \times 10^{10}$. In the third column is given, in microns, the distance $q-p$ by which the plane of the plate is below the plane of the ring; in the next column is given, in microns, the maximum distance δ of the rim of the collector plate from the imaginary plane through its center and parallel to the movable plate; at least, this would be the value of δ were the lower plate plane; as it is somewhat cylindrical in form the value of δ is greater than that given by reference to the equivalent mean plane of the lower plate. In the following column is given the effective inclination of the plates, $\theta = \tan^{-1} \frac{\delta}{R}$

The numbers at the heads of the sixth and succeeding columns denote the approximate distance, in mm, between the plates. Each of these columns is divided into three parts; in the first (O) is contained the fourth and fifth significant figures of the values of $\mathbf{v}$ as deduced from the observations without applying the correction for the inclination and the displacement of the collector plate; the middle portion contains the correction (C) due to these causes, and the third portion contains the fourth and fifth significant figures of the corrected values of $\mathbf{v}$. The first three significant figures of $\mathbf{v}$ are in every case 299. At the bottom of the table are given the averages of the corrected values corresponding to each distance between the plates, values marked with an asterisk being omitted. These means agree very well with one another, though the capacities measured varied in the ratio of 1 to 10. Their mean is $\mathbf{v} = 2.99630 \times 10^{10}$.

From this table it is easy to see the advantage of the pivoted lever (page 454) in the adjustment of the plates. In May the plate was adjusted by the use of the eye and fingers only, and the displacement of the plate was 170 microns, and the inclination was $5' 32''$. On October 3, January 5, 24, and 28, the plate was displaced and readjusted by the use of the lever. Under these conditions the displacement varied but from 3 to 13 microns, and the inclination from $1' 17''$ to $2' 31''$.

That the values obtained from the smaller capacities are somewhat erratic is not surprising. If we assume that the uncertainty in the electromagnetic measurements is 0.01 micro-microfarad (this is but two-thirds of the average deviation of the measured capacity of the cylindrical condenser from its mean on the same day, page 591), then the uncertainty introduced by this cause alone is as follows:

Distance between plates	1	5	10	15	20	25	30 mm.
Uncertainty in ν	0.5	2.7	5.4	8.4	10.8	13.5	15.3×10^6 .

That is, at 20 mm distance between the plates the values found for ν may vary between 2.9974 and 2.9952, even when the uncertainty in the electromagnetic measurements amounts to but 0.01 micro-microfarad, or a capacity of 90 microns.

Though the results obtained from the plates are of no value in establishing the value of ν , they are not at all discordant with the results obtained by the use of the other condensers. Realizing now the importance of these adjustments we shall take pains when we return to this work to reduce these errors to a minimum, and trust then to be able to obtain, by the use of plates, results of positive value.

57. DISCUSSION OF RESULTS.

From what has been already stated it is evident that the results obtained by the use of the plates and of the cylinders without guards are of a negative value—they do not contradict the conclusions that may be drawn from the results obtained by the use of the other condensers.

The results obtained by the use of the spheres and of the guard cylinders and contained in Tables XXI, XXII, and XXVII are shown graphically in Fig. 44. Each value of ν given in these tables is represented by a dot, of which the size denotes the weight attached to this value, and the ordinate gives the value of ν corresponding to it. The dots are in order of the corresponding determinations, from left to right. The dotted lines represent the values of ν corresponding to the weighted means of the various groups.

These results are also summarized in Table XXX, where Δ is the average variation of the members of each group from their mean,

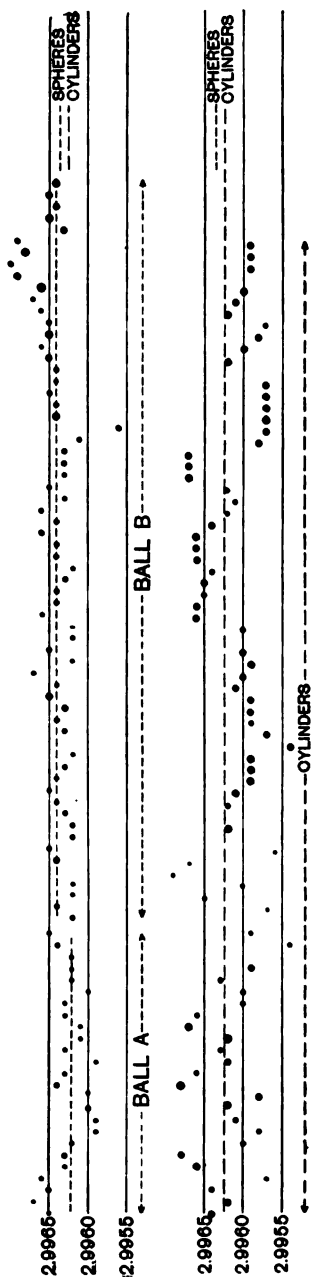


Fig. 44.—Graphical Summary of Results.

Δ_1 is the corresponding average variation from the weighted mean of all (2.99629), N is the number of independent determinations.

TABLE XXX.

Date.	Condenser.	V	Δ	N	Δ_1
1905	A	2.99621×10^{10}	0.00018×10^{10}	64	0.00018×10^{10}
1905	B	2.99637 "	0.00011 "	146	0.00013 "
1906	B	2.99659 "	0.00017 "	89	0.00030 "
1906	Cylinders	2.99624 "	0.00027 "	603	0.00028 "
Weighted Mean		2.99629 "	Mean		0.00022 "

The abnormally high value obtained by the use of ball B in 1906 has already been noticed. Being based upon the observations of but five days, it is not to be given undue importance, but nevertheless it does indicate a possible outstanding uncertainty of some kind.

From the concordance of the other observations we believe that the mean value (obtained from a series of observations extending over several months) of the square root of the ratio of the electrostatic to the electromagnetic capacity of an air condenser is

$$v = 2.9963 \times 10^{10} \left[\frac{\text{cm Int} \Omega}{\text{sec}} \right]^{\frac{1}{2}}$$

with an uncertainty of not more than 1 in 10,000.

Since the mean temperature at which the work was done was about 20°0, this gives us for the value reduced to vacuo (assuming the dielectric constant of air at 20°0 C and 760 mm pressure to be 1.00055)

$$v_0 = 2.9971 \times 10^{10}$$

We have given above a very full discussion of the sources of error in the determination of capacity in electrostatic and in electromagnetic measure, partly for the sake of the present work and partly because we think it desirable that the accuracy of the work be carried further in the near future. As we have already stated, the accuracy we expected to attain when the work was taken up nearly three years ago was considerably short of what we believe we have actually accomplished. We then regarded errors negligible

which we later took into account, and much of the theoretical investigation of the sources of error has been made since the observations were completed. Hence we were satisfied to begin the work with condensers much less perfect mechanically than they can be made. We are unable to pursue the subject any further at the present time, on account of other urgent work. Indeed, owing to the uncertainty in the absolute value of the ohm, there is no present need of greater accuracy than we believe we have attained. When, however, the ohm becomes known, to, say, 1 part in 15,000, as it doubtless will at an early date, we believe it would be a profitable undertaking to carry out a new determination of v , using spherical, cylindrical, and plate condensers, constructed with all possible accuracy, and to try to obtain a result trustworthy to within one unit in the fifth place—that is, to 1 part in 30,000. That would of course require improvements in the determination of electromagnetic capacity as well as improvements in the design and construction of the condensers. Such improvements are practicable, and such an accuracy, we think, is attainable. We hope we may be able to return to this problem in the near future.

58. PREVIOUS RESULTS AND THE VELOCITY OF LIGHT.

A résumé of all former work upon this subject was prepared by H. Abraham²¹ and presented before the International Congress of Physics which met in Paris in 1900. At the conclusion of this report he states that he considers the most accurate results to be the following:

Himstedt.....	3.0057×10^{10}
Rosa.....	3.0000×10^{10}
Thomson and Searle.....	2.9960×10^{10}
H. Abraham.....	2.9913×10^{10}
Pellat.....	3.0092×10^{10}
Hurmuzescu.....	3.0010×10^{10}
Perot and Fabry.....	2.9978×10^{10}
Mean.....	3.0001×10^{10}

Abraham stated that it was *probable* that this mean differs from the true value by not more than 1 in 1,000, but that this is not cer-

²¹ Congrès International de Physique, 1900; Rapports II, pp. 247-267.

tain. All of the above results are referred to air. Of these the result obtained by Thomson and Searle is very near the value given by the present work, which is a little more than 1 in 1,000 less than the above mean.

Turning to the question of the velocity of light, we have the results of Michelson²², obtained by the revolving-mirror method and giving

$$v = 299853 \text{ km} \pm 60 \text{ km},$$

that of Newcomb²⁴, obtained by the same method and giving

$$v = 299860 \text{ km} \pm 30 \text{ km},$$

and that by Perrotin²⁵, obtained by the toothed-wheel method and giving

$$v = 299860 \text{ km} \pm 80 \text{ km}$$

also, Weinberg²⁶ in discussing the results reaches the conclusion that the most probable value of the velocity of light in vacuo is

$$v = 299852 \text{ km} \pm 24 \text{ km}$$

From which we may conclude that the velocity of light in vacuo is

$$v = 2.9986 \times 10^{10} \frac{\text{cm}}{\text{sec}}$$

with an accuracy of 1 in 10,000.

This differs from our value for v_0 by 5 parts in 10,000, with a possible uncertainty of 2 parts in 10,000. Whether the remainder of the difference (1 in 3,000) is due to uncertainty in the value of

²² Astron. papers of the Am. Ephemeris, 1, pp. 112-145, 1880; 2, pp. 231-258, 1885.

²⁴ Astron. papers of the Am. Ephemeris, 2, pp. 107-230, 1883.

²⁵ Comptes Rendus, 185, pp. 881-884, 1902.

²⁶ Beibl. 28, p. 613, 1904.

the international ohm, or to greater errors in our determination of v and in the measurement of the velocity of light than we have supposed, or whether there is really a small difference between the velocity of light and the ratio of the units as measured by the method and at the frequencies we have employed, remains for future work to explain.

WASHINGTON, May 20, 1907.

A COMPARISON OF THE VARIOUS METHODS OF DETERMINING THE RATIO OF THE ELECTROMAGNETIC TO THE ELECTROSTATIC UNIT OF ELECTRICITY.

By E. B. Rosa and N. E. Dorsey.

In the masterly report to the British Association in 1863 "On the Elementary Relations Between Electrical Measurements," prepared by Prof. J. Clerk Maxwell and Mr. Fleming Jenkin,¹ occur the following paragraphs concerning the experimental determination of the ratio, ν , between electromagnetic and electrostatic measures of quantity:

"In order to determine the value of ν , it is necessary and sufficient that we should obtain a common electrostatic and electromagnetic measure of some one quantity, current, resistance, electromotive force, or capacity. There are thus five known methods by which the value can be obtained:

"First. By a common measure of *quantity*. Let a condenser of known capacity, s , be prepared. Let it be charged to a given potential P . Then the quantity in the condenser will be sP in electrostatic measure. The charge can next be measured by discharge through a galvanometer in electromagnetic measure. The ratio between the two numbers will give the ratio ν . The only difficulty in this method consists in the measurement of the potential P , entailing the measurement of an absolute force between two electrified bodies. The method was proposed and adopted by Weber.

"Second. By a comparison of the measure of *electromotive force*. The electromotive force produced by a battery, in electrostatic measure, can be directly weighed. Its electromotive force, in electro-

¹ Constituting Appendix C of the Report of the Committee on Standards of Electrical Resistance, B. A. Report, pp. 130-163; 1863.

magnetic measure, can be obtained from the current it produces in a given resistance. The ratio of the two numbers will give the value of v .

"Third. By a common measure of *resistance*. We know how to measure resistances in electromagnetic and in electrostatic measure. The ratio between these measures is equal to v^2 . The measure of resistance in electrostatic measure is not as yet susceptible of great accuracy.

"Fourth. By a comparison of *currents*. The electromagnetic value of a current produced by a continuous succession of discharges from a condenser of capacity s can be measured. The electrostatic value of the current will be known if the potential to which the condenser is charged be known. The ratio of the two numbers is equal to v .

"Fifth. By a common measure of *capacity*. The two measurements can be effected by the methods given. The ratio between the two measurements will give v^2 . This method would probably yield very accurate results."

During the nearly forty-four years that have elapsed since the above was presented to the British Association, many determinations of the ratio of the electrical units have been made. We present in this number of the Bulletin the results of our work by the last of the methods given by Maxwell and Jenkin, in which we have striven to carry the precision of the determination much further than has been done before. It seems a proper time to review the various methods that have been employed, or that might be employed, in order to show why we have chosen these particular methods and also to see whether in any future work other methods could be used with as good promise of obtaining an accurate value of v .

In the first stage of the work of finding v , extending up to about 1886, no value was found that could be depended on to nearer than 1 per cent, and most of the values were several per cent in error. In the second stage, during which a number of determinations were made between 1886 and 1898, the best results could be depended on to within one or two tenths of 1 per cent. We have now entered the third stage in the development of the work, in which results must be reliable to within one or two hundredths of 1 per cent in

order to be satisfactory in fixing the value of v . The question now is: How many of the various methods that have been employed or that might be employed would probably yield results to this order of accuracy if they were carried out under the most favorable circumstances and with the facilities available in a well-equipped, modern laboratory? The velocity v is so fundamental, and a knowledge of its value is so important, not only because of its theoretical identification with the velocity of light, but also because its numerical value is required in many electrical measurements and calculations, that it should be determined with all possible accuracy.

THE QUANTITY METHOD.

A determination by this method was carried out by Weber and Kohlrausch in 1857 and by Rowland in 1879. Practically every other method involves the use of a resistance, the value of which in absolute measure must be known. Hence this method, which does not require a knowledge of resistance, appealed to the early investigators, when resistances were not well known. The method is, however, complicated and difficult. In the first place, in common with all methods except the second (the electromotive-force method), it requires a condenser, the value of which is known in electrostatic measure, C . This is the easiest condition to meet, as a condenser can be built and measured so as to give its electrostatic value with high precision. The electromotive force to which it is charged must then be measured by an absolute electrometer. This is easy to do with a precision of a few tenths of 1 per cent, may perhaps be done to a tenth of 1 per cent, but is extremely difficult to do much better than this. The guard ring form has serious intrinsic difficulties, requiring as it does that the plane of the disk coincide closely with that of the ring. The force to be measured is very small unless the distance between the two plates is reduced unduly, and to get the precision of adjustment necessary while keeping the attracted plate free to move, it being at the same time in unstable equilibrium, is a matter of great difficulty. It is possible that a cylindrical electrometer can be designed that will avoid this trouble, but the latter has other difficulties of its own.

The electrostatic quantity, Q , becomes known when C , and E , are determined. This quantity of electricity is then measured elec-

trromagnetically by passing it through a ballistic galvanometer. To get Q_m with sufficient accuracy requires not only very accurate readings of the galvanometer deflections, but the determination of the constant of the galvanometer to a high order of accuracy. This can not be done satisfactorily by getting the strength of the earth's field in the usual way and calculating G from the dimensions of the galvanometer. The earth's magnetic field must be eliminated, as was done by Rowland, by the use of an absolute electro-dynamometer. This makes the experiment very difficult and complicated and renders the high precision we have specified practically impossible.

THE ELECTROMOTIVE-FORCE METHOD.

In this method no condenser need be employed. A certain constant electromotive force of considerable magnitude is measured by means of an absolute electrometer in electrostatic units E_s , and then it is obtained in electromagnetic measure by measuring the current I_m , flowing under this same electromotive force through a known resistance R_m .

$$E_m = I_m R_m, \text{ and then } \mathfrak{v} = \frac{E_m}{E_s}$$

If we assume that the value of the international ohm in absolute measure is known, then there are two measurements to be made in this method of determining $\mathfrak{v}$, namely, E_s by an absolute electrometer and I_m by an absolute electro-dynamometer. We have spoken above of the difficulty of measuring E_s with precision. The measurement of I_m is not so difficult, although by no means simple. It is doubtful whether this method of finding $\mathfrak{v}$ can compete in accuracy with the capacity method discussed below, and we believe that the most useful application of this method may be to *reverse the process* and, taking the value of $\mathfrak{v}$ as given by other methods, find $E_m (=E_s \mathfrak{v})$ by measuring E_s by the absolute electrometer. This will give a check on the value of the standard cell as determined by means of absolute electro-dynamometers. Inasmuch as $\mathfrak{v}$ is now probably as well or better known than the standard cell, this would be a valuable contribution to our knowledge of the value of the standard cell, provided the absolute electrometer measurements are sufficiently accurate. If they are not, they would be worth nothing in fixing the value of $\mathfrak{v}$.

Maxwell made the first determination (1868) by this method, combining the two instruments, the absolute electrometer and the absolute electrodynameometer, in one, so as to make it a null method. The apparatus was complicated and the experimental difficulties serious. As a pioneer experiment it was a success, although the result was 3 per cent too small. Hurmuzescu in 1896 published a determination of ν by the method of Maxwell; that is, combining the electrometer and electrodynameometer in one instrument. The former was a double cylindrical electrometer, and suspended inside a horizontal solenoid in the axis of rotation of the electrometer was the moving coil of the electrodynameometer. The instrument was constructed, and a long series of observations was made. The results were about one-tenth per cent too large. This seems to be one of the best determinations that has been made.

In the series of experiments carried out by Thomson and King (1869), McKichan (1874), Shida (1880), Thomson, Ayrton and Perry (1888), the electromotive force was measured by a Thomson absolute electrometer and the current by an absolute electrodynameometer or a tangent galvanometer. None of these determinations were very exact, the values being several per cent too small in every case.

The very careful determination by Pellat (1891) was made, using a Thomson electrometer and a Pellat absolute current balance. Full particulars of the work were not published, but the author claims no better accuracy than 1 in 500 for the electrometer. The result was nearly one-half per cent too high.

Perot and Fabry (1898) used an absolute electrometer of parallel plates, consisting of glass disks worked to optical surfaces and lightly silvered, the small distance apart being measured by the method of interference. The current through a standard resistance was measured in terms of the electrochemical equivalent of silver. The result is affected, as Abraham points out, by the uncertainty in the electro-chemical equivalent of silver, and the approximations used in calculating the constant of the electrometer. The result happens to be, however, only very slightly (0.1 per cent approximately) too high.

Some at least of the above experiments have been carried out by the most skilled observers working with excellent facilities. It seems doubtful whether very great improvement could be made by a repetition of the work.

THE RESISTANCE METHOD.

Assuming as in the previous method that the value of the international ohm is known in absolute measure, the problem in this method is to find the value of some particular resistance in this and in electrostatic measure. Since the ratio $\frac{R_e}{R_i} = v^2$ the uncertainty

in the absolute value of the ohm enters by only half its value in v . In this respect this method is superior to the preceding. If the resistance be a well-constructed coil of manganin wire, or a series of such coils, so mounted as to permit accurate comparisons with standard resistances, the determination, with the required precision, of their values in international ohms is of course not a difficult matter. Practically the whole problem lies in measuring the resistance in electrostatic units.

If a condenser of capacity C_e charged to a potential V_1 , discharge itself through a resistance R_e , the current at any moment is

$$i_e = \frac{V_e}{R_e} = -C_e \frac{dV_e}{dt}$$

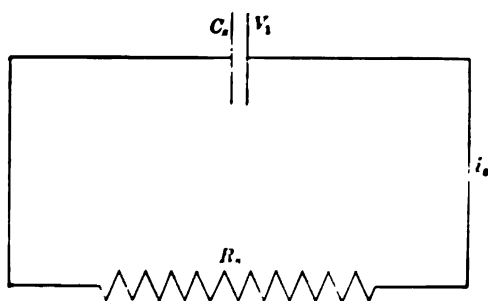


Fig. 1.

$$\therefore C_e \frac{dV_e}{dt} + \frac{V_e}{R_e} = 0 \quad (1)$$

$$\int_{V_1}^{V_2} \frac{dV_e}{V_e} = - \int_{t_1}^{t_2} \frac{dt}{C_e R_e}$$

$$\log \frac{V_1}{V_2} = \frac{t_2 - t_1}{C_e R_e}$$

$$R_e = \frac{t_2 - t_1}{C_e \log \frac{V_1}{V_2}} = \frac{t}{C_e \log \frac{V_1}{V_2}} \quad (2)$$

Thus, if a condenser of electrostatic capacity C_e discharges itself in t seconds through a resistance R_e from an initial potential V_1 , to a

lower potential V_2 , the value of this resistance in electrostatic measure is given by equation (2). Substituting $R_s = \frac{R_m}{\nu^2}$, we have

$$\nu^2 = \frac{C_s R_m \log \frac{V_1}{V_2}}{t}$$

Suppose that $V_2 = \frac{V_1}{2}$, that is, that the final potential is exactly one-half the initial potential, then

$$\nu^2 = \frac{C_s R_m \log_e 2}{t}$$

To get ν , therefore, we must have in addition to the value of the resistance in absolute electromagnetic measure two quantities: (1) The capacity in electrostatic units of the condenser; (2) the time t in which the condenser discharges to one-half its initial potential.

In order to find the approximate magnitude of the quantities C_s , R , and t , we may use equation (3), assuming ν known. The condenser must be a very well-constructed air condenser; suppose its capacity is 0.1 microfarad, which is rather large for an air condenser, but readily attainable. Let $t = 0.1$ sec. and take $\nu = 3 \times 10^9$. Substituting these values in equation (3) we find, since $\frac{C_s}{\nu^2} = C_m$

$$R_m = \frac{t}{C_m \log_e 2} = \frac{0.1}{0.1 \times 10^{-6} \times .693} = 1,443,000 \text{ ohms.}$$

This is a reasonable resistance, and hence we see that the experiment is perfectly practicable if we can find a way of measuring t with the required accuracy, and getting the value of the condenser in electrostatic measure with sufficient accuracy.

The electrostatic capacity can of course be determined only by comparing the condenser with another condenser, the value of which can be computed from its dimensions. Such a condenser will have a value about 1,000 times less than the condenser proposed to be used, that is, about 0.0001 inf. The difficulties of making directly an accurate comparison of two condensers of so different values are very great. It would probably be better to measure each absolutely by means

of the Maxwell bridge, and so obtain their ratio. If one were to attempt to work directly with the small air condenser whose capacity is determined from dimensions, and discard altogether the large air condenser of 0.1 mf, he would be obliged to use very great resistances, 1,000 megohms or more, or very short intervals of time. In either case the difficulty of getting accurate results would be increased. Supposing, however, that the electrostatic value of the large air condenser has been satisfactorily measured, let us see how to get the

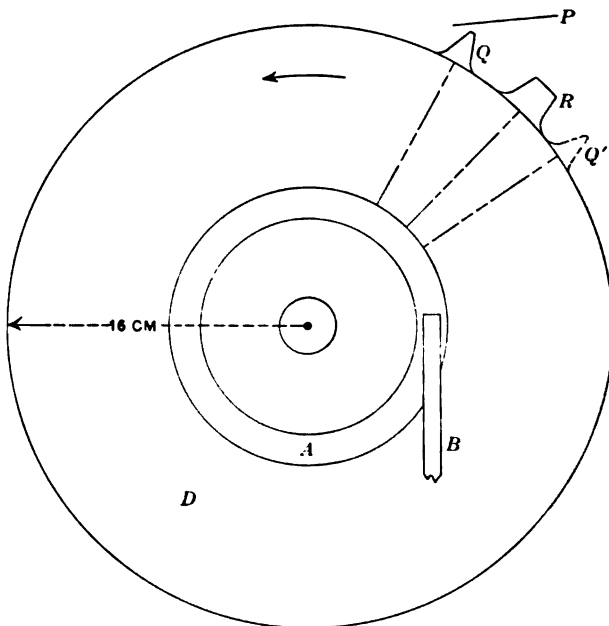


Fig. 2.

time t , during which it is discharged to one-half its initial potential, through the large resistance R .

Let D (Fig. 2) be a disk of hard rubber 32 cm in diameter mounted on a shaft and driven by an electric motor at a uniform speed of 600 revolutions per minute, or any other desired speed. At Q is a boss of phosphor-bronze which makes contact with the brush P for an instant only. R is a second boss making a longer contact. By contact rings and brushes one on each side of the disk, Q is joined to the middle of the charging battery and R to the end (Fig. 3). The

brush P is joined to the terminal B of the condenser. The condenser is then charged as P makes contact over R, and discharges itself through the high resistance during nearly one whole revolution of the disk, occupying 0.1 second, until the brush touches R again, when it is recharged. If as P touches Q the potential of the terminal B of the condenser is at a different potential from the middle point of the battery which is joined to Q, the galvanometer (or electrometer) will give a deflection. By adjusting the speed or the resistance or the potential of Q or R, the galvanometer deflection could be made zero. The simplest way, of course, would be either to keep the ratio of the potentials exactly 1 to 2, by means of a potentiometer and adjustable rheostat, and the resistance constant, and then adjust the speed of the contact disk so that the deflection is zero on the average for several minutes, recording the speed on a chronograph; or else, hold

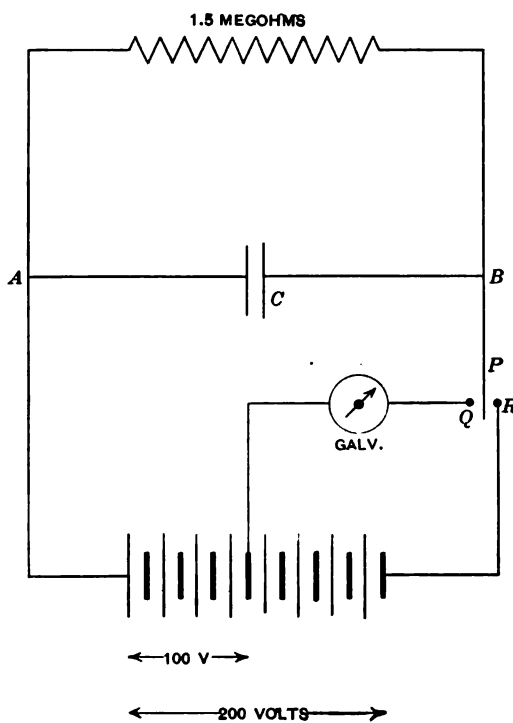


Fig. 3.

the speed steady by an auxiliary bridge and adjust the resistance to give zero deflection. In practice both methods should be used.

In order to determine the exact time between the breaking of contact at R and the middle of the contact at Q one could measure the angle by which Q must be turned back (to the position Q') in order to show a very small fall in potential, say one-half per cent, and calculate the time from this measured fall in potential between the breaking of R and the contact at Q'. This would give, know-

ing the angle QQ' exactly, the effective angle between the break of R and the middle of Q . This effective angle θ would depend on the adjustment of the brushes. Perhaps two exactly similar studs, QQ' , offset from R , and two brushes would do better. One brush would charge the condenser and one would make contacts on Q and Q' . A switch outside would put Q and Q' in service alternately. Q would show a drop of 100 volts; Q' of only about 1 volt. Then, without stopping the commutator or touching the brushes, the two observations would give a very exact value of the effective angle, θ , and from this and the speed, t would become known. This null method of working, with a large capacity (which would make the capacity of the lead wires, commutator, and brushes of little consequence) and a fairly high voltage which would reduce the disturbance due to electrostatic charges would probably give very accurate results. Possibly a revolving brush, P , with a stationary ring carrying Q and R would be better. Then Q could be moved back through a known angle, α , and Q' would be represented by Q in its new position, now, of course, joined to a different point of the battery, 1 volt (say) from the end.

It would seem that this experiment could be carried out with success, and give an accurate value of R_s and so of v . As already stated, however, it is a serious undertaking to obtain C_s , the electrostatic capacity of the large condenser. If we do it by getting electromagnetic capacities of two condensers and taking their ratio, we have v^2 in the ratio of C_s to C_m for the one absolute condenser. Hence all the work in getting R_s is *really unnecessary, except as affording by a roundabout way a value of v more easily obtained by the method of capacities*. By constructing an absolute air condenser having a capacity 100 times as great as assumed above (i. e. 0.01 mf) the experiment of getting R_s might be carried out on the absolute condenser itself. This would greatly simplify the work, but it would be a difficult matter, if not impossible altogether, to construct such a condenser so that its capacity could be certainly measured to 1 part in 10,000 by measurements of its dimensions alone. It would probably be a multiple-plate condenser, and the connecting wires would offer a serious difficulty in getting the capacity by calculation. This method was attempted by Klemenčič at Vienna in 1886, but without much success.

THE CURRENT METHOD NOT AVAILABLE.

There is no way of measuring directly the value of a steady current in electrostatic units. The method suggested by Maxwell, of charging and discharging a condenser rapidly, sending the charges or discharges through a galvanometer and measuring the mean value of such an interrupted current, as though it were a continuous current, requires a condenser of known value and is really included in the method of capacities. Hence we may better regard the current method as unavailable and include the plan proposed by Maxwell under the method of capacities.

THE METHOD OF CAPACITIES.

This is, by all means, the best of the four methods available. Methods 1 and 3 (the quantity method and the resistance method) both involve the use of capacities of known values in electrostatic measure. Hence they also might have been included under this method. The method of electromotive forces is therefore the only one that does not involve the use of a capacity, and that method is not capable of yielding results at all comparable in accuracy with the method of capacities. Hence we see that a precise knowledge of the velocity ν must come through the use of capacities.

There are several different ways of obtaining ν by this general method. In every one we must obtain the electrostatic capacity C , by calculation from the dimensions of the condenser. The difference comes in the various ways of getting C_m , but in all we require to know the value of a resistance in absolute measure. We shall specify seven different methods of finding C_m , the third and fourth of which are the simplest and most direct and capable of the highest accuracy.

(a) **By ballistic galvanometer.**—In this method the condenser of known electrostatic capacity C , is charged by a certain emf. E and then discharged through a ballistic galvanometer, precisely as in the quantity method. The difference between the two cases is this: In the first or quantity method the emf. E , must be measured by an absolute electrometer, giving $Q_s = C_s E_s$, and the ballistic galvanometer must be calibrated by a current measured by an absolute electro-dynamometer or tangent galvanometer; the resist-

ances are not assumed to be known in absolute measure. In this method the same battery is used in charging the condenser and in calibrating the galvanometer, so that it is $\frac{Q_m}{E_m}$ that is measured by the galvanometer, and that is C_m directly, which is of course the quantity sought. In thus eliminating E the value of a resistance is introduced, so that while avoiding the absolute electrometer and electrodymanometer, it is necessary to know resistance in absolute measure.

This method was employed by Ayrton and Perry in Japan in 1879 and by Hockin in the same year. While simple and direct, this method can not compare in accuracy with the null methods (c) and (d) below.

(b) **By steady deflection of a galvanometer**, using a fork or rotating commutator to charge and discharge the condenser periodically. The charges or the discharges may be allowed to pass through the galvanometer, and the steady deflection produced is then compared with that due to a steady current through a known resistance from the same battery. The number of charges and discharges per second must be known as well as the resistance employed in getting the steady current deflection.

This method was employed by Stoletow at Moscow in 1881 and by Klemenčič at Vienna in the same year. Stoletow used a revolving commutator to effect the successive charges and discharges of the condenser. Klemenčič used a tuning fork for this purpose. This was an important improvement over sending a single charge or discharge through the galvanometer.

(c) **By the Maxwell bridge, getting C_m in terms of R and t .**—This is the best-known capacity method, and has yielded excellent results. The condenser and commutator are placed in one arm of the bridge, and the bridge balanced by varying either a resistance or the speed of the tuning fork or commutator. Being a null method, the bridge may be made very sensitive by using high voltages and high frequencies of charging. As this and the next method (d) have been discussed at length in the foregoing paper we say very little about either at this time.

This method was first employed for finding ν by J. J. Thomson at Cambridge in 1883. It has since been employed by Himstedt at

Darmstadt (1887), by Rosa at Baltimore (1888), by Thomson and Searle at Cambridge (1890), and by the present writers at Washington 1904-1907.

(*d*) **By the differential galvanometer.**—This is a modification of (*b*), the interrupted current from the condenser and the steady current from the battery passing simultaneously through the two coils of a differential galvanometer, and so making it a null method. A greatly increased sensibility is thus secured and errors due to measuring deflections eliminated. This method possesses the same possibilities of high accuracy as (*c*), and is subject to practically the same limitations.

The method was first employed by Klemenčič at Vienna in 1884. It has since been used by Himstedt at Freiburg (1886) and Darmstadt (1888), by Abraham at Paris (1892), and by the present writers at Washington (1905-1907).

(*e*) **By oscillatory discharges of a condenser.**—If a condenser of capacity C_m be placed in series with inductance L_m , the period of its oscillatory discharge is given by the expression

$$t = 2 \pi \sqrt{C_m L_m}$$

when the resistance is small. By determining t , one obtains a measure of the product $C_m L_m$, and by then measuring L_m the capacity of the condenser in electromagnetic measure becomes known. From C_s and C_m , ν becomes known as in the other capacity methods. There are thus three quantities to measure— t , C_s , and L_m .

This method was employed by Colley at Kasan (1886), by Webster at Clark University (1898), and by Lodge and Glazebrook (1899).

The time of oscillation may be determined in various ways. Colley used a galvanometer of very high frequency, Webster employed an electrometer, Lodge and Glazebrook photographed by a revolving mirror the spark produced in the discharge.

Although the accurate determination of the period t is not without difficulty, the chief sources of error probably lie in getting C_s and L_m . In order to obtain t accurately, the oscillations must not be very rapid. This requires a relatively large condenser and a large inductance coil. If the condenser consists of a large number of parallel plates very close together (Lodge and Glazebrook), it is

impossible to get their capacity with precision by measurement and calculation. If the capacity be determined by comparison with a standard condenser of suitable design for accurate calculation of the capacity, the experimental comparison of the two involves greater errors than the absolute measurement of C_m for the small condenser. Similarly, the inductance must be larger than can be accurately calculated from its dimensions, and its electrostatic capacity offers a troublesome factor to obtain except by experiment. We believe that one-tenth of 1 per cent is about the limit of accuracy possible

to obtain at present by this method, which is a much higher accuracy than has yet been obtained by the method.

(*f*) **By rotating a magnet within a suspended coil.**—This method, which has been suggested by Gray,² has never been carried out. A magnet is to be rotated rapidly within a coil, the latter being suspended in a vertical plane by a bifilar so as to turn freely about its axis. The induced current deflects the coil through an angle θ , the value of which depends on the value of the capacity and induc-

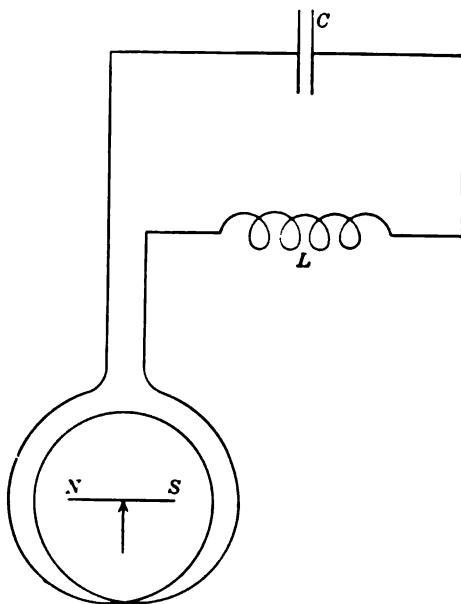


Fig. 4.

tance in series with the coil, as well as upon the coil itself, the suspension, the magnet, and the speed with which it is rotated. The maximum deflection is given by such a speed that $t = 2\pi\sqrt{CL}$, where t is the time of one revolution of the magnet. This requires the product CL to be relatively large—that is, one would have to employ a large capacity and a large inductance. By using three different angular velocities— n_1 , n_2 , n_3 ,—and measuring three dif-

² Absolute measurements, etc.

ferent deflections— D_1 , D_2 , D_3 —the value of $C_m L_m$ is obtained from the following equation:

$$C^3 L^3 = \frac{1}{n_1^3 n_2^3 n_3^3} \cdot \frac{\frac{n_1^3}{D_1}(n_2^3 - n_3^3) + \frac{n_2^3}{D_2}(n_3^3 - n_1^3) + \frac{n_3^3}{D_3}(n_1^3 - n_2^3)}{\frac{n_1}{D_1}(n_2^3 - n_3^3) + \frac{n_2}{D_2}(n_3^3 - n_1^3) + \frac{n_3}{D_3}(n_1^3 - n_2^3)}$$

This equation results from the three equations given by the three sets of observations, after eliminating the resistance and the constants of the coil.

It is obvious that this experiment, while interesting theoretically, can not afford a means of determining ∇ with precision. The three speeds and the three deflections would have to be determined with extraordinary precision to give an accurate value of $C_m L_m$. When that is determined it still remains to find C_s and L_m as we obtain C_m , by dividing $C_m L_m$ by L_m , and then get ∇^2 from C_s and C_m . But since C_m and L_m are both necessarily large, we can not calculate them directly from their dimensions, but must compare them with absolute condensers and coils that have been so determined or measure L_m in terms of resistance. Thus the method is very elaborate and offers many sources of error.

An idea of the magnitude of C and L can readily be obtained from the condition that for maximum deflection $t = 2\pi\sqrt{CL}$; $\frac{2\pi}{t} = n =$ angular velocity, t being the time of one revolution of the magnet. Suppose $n = 100$, corresponding to about 1,000 revolutions per minute. Then

$$n^2 = 10,000 = \frac{1}{CL}$$

$$\text{Let } C = 10 \text{ microfarads} = \frac{1}{10^6} \text{ farads}$$

$$L = 10 \text{ henrys}$$

$$\text{Then } CL = \frac{1}{10^4} = \frac{1}{n^2}$$

Evidently a capacity of 10 microfarads and an inductance of 10 henrys are far too large to obtain directly from their dimensions. Both can be measured in terms of ohms, but to get C , requires, as in the preceding method, a comparison with an absolute condenser.

(g) **By comparing a capacity and a resistance by use of an alternating current.**—This method was employed by Miss Maltby in 1897, as follows:

An alternating current flows through ABD, the air condenser C being in series with a noninductive resistance W. The applied electromotive force is generated by the rotation of a magnet within a

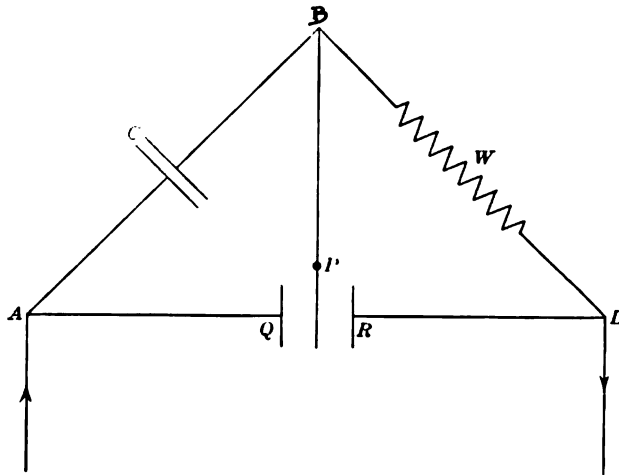


Fig. 5.

coil, and is nearly a sine wave. The electrometer PQR is used to indicate by its balance that the effective emf. on AB is equal to that on BD. In this case

$$W' = \frac{I}{2\pi n C}$$

$$\text{or } T = \frac{I}{2n} = \pi C W'$$

T being the time of one-half cycle of the alternating emf. Knowing T from the speed of the rotating magnet and W' , the capacity C becomes known in electromagnetic measure. It is to be observed that the measured capacity C_m of the condenser will be too large if correction is not made for the capacity of the electrometer.

This simple and interesting method is evidently not suitable for precision measurements, unless the wave form is determined very accurately and allowance made for any harmonics found.

There are other alternating current methods of obtaining capacity in terms of resistance, those of Wien using a bridge and tuned galvanometer being especially favorable to accuracy as the effect of harmonics is eliminated by the tuned galvanometer, and no error is produced by a wave form varying from a sine curve. The chief difficulty in any case is to obtain the electrostatic capacity of the condenser if the latter is large, or the electromagnetic capacity with sufficient accuracy if the condenser is small, the difficulty common to most of the condenser methods.

(*h*) **By comparing C_m with L_m by means of a bridge.**—We can measure the electromagnetic capacity of a condenser in terms of an inductance by means of the bridge methods of Maxwell, Anderson, Wien, or Rosa, using alternating currents. Having thus obtained C_m , we get ν as soon as C_s is determined. This appears a simple and direct method of getting the ratio of the units. The difficulty lies chiefly in getting the electrostatic capacity C_s of a condenser large enough to use in the bridge methods for finding C_m . It is the same difficulty noticed already in other methods. There is no way of directly measuring a small capacity (such as that of an accurate absolute condenser) in terms of an inductance with sufficient precision for this work. Hence it is necessary to get the value of a large air condenser by comparing it with a small absolute air condenser, and then compare the former with the known inductance. This of course can be done with considerable precision, but it is less direct and far less likely to give a result of high precision than methods (*c*) and (*d*) above described, and it involves precisely the same kind of errors that are involved in those methods, with others in addition. We do not mean to say that the last three capacity methods are without value. What we wish to emphasize is that not one of them permits as direct a determination of the capacity of an absolute condenser in electromagnetic measure as do methods (*c*) and (*d*), and that while they all possess serious disadvantages as compared with (*c*) and (*d*), none of them possesses any single compensating advantage.

Our conclusions are as follows:

1. Of the four general methods available, the quantity method is most complicated and least accurate.
2. The electromotive force method is better than the first, but can perhaps be of more service in giving us a more accurate knowledge of the absolute value of the standard cell, taking ∇ as known, inasmuch as ∇ is as well or better known than the value of the standard cell.
3. The resistance method is roundabout, and could not be expected to give as accurate results as the more direct of the capacity methods.
4. Of the seven capacity methods that have been used or suggested, the first two (*a*) and (*b*) have been superseded by the next two (*c*) and (*d*), which are the best of all; the last three are complicated and incapable of high accuracy.
5. With the use of several different forms of absolute condensers, constructed as perfectly as possible, and measured by the capacity methods (*c*) and (*d*), we believe that thoroughly reliable results can be obtained without further use of any of the less direct and more difficult methods.

WASHINGTON, June 1, 1907.

PRELIMINARY SPECIFICATIONS FOR CLARK AND WESTON STANDARD CELLS.

By F. A. Wolff and C. E. Waters.

Directions for the preparation of materials used in Clark and Weston standard cells, and for their construction, have been published from time to time.¹

The following specifications are based upon these directions and upon work done at the Bureau of Standards. They are submitted in more detail than might be considered necessary, in order to secure the fullest possible cooperation of other workers in this field in determining the accuracy of reproduction attainable.

Definitions.—The *Clark standard cell* will be understood to mean a voltaic combination having as its positive electrode pure mercury covered with a paste consisting of pure mercurous sulphate mixed with finely divided mercury and pure zinc sulphate crystals and solution, and as its negative electrode an amalgam containing 10 per cent by weight of pure zinc, covered with a layer of pure zinc sulphate crystals, the electrolyte being a solution of pure zinc sulphate in saturation equilibrium with the other constituents of the cell.

The *Weston standard cell* will be understood to mean a similar combination in which the zinc sulphate is replaced throughout by cadmium sulphate and the zinc amalgam by an amalgam containing 12.5 per cent by weight of pure cadmium.

¹ Kahle: Zs. Instrk. **13**, 191; 1893; Wied. Ann., **51**, 203; 1894.

Jaeger & Wachsmuth: Electrotech. Zs., **15**, 507; 1894; Wied. Ann., **59**, 575; 1896.

Carhart & Hulett: Trans. Am. Electrochem. Soc., **4**, 59; 1904.

Committee Am. Inst. El. Eng., Trans. A. I. E. E., **23**, 827; 1904.

Smith: Electrician (Lond.), **55**, 857; 1905.

It is impossible at present to obtain all the materials employed in the construction of standard cells of sufficient purity for use in *primary* standards, and they should, therefore, be prepared according to the methods described below.

On account of the large electromotive differences found in samples of mercurous sulphate purchased as chemically pure, it is always advisable to subject it to further treatment or to prepare it specially, even when it is intended for use in *secondary* standards. For the latter purpose, C. P. materials may be used without further treatment, unless the zinc and cadmium sulphates are found to be acid to Congo red, in which case they should be recrystallized. Cells set up with ordinary C. P. materials and specially prepared or treated mercurous sulphate should not differ from those in which the most carefully prepared materials are used by more than 0.01 per cent.

PURIFICATION AND PREPARATION OF MATERIALS.

Mercury.—Mercury, as purchased, may contain varying amounts of different metals, such as copper, silver, iron, etc., and by use in the laboratory others, such as zinc, cadmium, and nickel, may be added. If very impure it “tails” when poured from the bottle. In this case it should be first subjected to a preliminary purification by method *a* or *b*.

(*a*) The simplest, though least efficient, method is to shake it for a long time with a solution of mercurous nitrate in dilute nitric acid, or in dilute nitric acid only. Instead of shaking, it may be allowed to fall in very fine drops from a funnel drawn out at the end, through either of the above solutions contained in a long, wide tube, at the lower end of which is sealed a smaller S-shaped tube which serves as a trap. Before introducing the solution some mercury must be poured into the trap, and the first portions that run out must be poured through the funnel again. Instead of one of the above solutions a dilute solution of ferric chloride, slightly acidified with hydrochloric acid, may be used. The treatment must be continued for a long time to completely remove the dissolved impurities, as the action of the solution is necessarily superficial and therefore slow.

(*b*) By making the mercury the anode and a piece of platinum foil the cathode, using 2 per cent nitric acid as the electrolyte, the more positive metals go almost completely into solution by electrol-

ysis, leaving in the mercury the less positive metals which exert only a minor influence on the e. m. f. and which may afterwards be removed, as described under *c* or *d*.

The current density should be about 0.1 amperes per dm², and the electrolysis should be continued, constantly stirring the mercury, for some hours after it no longer tails. The time required depends on the amount and the original condition of the sample. The mercury deposited on the cathode, and possibly containing some of the electropositive impurities, is prevented from dropping back into the anode mercury by suspending under the cathode a small beaker hung from the side of the battery jar by means of a support made of glass rod. The combination anode and stirrer described below may be conveniently used. The depth of the mercury should be great enough to cover completely the blades of the stirrer when rotating, and the volume of dilute acid employed should be at least four times that of the mercury.

Ordinary mercury, or that which has been subjected to either of the preliminary treatments described above, must be further purified by one of the following methods:

(*c*) By double distillation in an ordinary vacuum still.

(*d*) By distillation in a current of air under reduced pressure. For this the form of still described by Hulett and Minchin² is recommended.

Zinc and Cadmium.—The traces of impurities ordinarily found in the C. P. metals, purchased from reliable manufacturers, have such a slight influence on the e. m. f. that they may be used without further purification in the preparation of the amalgams for *secondary* standards. For *primary* standards the amalgams may be made by electrolysis, as described below, thus rendering unnecessary the preparation of the pure metals, methods for which are, however, given.

Zinc.—The pure metal may be obtained:

(*a*) By distillation under diminished pressure according to the method of Morse and Burton.³ A large tube of Jena or Kavalier's combustion glass is closed at one end. The metal is introduced and the tube is then drawn out as close as practicable to the metal. Sev-

³Phys. Rev., 21, 388; 1905.

²Am. Chem. Jour., 10, 311; 1888.

eral centimeters farther along, according to the amount of zinc used, the tube is again drawn out. As the object of this is to make a reservoir into which the metal is distilled, the tube should not be drawn out symmetrically, but as much to one side as possible, to prevent the melted metal from flowing from one section of the tube to another. The tube is placed in a combustion furnace, connected by means of a rubber stopper and heavy tubing to a mercury or Geryk pump and exhausted. The pump should be kept running not only during the distillation, but afterwards, until the tube has cooled. The tube is first heated at the closed end and by smaller flames under the second section. The flames must be regulated so that the tube does not soften and collapse, and also so that as little zinc as possible is carried over beyond the second section. When about three-fourths of the zinc has distilled off the flames are extinguished and the tube allowed to cool. Air must not be admitted until the metal is perfectly cold. If the tube is sufficiently long, it may be divided into three sections and the zinc distilled from the second into the third, thus effecting a double distillation in practically one operation. The flames under the closed end of the tube should be turned low, but not extinguished, to prevent cooling and consequent cracking. The distilled metal adheres strongly to the glass, which may be removed by hammering with a porcelain pestle.

(b) Pure zinc may be obtained by electrolysis of a zinc sulphate solution purified as described below.⁴ The 40 per cent solution is electrolyzed, using two platinum cathodes and a pure sheet zinc anode placed between them, but separated by silk partitions. Some zinc oxide is placed around the anode and the solution is continuously neutralized by adding ammonia. In the cell with the cathodes is placed a small platinum anode to keep the solution acid. The metal thus obtained may be distilled or used directly to prepare the amalgam.

Cadmium.—The pure metal may be obtained:

(a) By distillation using the same apparatus as described for zinc.⁵ It distills at a much lower temperature.

⁴ Mylius and Fromm: *Zs. anorg. Chem.*, **9**, 144 (1895).

⁵ Morse and Jones: *Am. Chem. Jour.*, **14**, 261; 1892.

(b) By electrolysis of the fused chloride⁶ or of a purified solution of the sulphate.⁷

Zinc Sulphate.—The salt is apt to contain sulphates of cadmium, iron, lead, etc., and free sulphuric acid. The last of these has the greatest effect upon the e. m. f. and promotes the formation of gas in the amalgam limb. The effect of such small quantities of the other impurities as are apt to be contained in the “chemically pure” salt is practically negligible. The following methods of purification, both giving the same results in the cell, may be used:

(a) Dissolve the pure salt, as purchased, in hot water, add an excess of pure zinc oxide, and keep at nearly the boiling point for several hours to throw down iron as completely as possible. If the iron is in the ferrous state, its precipitation is greatly hastened by adding pure hydrogen peroxide, a very small quantity of which will suffice to effect its oxidation. Filter, acidify very slightly with dilute sulphuric acid, and evaporate until the greater part of the zinc sulphate crystallizes out on cooling with ice (5° or lower). Stir frequently while cooling so as to obtain small crystals. These are filtered off, using a platinum cone, then washed once or twice with very little ice-cold water, using suction, redissolved in a little hot water, and recrystallized as before. Further crops of crystals can be obtained from the first and second mother-liquors. Those from the first are recrystallized once, those from the second mother-liquor twice. The three lots of crystals are then combined and dissolved in sufficient water to form a saturated solution at room temperature. The solution is filtered if necessary and set aside to crystallize by spontaneous evaporation in dishes covered with filter paper. If desired, a saturated solution may be made at not more than 35°, cooled, with constant stirring, by surrounding the beaker with ice, and the crystals filtered off at once. In the final recrystallization the solution must not be heated at any stage above 35°, on account of the danger of forming a lower hydrate, which takes place at 39°. In either case the crystals are washed with two or three small portions of ice-cold water, air-dried, and placed in well-stoppered bottles before they begin to effloresce.

⁶ Lorenz: *Zs. anorg. Chem.*, **11**, 49; 1896.

⁷ Mylius & Funk: *Ibid.*, **13**, 157; 1897.

The zinc oxide employed is prepared by adding ammonia to a solution of zinc sulphate until the precipitate dissolves. It is then filtered into a large volume of water, allowed to settle, the supernatant liquid decanted and the precipitate thoroughly washed on a Büchner funnel (hardened filter paper), removed from the filter paper and ignited in a platinum crucible inclosed in one of porcelain to prevent access of reducing gases.

(*b*) The zinc sulphate may also be purified by electrolysis.⁸ After removal of iron, as described under *a*, a weak current (about 0.1 ampere per dm²) is passed through a solution of zinc sulphate containing suspended zinc oxide to keep it slightly basic. Platinum electrodes are used and the solution is kept stirred. The electrolysis is continued until a clean anode is no longer coated with lead peroxide. The solution is sufficiently pure, though it may still contain traces of other metals. It is filtered, acidified slightly, and the salt recrystallized as described above.

Cadmium Sulphate.—The commercial “chemically pure” salt may contain zinc, lead, ferrous and ferric iron, and occasionally nickel. The salt can be purified by dissolving in an excess of water at about 70°, filtering if necessary, adding an excess of pure cadmium oxide or basic cadmium sulphate and heating for some hours. As much of the iron may be in the ferrous state, a little pure hydrogen peroxide should be added with the cadmium oxide to oxidize the iron. The solution is filtered from precipitated iron and basic cadmium sulphate, acidified slightly, and evaporated nearly at its boiling point in a large porcelain dish resting on a pipestem triangle on a hot plate or supported some distance above the flame of a gas stove or ring burner. Even then the flakes of the lower hydrate of cadmium sulphate which are formed fall to the bottom and cause violent bumping unless they are frequently removed. They are allowed to drain in a funnel with a platinum cone. When the solution has been evaporated to a small volume, it is poured, while still hot, through the funnel, the crystalline flakes are packed down with a pestle, allowed to cool, and washed twice with a little cold water, using suction. They are then redissolved and the solution evaporated down to a small volume as before. After this operation the crystals are dissolved, with mechanical stirring, in an equal weight of water at room temperature, fil-

⁸ Mylius and Fromm: *Zs. anorg. Chem.*, **9**, 144; 1895.

tered if necessary, and set aside in shallow layers (2 to 3 cm) in crystallizing dishes covered with filter paper. Heat should not be employed to hasten the final crystallization on account of the danger of transforming the crystals into a lower hydrate, which takes place at about 70° . If the solution is saturated when placed in the dishes, especially if it be a mother-liquor from which a crop of crystals has been removed, the salt is almost certain to come down in a few hours as a crust over the bottom, instead of isolated crystals. Most of the crystals are cloudy, but those from a purified solution have been found to give as good results in the cells as the perfectly clear ones. The crystals, especially the clear ones, adhere so firmly to the bottom of the dish that they are apt to be broken in removing them. This may be obviated by decanting the mother-liquor and pouring a little pure water over the crystals. In a few moments they will be loosened, without going into solution to any great extent, and may be removed with a spatula. They are to be washed two or three times with water and preserved air-dried in a bottle. Further crops of crystals may be obtained from the mother-liquor, which should first be diluted with about one-tenth its volume of water.

Zinc and Cadmium Sulphate Solutions.—A *saturated* zinc or cadmium sulphate solution is required for making up the paste and for filling the cells. It is prepared by agitating an excess of purified salt with distilled water. In the case of zinc sulphate, shaking with water previously heated to not more than 30° for at least a half hour is sufficient, while for cadmium sulphate mechanical stirring for three or four hours is required on account of the slowness with which it dissolves. In the case of zinc sulphate the mother-liquor, from which a crop of crystals has been removed by cooling, should not be used, as this is unsaturated at room temperature.

Amalgams.—*Zinc amalgam* containing 10 per cent, by weight, of zinc may be prepared by the following methods:

(a) By dissolving a weighed amount of pure zinc in 9 times its weight of pure mercury. The latter is heated gently in a porcelain dish on a hot plate or sand bath. The zinc, previously treated with very dilute sulphuric acid to remove the film of oxide, and then washed with water and dried, is placed upon the hot mercury and frequently stirred to hasten solution. The heat is to be increased whenever the amalgam shows a tendency to solidify.

(b) A better method, which obviates the preparation of the pure metal, consists in the electrolytic deposition of the zinc from a purified solution of the sulphate, slightly acidified with sulphuric acid, using a weighed amount of mercury as cathode and a rod of the purest commercial zinc as anode. The current strength is measured and the electrolysis continued until somewhat more than the required amount of zinc has been deposited. The amalgam is then removed from the solution, washed thoroughly, dried and weighed, and enough mercury added to bring the amalgam to the required percentage. The anode should be inclosed in filter paper to prevent impurities contained in it from contaminating the amalgam.

Cadmium amalgam, containing 12.5 per cent by weight of cadmium, may be prepared by the same methods as zinc amalgam. On account of its relatively low melting point the temperature of the steam bath is sufficient.

Oxidation of the amalgams.—On exposure to the air the surfaces of the amalgams are slowly tarnished by oxidation. As considerable changes in the composition on the amalgams have no appreciable influence on the electromotive force, this is of no importance. If desired, the oxide may be removed by straining the melted amalgam through a test tube drawn out at the bottom to a small opening, and heated gently to keep the amalgam from solidifying.

Mercurous Sulphate.—The “chemically pure” mercurous sulphate at present obtainable on the market may contain as impurities basic mercurous sulphate, basic mercuric sulphate, traces of nitrate, etc., according to the method of preparation. In addition, the size of grain of the commercial samples, usually prepared by rapid precipitation, may also have an influence on its electromotive properties. Such materials can, therefore, not be employed if the highest accuracy of reproduction is sought. In addition, pure mercurous sulphate hydrolyzes in the presence of water, or even dilute sulphuric acid, so that the washing with water as called for by the old specifications should be omitted. Commercial samples give a higher e. m. f., which decreases for a considerable period after the cell is set up. When, however, mercurous sulphate prepared by one of the following methods is employed, the cells assume their normal values within a few days. As mercurous sulphate is acted upon by light, it should be prepared in a dimly-lighted room and preserved in the dark.

(a) The electrolytic method :⁹ Pure mercury, prepared as specified above, is placed in a battery jar of convenient size to a depth of 3 to 4 cm and covered to a depth of 10 to 15 cm with chemically pure 1:6 sulphuric acid. It is then connected to the negative terminal of a battery or lighting circuit in the usual manner by a platinum wire protected from contact with the solution by a glass tube. The cathode consists of a small piece of platinum foil immersed in the upper part of the solution. The current is regulated by means of a suitable rheostat to 2 to 5 amperes per dm² of exposed anode surface.

To avoid the possible formation of mercuric sulphate, the mercury and the solution should be vigorously and continuously stirred, without, however, breaking up the mercury into small globules. The stirrer may be a glass T-tube with a cross arm of suitable size for the battery jar employed, rotating close to the surface of the mercury. It may

be fastened at the open end of the long arm to a small brass cylinder fitting the sleeve on the shaft of the stirring apparatus. One end of the cylinder should be turned down so as to fit loosely into the glass tube, to which it is fastened by shellac or hard Khotinsky cement.

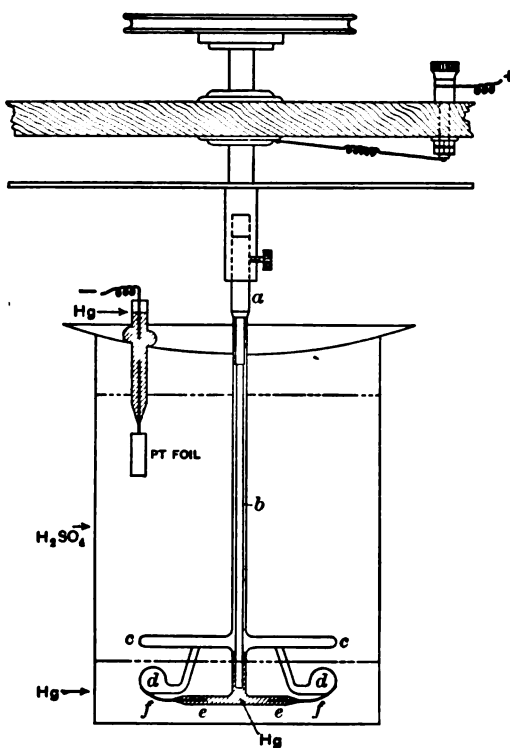


Fig. 1.

⁹ Carhart and Hulett: *Trans. Am. Electrochem. Soc.*, 5, 59; 1904. Wolff: *Ibid.*, 5, 49; 1904.

A still more efficient form of stirrer has a second cross arm about 3 cm above the first, so that both the mercury and the solution in contact with it may be stirred simultaneously.

The most satisfactory form is a combination stirrer and anode connection. The two cross arms, figure 1, *cc* and *ee*, are about 3 cm apart, and the lower arm has short pieces of platinum wire, *ff*, sealed into the ends. The efficiency of the stirrer is increased by the paddles, *dd*, made of glass rod 2 to 2.5 mm in diameter, flattened at the end and sealed to the cross arm, *cc*, and the platinum terminals, thus also making the construction more rigid and preventing the platinum wires from being broken off during cleaning. Contact is made with the bearing of the rotating apparatus by means of a copper wire, *b*, soldered into the lower end of the brass cylinder, *a*, and extending almost to the cross tube, *ee*, the latter and the lower part of the stem being filled with mercury. The soldered joint should be protected from contact with the mercury by a coat of shellac or Khotinsky cement.

The object of this construction is to secure a vigorous stirring of the mercury, so as to continually expose a fresh surface, to prevent the mercurous sulphate formed from heaping up around the usual anode tube and to keep it in suspension. The operation should be watched, as the speed of the stirrer may decrease after the formation of sulphate begins, on account of the greater resistance to be overcome.

The best position of the stirrer can readily be determined by trial, and is that position in which the mercury is vigorously rotated without being broken up into small globules. A speed of 100 to 200 revolutions per minute may usually be employed. The depth of the mercury should be sufficiently great so that the platinum terminals will not be exposed to the action of the electrolyte even during rotation.

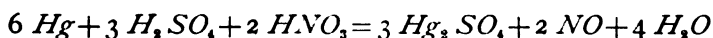
The product formed is light to dark gray, from the presence of finely divided mercury, depending upon the current density, strength of acid, and rate of stirring employed, all of which may be varied between wide limits without appreciably affecting the electromotive properties of the product. To minimize the possible influence of size of grain the stirring is preferably continued for some hours after the current is interrupted.

The mercurous sulphate, separated from the excess of mercury, which interferes with its subsequent washing, by means of a separating funnel (using no grease on the stopcock) or by means of a pipette, should then be transferred to a clean glass-stoppered bottle and preserved in the dark under 1:6 sulphuric acid.

The theoretical rate of formation is 5 grams per ampere hour. Where a considerable quantity is formed, sufficient sulphuric acid may be added from time to time to make up for that which has combined with the mercury, or 1:5 sulphuric acid may be used in place of that specified above.

(b) Mercurous nitrate is prepared by the action of strong, C. P. nitric acid on an excess of pure mercury and the concentrated solution diluted with five volumes of distilled water, adding, if necessary, a few drops of nitric acid to clear up any cloudiness due to the formation of basic nitrate. This solution is then added drop by drop, with constant stirring, to 5 volumes of 1:4 sulphuric acid. Wash several times by decantation with 1:6 sulphuric acid and preserve in the dark in a glass-stoppered bottle with mercury and acid of the same strength.

(c) A simpler method, in which the preparation of mercurous nitrate is avoided, consists in the treatment of mercury with sulphuric acid containing a small percentage of nitric acid. The reaction which takes place is that involved in the Lunge method for the estimation of nitrates, so that practically all the nitric acid is eliminated from the solution.



The rate of the reaction depends on the concentration of the sulphuric acid, the amount of nitric acid present, and the temperature. The mercury, covered with a liter of hot 1:2 sulphuric acid, is contained in a beaker on a hot plate. Five cc concentrated nitric acid are then added and the whole stirred vigorously for some time after the reaction is completed, as shown by the disappearance of the brown fumes of nitrogen peroxide. On account of these fumes a well-ventilated hood is desirable. The product formed, which is quite gray from the presence of finely divided mercury, is washed several times by decantation with 1:6 sulphuric acid and preserved in the dark in a glass-stoppered bottle with acid of the same strength.

(*d*) By reduction of mercuric sulphate with mercury. Twenty-five grams of mercury are added to 50 cc boiling concentrated sulphuric acid in a porcelain dish, continuing the heating for some time after all the mercury has dissolved. Cool, pour in 500 cc of 1:6 sulphuric acid, and add the diluted solution, drop by drop, to a liter of the same acid contained in a large beaker with a 2 cm layer of mercury, vigorously stirring both the solution and the mercury. The stirring should be continued for some hours after all the mercuric sulphate has been added. The dark-gray product is washed and preserved as under (*c*).

(*e*) By recrystallization of commercial mercurous sulphate.¹⁰ Fifty grams of commercial mercurous sulphate and 25 grams of mercury are added to 200 cc concentrated sulphuric acid previously heated to 150°. Stir about ten minutes, allow to settle if necessary, and pour slowly and with constant stirring into about 10 volumes of 1:6 sulphuric acid. To avoid spattering, the beaker containing the dilute acid is covered with a perforated watch glass, and the hot solution is introduced through a narrow-stemmed funnel extending almost to the surface of the acid.

(*f*) Ordinary commercial samples, differing widely in their electromotive properties from samples prepared by the preceding methods, may be brought into sufficiently close agreement for use in secondary standards by digestion with hot dilute sulphuric acid. Fifty grams of the salt are finely ground, preferably in an agate mortar, with enough 1:4 sulphuric acid to make a thin paste. This is poured into a beaker containing 250 cc of 1:4 acid, the finer particles being separated by decantation. The coarser particles are to be retreated in the same way three or four times, regrinding if necessary. After adding 20 cc of mercury, the whole is heated for three hours to 100°, with vigorous stirring, which should be continued until the acid has cooled. The stirring may be interrupted from time to time to wash down any particles of mercurous sulphate which may collect on the sides of the beaker above the level of the liquid and therefore become exposed to the combined action of water condensing on the walls and oxygen from the air.

The Paste.—This should be freshly prepared before setting up the cells. It is obvious from the irregular, pitted character of the

¹⁰ Smith: Electrician (London), 55, 857; 1905.

mercurous sulphate crystals, as shown under the microscope, that the greatest care must be taken in washing the sample to completely remove all traces of acid.

This is most easily done in a Gooch crucible of about 25 cc capacity with a disk of hardened filter paper at the bottom. The disks are cut to size with a cork-borer, warmed for some time with dilute nitric acid, washed with hot distilled water until acid-free, and dried. A sufficient quantity of the sulphate, shaken up with the acid under which it is preserved, is poured into the crucible, care being taken not to transfer any mercury, which interferes with the washing, the acid removed by suction, and the salt washed twice with small portions of 1:6 sulphuric acid to remove possible traces of mercuric sulphate formed by the action of air in the bottle. The acid is removed by washing with five or six small portions of redistilled absolute alcohol, care being taken to wash down the sides of the crucible. To completely remove the alcohol the sulphate is then washed three or four times with a saturated solution of $\left\{ \begin{array}{c} \text{zinc} \\ \text{cadmium} \end{array} \right\}$ sulphate, taking the same precautions as with the alcohol. If the alcohol is not available, two or three additional washings with the sulphate solution will suffice. If the paste cracks or separates from the sides of the crucible, more of the wash liquid is to be added and the contents of the crucible thoroughly stirred up before again applying suction. After scraping off the upper layer, the mercurous sulphate is transferred to a small, clean, dry beaker or crucible, mixed with one-third to one-half its volume of finely powdered $\left\{ \begin{array}{c} \text{zinc} \\ \text{cadmium} \end{array} \right\}$ sulphate crystals and sufficient saturated $\left\{ \begin{array}{c} \text{zinc} \\ \text{cadmium} \end{array} \right\}$ sulphate solution to make a moderately thin paste. A large excess of $\left\{ \begin{array}{c} \text{zinc} \\ \text{cadmium} \end{array} \right\}$ sulphate crystals in the paste should be avoided so that practically every part of surface of the mercury electrode will be in contact with mercurous sulphate, thus securing a rapid attainment of saturation equilibrium. With white samples of mercurous sulphate one-third their volume of mercury should be added in making up the paste. To eliminate possible influence of size of grain, the mercurous sulphate should not be ground, and the paste should be stirred as little as possible in its preparation.

The Cells.—For facility in filling and sealing the H type is recommended. The size and dimensions, although not affecting the e. m. f., determine the polarization produced by the passage of a current and the rapidity with which the cell assumes the temperature of its surroundings. Figure 2, drawn to half scale, gives the dimensions of the cells made at the Bureau, and which have been found entirely satisfactory. Especial care must be taken in sealing in the platinum wires (B. & S. No. 32) and subsequent annealing. As recommended by Hulett, the platinum wire inside the cell may be covered with a thin layer of glass to within 1 mm or less from the end.

Clark cells frequently crack at the point where the platinum terminal is sealed into the amalgam limb. This may be avoided by a

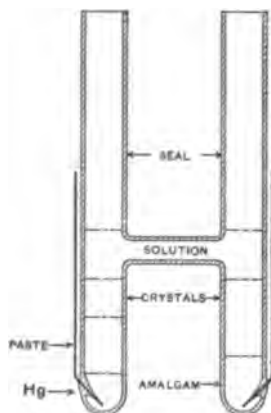


Fig. 2.

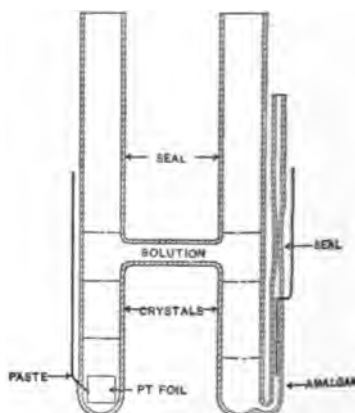


Fig. 3.

construction recommended by Kahle and shown in figure 3, in which the platinum terminal of the amalgam limb is sealed into a side tube, contact being made by sucking the amalgam up into this tube while it is still liquid. The platinum wire should extend downward about 2 cm below the point at which it is sealed into the tube and the amalgam should cover only its lower half, thus preventing the glass from cracking at the seal. This construction also reduces the chance of contact between the platinum and the electrolyte and for this reason is preferable for portable cells.

As the cells are to be sealed off above the cross arm after filling, they may be slightly constricted beforehand, but if the wall thick-

ness is not greater than 0.75 mm this will not be found necessary. The length of the tube above the cross-arm as shown in figures 2 and 3, although greater than absolutely necessary, facilitates the sealing but at the same time somewhat increases the difficulty of filling.

Cleaning the cell.—If the cells and filling tubes are badly contaminated with grease, etc., they should be covered with chromic acid mixture and allowed to stand over night. Too long a contact of the cells with this cleaning mixture should be avoided on account of the danger of forming lead chromate from the lead sealing-in glass employed. The cells should then be washed with distilled water, filled with aqua regia, which is allowed to remain in them for thirty minutes, and repeatedly washed with distilled water. The action of the aqua regia on the platinum wires facilitates subsequent amalgamation. Ordinarily the cleaning with chromic acid mixture may be omitted.

Amalgamation of the platinum terminals.—In order to minimize the possibility of contact between the electrolyte and the platinum terminals, especially from shaking in transport, they should be amalgamated with a solution of pure mercurous nitrate in dilute nitric acid. The amalgamating solution is introduced into the cell and a weak current is passed through it from a platinum wire anode to the platinum terminal externally connected to the negative pole of a battery. Sufficient mercury is deposited in a short time.

For portable cells the limb intended to receive the paste may be provided with a platinum foil electrode welded to the platinum wire. (Fig. 3.) In this case the whole surface must be thoroughly amalgamated, for which a proportionately longer time is required. Especial care must be taken, as the amalgamated foil is employed to replace the mercury.

To remove every trace of the amalgamating solution the cell is next washed several times with dilute nitric acid and finally repeatedly rinsed with distilled water. The amalgamated surface may be "rinsed" with a small quantity of pure mercury and the cell then dried in an air bath at 110°.

Introduction of the materials.—The materials are most easily and neatly introduced by means of filling tubes. Great care should be taken not to allow them to come in contact with the walls of the cell.

The amalgam.—The zinc or cadmium amalgam, prepared as

described above, is heated slightly above its melting point and a quantity of it sufficient to cover the platinum terminal to a depth of at least 10 mm is transferred to the cell by means of a previously heated, clean, dry pipette. A pipette with a rounded end, at the center of which is a small hole, will be found more satisfactory than one drawn out to a capillary, which the amalgam tends to clog up owing to cooling. After heating the pipette, which is provided at the upper end with a rubber tube, it is introduced below the surface of the amalgam, meanwhile blowing through it to avoid the introduction of any of the film of oxide on the surface. A suitable amount of amalgam is then drawn into the pipette by gentle suction, the suction being released while the pipette is removed from the amalgam, and then applied again sufficiently to prevent any more of the amalgam from running out. Any particles of the amalgam adhering to the outside are then removed by wiping with filter paper and the pipette quickly introduced into the cell to within 2 cm of the bottom. On releasing the suction the amalgam will run out freely, the amount introduced being regulated by again applying suction at the proper moment. The pipette is then withdrawn without touching the walls of the cell, which, if necessary, may be protected by means of a thin glass tube of slightly less diameter than the cell. The amalgam may also be introduced by means of a narrow-necked funnel previously heated.

Particles of amalgam on the cell wall can not exert any influence on the e. m. f. except when forming an integral part of the amalgam electrode and not fully covered with crystals. They can be removed by the aid of a glass rod, but great care must be taken to prevent them from reaching the other limb and contaminating the mercury, which would of course seriously influence the e. m. f.

Introduction of the mercury.—The mercury may be introduced into the cell in a similar manner. If a pipette be used it must, however, be drawn out to a *very* fine capillary. The simplest method consists in pouring into the cell sufficient mercury to cover the platinum terminal to a depth of at least 10 mm, care being taken not to introduce any into the amalgam limb. By cautious tilting air bubbles entrapped under the mercury can easily be removed.

Introduction of the paste.—The paste, prepared immediately before setting up the cell, as described above, is most neatly introduced by means of a pipette with an opening 3 to 4 mm in diameter. Its consistency should be such that it can be readily drawn up into and flow from the pipette; and, therefore, finely-crushed $\left\{ \begin{array}{c} \text{zinc} \\ \text{cadmium} \end{array} \right\}$ sulphate crystals should be employed. After filling the carefully cleaned and dry pipette with paste, it should be wiped with clean filter paper, and a small amount of the paste allowed to flow out. The end of the pipette is then brought close to the mercury and paste allowed to flow into the limb to a depth of 1.5 to 2 cm, care being taken to keep the cell walls clean and to avoid trapping air bubbles.

The paste may also be introduced by means of a thistle tube extending to within 2 cm of the surface of the mercury. If necessary the walls of the cell may in addition be protected by a guard tube.

Introduction of the crystals and solution.—The amalgam and paste are next covered to a depth of about 1 cm with *saturated* $\left\{ \begin{array}{c} \text{zinc} \\ \text{cadmium} \end{array} \right\}$ sulphate solution, after which pulverized $\left\{ \begin{array}{c} \text{zinc} \\ \text{cadmium} \end{array} \right\}$ sulphate is introduced, a little at a time to avoid trapping air bubbles, to a depth of 1.5 to 2 cm by means of a wide-stemmed funnel. The cell is then filled slightly above the cross arm with *saturated* $\left\{ \begin{array}{c} \text{zinc} \\ \text{cadmium} \end{array} \right\}$ sulphate solution.

In order to maintain a concentration equilibrium at all temperatures to which the cell is likely to be exposed, in all cases a saturated solution of zinc or cadmium sulphate and a sufficient excess of the corresponding crystals over the amalgam and in the paste should be employed. The layer of crystals in the mercury limb is intended as a further precaution against their leaching out of the paste, as sometimes happens when the solution is not completely saturated.

Sealing.—The cell is now sealed by the aid of two small horizontal blowpipe flames applied to the cell wall from opposite directions. One limb of the cell is closed by a cork and the other by a cork nicked at one side, through which passes a glass rod to serve as a

handle in drawing it out. The cell is gradually heated 2 to 3 cm above the level of the liquid until the danger of cracking has passed, and then held in the flame, rotating meanwhile, until the tube almost collapses, after which it is closed by drawing it out slowly. By judicious heating the seal may be nicely rounded by the expansion of the enclosed air. The second limb is sealed in a similar manner.

Whenever, in filling the cell, the materials accidentally come in contact with the part of the glass which is to be heated in sealing, it is first cleaned by wiping with filter paper slightly moistened with distilled water and then with dry paper.

CALORIMETRIC RESISTANCE THERMOMETERS AND THE TRANSITION TEMPERATURE OF SODIUM SULPHATE.

By H. C. Dickinson and E. F. Mueller.

I. CALORIMETRIC RESISTANCE THERMOMETERS.

1. INTRODUCTORY.

Of the methods in use for temperature measurement, the variation in resistance of metallic wires affords probably the most accurate means of obtaining differences of temperature over a limited range. When the resistance material is a pure metal, resistance is in most cases a simple function of temperature. The metal most generally used has been platinum, because of its high specific resistance, high melting point, and freedom from oxidation. The platinum resistance thermometer has been developed and carefully studied by a number of able experimenters.¹

The form of resistance thermometer most generally used consists of a coil of platinum wire wound on an open frame of mica, connected to heavy platinum leads and surrounded by a tube of glass or porcelain from 1.0 to 1.6 cm in diameter. When intended for use with a potentiometer, four leads are used—two current leads and two potential leads. If for use with a Wheatstone bridge, the coil is connected to one pair of leads, and another similar pair in the form of a simple loop is placed beside them and connected in the adjacent arm of the bridge to compensate for varying depth of immersion.

¹Callender, Phil. Trans. A, 178, p. 161; 1887. Callender and Griffiths, Phil. Trans. A, 182, p. 119; 1891. Harker and Chappuis, Phil. Trans. A, 194, p. 37; 1900. Harker, Phil. Trans. A, 203, p. 343; 1904. Dewar, Proc. Roy. Soc., 73, p. 244; 1904.

The present investigation has to do with the adaptation of the resistance thermometer to calorimetric and other uses for which the ordinary form is not suited. For use in a calorimeter, or where temperatures are changing rapidly, a resistance thermometer must satisfy some special conditions: (a) It must quickly assume the temperature of its surroundings, so that its indications may give the true temperature at the time of observation; (b) it must have small heat capacity and must conduct little heat away from the bath in which it is immersed; (c) thermo-electromotive forces must be avoided so far as possible; (d) the resistance should be relatively high; (e) conduction of heat by the leads, affecting the temperature of the resistance coil, must be avoided; (f) as in any resistance thermometer, the leads must be so adjusted that the indications are independent of the depth of immersion, i. e., the resistance of the compensating leads must equal that of the coil leads when a bridge method is used; (g) insulation resistance must be high to avoid leakage.

In some special cases it has been possible to use bare platinum wires without any protecting sheath—as for the measurement of the temperature of gases under certain conditions—but for most purposes it is necessary to provide some protection. To provide such protection and still allow rapid equalization of temperature requires some special device. Evidently the best results will be obtained by the use of a thin sheath of metal with a layer of the thinnest possible insulating material between it and the resistance coil. Jaeger and Steinwehr¹ have described a resistance thermometer made after this plan which they have used in several investigations. The thermometer described below has been developed along somewhat different lines. It was desired to build an instrument having the resistance coil in small compass, so that it could be used to determine the temperature, at a definite point, of a liquid flowing in a tube. As it is important to have a large external surface, the use of a flat coil and thin flat sheath suggested itself. To get a moderately high resistance, the use of very fine wire was tried, and, after a number of preliminary experiments, a wire of about 0.02 mm diameter was found to serve the purpose.

¹Jaeger und Steinwehr, *Zs. für Instrk.*, **26**, p. 237; 1906.

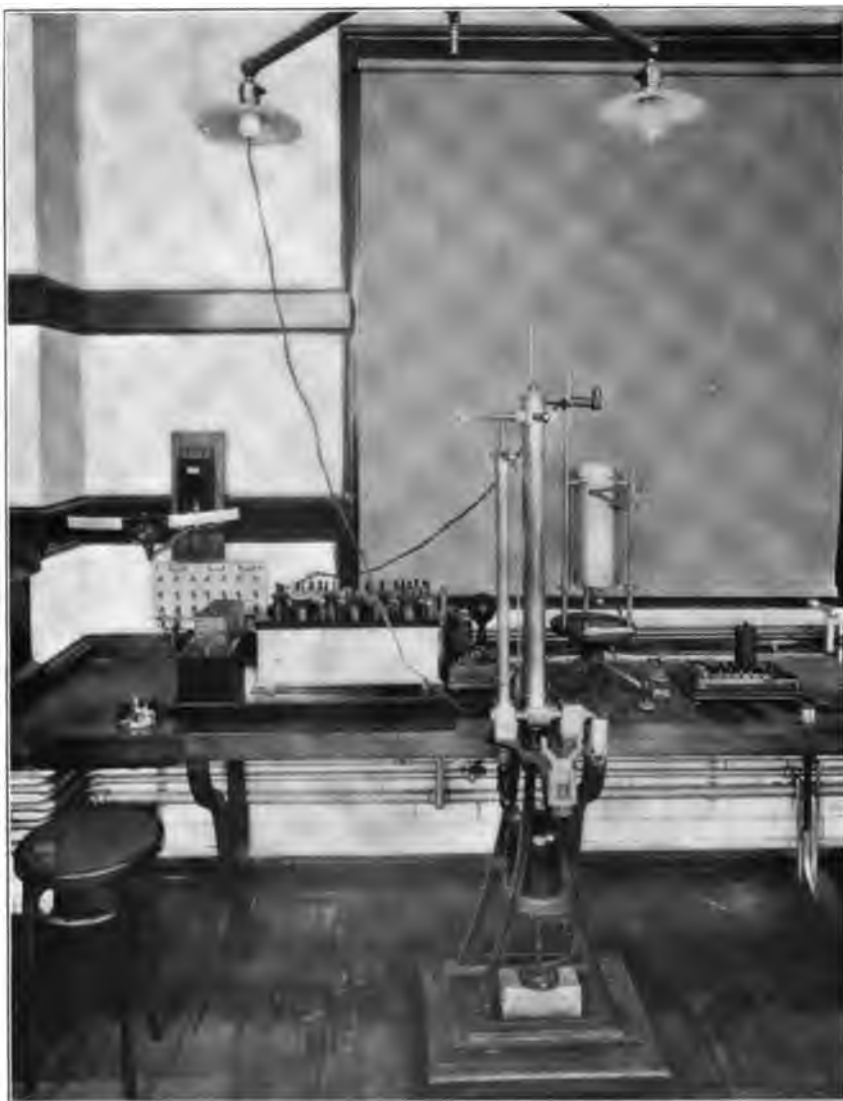


Fig. 1.—Apparatus used in *Fundamental Interval Determinations*.

2. DESCRIPTION OF THE THERMOMETERS.

The accompanying cut (Fig. 2) shows the construction finally adopted. The four leads (*L*) are of thin copper strip about 4 mm wide and 0.12 mm thick. To each of these is fused a short piece (*T*) of 0.1 mm platinum wire about 10 mm long. The leads are then laid side by side, separated by thin mica strips (*M*), and one pair is connected to the ends of a coil (*C*) of about 10 cm of 0.02-mm platinum wire wound in a flat coil on one of the mica strips; the other pair of leads are connected by fusing together the ends of the short platinum terminals.³ These parts are shown separated in Fig. 2 b.

The use of these thin platinum terminals on both the main and the compensating leads was found to be necessary to compensate for the conduction of heat by the copper leads which would affect the temperature of the resistance coil. After the adoption of this method of connecting the coil to the copper leads, no further trouble was experienced. The leads and coil thus prepared were enclosed in a flat sheath (*S*) of thin copper (0.12 mm thick) and insulated from it by thin mica. This sheath was soldered along the side and copper-plated to strengthen it somewhat and then gold-plated to protect from oxidation. The sheath when finished was about 7 mm wide and 1 mm thick, somewhat thinner at the coil. Two thermometers thus constructed were sealed into heads (*H*), of fiber as shown in Fig. 2 a, but it was found that moisture entered these heads and the insulation between leads and outer sheath was impaired. To avoid this trouble, heads of hard rubber were substituted and the covers (*D*), containing drying salt were also added. These covers open into *H* through the small glass tube (*G*).

The thermometers here described fulfill the requirements outlined

³In making up these thermometers it was found impracticable to wind the coils after fusing to the leads, and to overcome the difficulty of fusing wires so fine as 0.02 mm in somewhat inaccessible positions an arc was used. The arc was formed between a small graphite pencil and one of the terminals to be fused, when the other terminal was brought up and the two connected. After a little practice this could be done quickly and neatly. This same method has since been used to advantage in fusing together wires of different materials and sizes. The platinum terminals also were fused to the copper leads by the same method. The fusing of copper wires is more easily effected when borax is used as a flux. It is necessary to regulate both the voltage and the resistance in series with the arc for different sizes of wire.

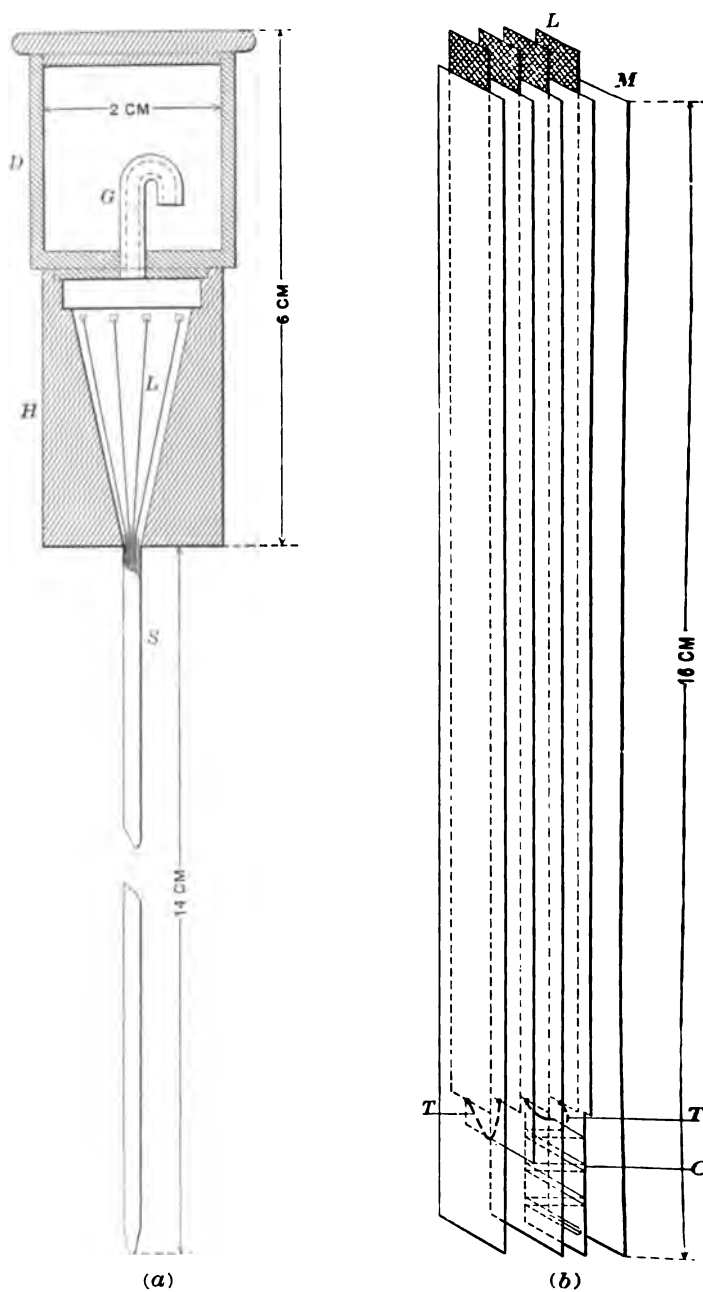


Fig. 2.—Diagram showing construction of thermometer.

above in a very satisfactory manner. (a) When immersed in ice or steam they assume the temperature to within 0.001° in about three seconds. This time constant has not, however, been very carefully determined, as the period of the galvanometer is much longer than the time lag of the thermometer. (b) The water equivalent of the thermometers, exclusive of the heads which are not immersed, is less than 1 gram. The heat conductivity of the leads is small, and as its effect can be eliminated in most cases, no determination has been made. (c) The junctions of the copper leads with the platinum coil are made within the thermometer sheath and are separated from each other by only a thin layer of mica, so that thermo-electromotive forces from these junctions are extremely small. The small thermo-electromotive forces observed seem to come entirely from the external circuit and are never troublesome. (d) The resistance of these thermometers is approximately 30 ohms in ice and 40 ohms in steam, so that a change of 0.1 ohm corresponds to 1° change in temperature. (e) The effect of heat conductivity of the leads in changing the temperature of the platinum coil has been so far avoided by the device described above (Terminals (T), Fig. 2) that no effect is noticeable when the immersion of the sheath is as much as 5 cm. (f) The resistance is made independent of the temperature of the exposed portion of the sheath by the use of compensating leads. (g) After adding the drying capsules to the heads of the thermometers the insulation resistance was found to be over 100 megohms, so that errors due to leakage are negligible.

In reference to the resistance coil of 0.02-mm wire it may be said that considerable care was exercised in winding and annealing. The wire was first glowed by passing a current through it for some minutes, heating to about $1,000^\circ$; then, after winding on the mica strip, the process was repeated at a somewhat lower temperature, and finally, after the sheath was put on, the whole was heated to about 400° in an oven before soldering up.

The thermometers described by Jaeger and Steinwehr⁴ showed a depression of R_0 (the resistance in melting ice) after heating to 100° , corresponding to about 0.1° or nearly the same as is shown by a good mercury-in-glass thermometer. With this fact in mind several

⁴ Loc cit.

special experiments were made to see if the same effect were present in this type of instrument, but the results failed to show the slightest trace of a change after heating.

A second pair of thermometers similar to those described was made later and showed some irregularities which, however, were soon traced to mechanical strain of the wire, and when the cases were slightly opened out around the bulb these irregularities disappeared.

3. RESISTANCE MEASUREMENTS.

Measurements of resistance were made by the Wheatstone bridge method using a bridge (Fig. 1) constructed for the Bureau of Standards three years ago especially for resistance thermometry. This bridge, shown diagrammatically in Fig. 3, has an adjustable resistance of 110 ohms—the lowest coils being 0.01 ohm. In place of a bridge wire, three slides are used giving steps of 0.001, 0.0001, and 0.00001 ohm, respectively. The slides each have ten steps and consist of a series of coils shunted over a portion of the resistance in the main bridge arm. This arrangement avoids the uncertainty of a bridge wire calibration since only moderate accuracy is required in adjusting the shunt coils, and in addition makes the bridge direct reading. Mercury contact links are used for all connections in the main circuit and their resistance which is about 0.0000 ± 6 ohm per link, has proved to be constant to better than 10%. Ratio coils of 10, 100, 1,000, and 10,000 ohms are provided, but the 1,000-ohm coils were used throughout this investigation. The ratio coils can be interchanged by moving two links. The coils are immersed in oil and the temperature is kept constant at 30° by means of a thermostat and stirrer. Reversing keys for battery and galvanometer and an Ayrton-Mather universal galvanometer shunt are provided. A Griffiths thermo-electric key immersed in oil was used so that the galvanometer circuit was always closed except at the instant of making the battery circuit.

The bridge has been calibrated four times with great care at intervals during the progress of the work, and all measurements are corrected to these calibrations. Throughout the present investigation the measurements were made with an electromotive force of 2.6 volts. With this arrangement the current in the thermometers was

kept nearly constant at 0.0025 ampere. Experiments showed that this current raised the temperature of the thermometer coils $0^{\circ}.0105$ when in ice and $0^{\circ}.0110$ when in steam. It thus appears that this heating effect is constant within the limits of observational error, and as the calibration and all measurements were made under the same conditions no correction need be applied.

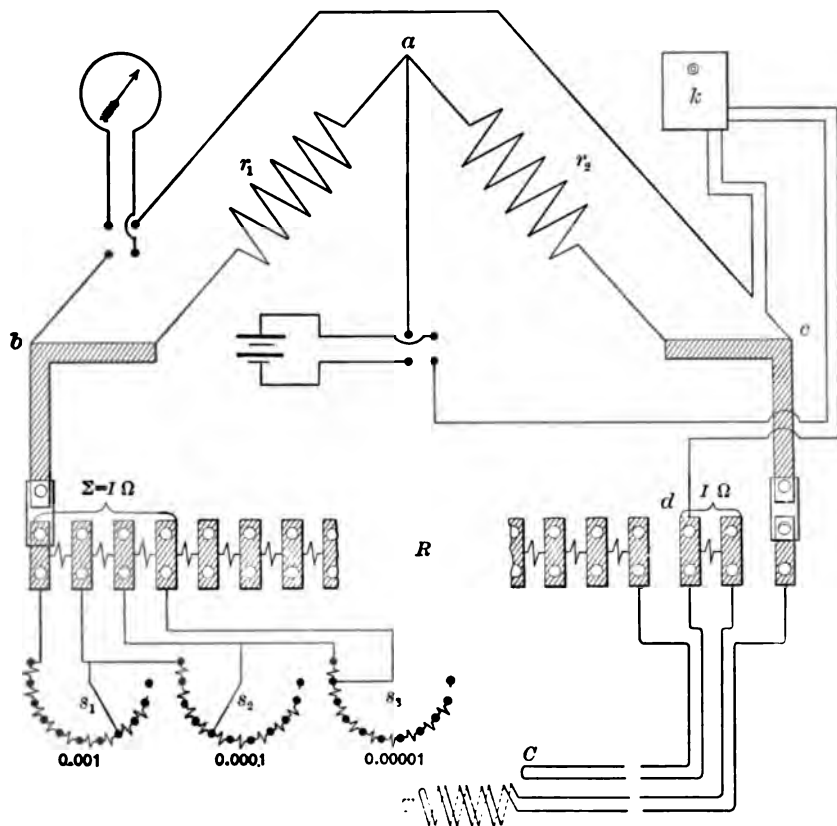


Fig. 3.—Diagrammatic sketch of bridge.

The galvanometer used was of the Broca type, with vertical magnets, having their consequent poles at the center of the coils, and was adjusted to a sensibility of about 600 megohms. With the scale placed at 50-cm distance, this gave a deflection of 1 mm for a change of 0.0001 ohm in the resistance to be measured, or, in other words, 1 mm deflection per $0^{\circ}.001$.

4. CALIBRATION OF THE THERMOMETERS.

A platinum temperature scale may be defined in terms of resistance. If R_0 is the resistance in melting ice and R_{100} the resistance in steam, under standard conditions, and R_t any observed resistance, then the corresponding platinum temperature is defined thus:

$$pt = \frac{R_t - R_0}{R_{100} - R_0} \times 100.$$

Callendar has given the relation between temperature "t" on the gas scale and "pt" as defined above in the form $t - pt = \delta(t/100 - 1)t/100$ where δ is a constant. This relation has been shown to hold within the limits of accuracy at present attained in gas thermometry between -100° and $1,100^\circ$ when δ is determined by observation at $0^\circ, 100^\circ$, and the boiling point of sulphur (444.7°).

The three constants of the resistance thermometer (R_0 , R_{100} , δ), are fixed by measuring the resistance in melting ice, in steam, and at one other temperature which is known in terms of the gas scale.

Determination of R_0 and R_{100} .—In the course of six months the resistance R_0 of the thermometers changed slightly, the one by 0.0006 and the other by 0.001 ohm while small variations of 0.0001 or 0.0002 ohm (0.0001° or 0.0002°) were observed from day to day. These small irregularities are near the limits of accuracy of measurement and can not be definitely ascribed to any one cause, while the secular change may be due to varying state of strain in the wire, or more probably to changes in the leads and terminals. The changes in the bridge itself have been much larger than the changes in the thermometers. It should be noted that to maintain R_0 constant to 0.001 resistances must remain constant to 1 in 300,000, but for the same constancy of temperature indication the coils representing the difference $R_{100} - R_0$ need remain constant to only 1 in 100,000. In the present instance nearly all the change observed was in a 20-ohm coil which was always in circuit, and therefore did not affect the value of $R_{100} - R_0$ or of $R_t - R_0$. For these reasons it is evident that the best results should be obtained from a series of independent determinations of $R_{100} - R_0$ referring measurements at other temperatures to the R_0 determined at the time. This method was followed throughout the present work. The results of the series of inde-

pendent determinations of the $R_{100}-R_0$ are given in the following table.

TABLE I.
Values of $R_{100}-R_0$.

Date	Thermometer A		Thermometer B	
	R_0	$R_{100}-R_0$	R_0	$R_{100}-R_0$
10- 6-06	29.00548	9.75060	28.94158	9.76746
		9.75068		9.76755
10- 8-06	29.00498	9.75079	29.94128	9.76777
		9.75097		9.76777
10-12-06	29.00530	9.75055	28.94160	9.76747
		9.75055		9.76709
10-18-06	29.00570	9.75063	28.94202	9.76720
		9.75062		9.76720
1-18-07	29.00579	9.75091	28.94230	9.76717
		9.75062		9.76728
		9.75019		
2-27-07	29.00603	9.75111	28.94260	9.76791
		9.75115		9.76800
Mean		9.75072 ± .00005	9.76749 ± .00006	

[NOTE.—*July, 1907.* After a new calibration of the bridge R_0 was found to be 29.00570 and 28.94200 for A and B, respectively, indicating that the changes in the thermometers have been less than 1 in 100,000.]

The apparatus used for the determination of ice and steam points is of the International Bureau form (Fig. 1). Pressures were read on a Fuess standard barometer. All of this apparatus, as well as the mercurial thermometers and comparator subsequently referred to, and the methods and precautions observed in their use, are fully described in another paper.⁵

Determination of δ .—As has been stated, the δ of the Callendar formula may be fixed by observing the resistance at a third known temperature. To determine its most probable value in the interval 0° to 100° , the platinum thermometers were compared at several points with a number of primary standard mercurial thermometers. It is evident that comparisons at temperatures near the fixed points

⁵ Waidner and Dickinson, this Bulletin 3, p. 663.

where $t - pt$ is small, are of little value in the determination of δ , so that observations are restricted to the middle part of the range 0° to 100° . Further, in the use of mercurial thermometers it is advisable to select points at which the calibration corrections have been most accurately determined. For these reasons the points 30° , 40° , 50° , 60° , and 70° were chosen.

The thermometers used at 30° , 40° , and 50° were Tonnelots Nos. 4332, 4334, 4335, and 4336; and at 60° and 70° Baudins Nos. 16016, 16017, 16018, and 15962. Since all of these had been included in the intercomparisons by Waidner and Dickinson they give in effect the mean scale of twelve thermometers at 30° and 40° , the mean of sixteen at 50° , and the mean of seven at 60° and 70° .

The mercurial thermometers were mounted on the holder and the platinum thermometer was inserted through the cover of the comparator.⁶ With this arrangement the platinum thermometer was about 50 cm above the bulbs of the mercurial thermometers. The complete absence of irregular variations in the reading of the resistance thermometers shows that there were no local variations in temperature in the comparator, so that any possible difference between the platinum and mercurial thermometers must have been constant. It may readily be shown that any such difference must be of negligible importance.⁷

Observations.—A whole day was required for the comparison at each temperature. The observations were carried out in the following order: (a) The resistance of the platinum thermometers was measured in ice, in steam, and again in ice, giving R_0 and $R_{100} - R_0$;

⁶Waidner and Dickinson, loc. cit., note 5.

⁷At 70° , the highest temperature used in these experiments, 300 watts were required to maintain the temperature constant. The loss of heat from the water was, therefore, 71.4 calories per second. The surface from which heat could be lost by the water between the bulbs of the mercurial and platinum thermometers was less than one-twentieth of the total. This surface was at least as well protected as the average, so that if the loss per unit surface be assumed uniform, this portion should lose 3.6 calories. The water contained in this space was about 5,000 grams, so that its temperature should fall 0.0007 per second. The time required for a given portion of water to pass from the bulbs of the mercurial thermometers to the platinum thermometer was less than one second. The largest difference of temperature, therefore, was probably less than 0.0007 . The time lag of the mercurial thermometers as compared with the platinum thermometers introduces another error of about the same magnitude and opposite sign.

(b) in the meantime the temperature of the four mercurial thermometers in the comparator had been brought to the comparison point and two ice-point determinations were made for each, both estimates and micrometer settings being made by each observer (a similar series of ice-point determinations was made after the comparison); (c) the mercurial and platinum thermometers were put in position in the comparator and the readings taken. Table II contains half the readings in the comparison of thermometer A at 40° and shows the procedure. Series one and four are micrometer observations and series two and three are estimates. The numbers in brackets in columns marked (*t*) are estimated readings used to check micrometer observations. The corrections to the reading of each thermometer are given at the bottom of the respective columns. The sixth and seventh columns contain the bridge readings and corresponding values of R_t . The last column contains pt , t , and the values of $t-pt$ and δ for each series.

A single series consisting of sixteen readings on the mercurial thermometers and eight resistance measurements took about five minutes, and the temperature was so controlled that a rise of from $0^{\circ}01$ to $0^{\circ}03$ was secured.

The values of $t-pt$ deduced from the observations, and the corresponding values of δ are given in Table III. The largest residual in terms of temperature is $0^{\circ}002$. In the course of other work the platinum thermometers have been compared with two mercurial standards at 5° , 15° , and 22° . Using the value of δ from this calibration the largest difference observed was $0^{\circ}005$, which is about the probable error of the mercurial thermometers. It appears from the results obtained that δ is constant between 0° and 100° to within the limits of observational errors of mercurial thermometers and that these platinum thermometers calibrated in this way will serve to determine temperatures on the International Hydrogen Scale to about $0^{\circ}002$. For calorimetric work, where the temperature difference is only a few degrees, the interval defined by these thermometers may be taken as representing the interval on the hydrogen scale to a much higher order of accuracy.

Divisions front	Tonnelot 4335			Tonnelot 4334			Tonnelot 4335			Tonnelot 4336			Colla	r ₁ r ₂	r ₁ r ₂	
	Div.	Mem.	t	Div.	Mem.	t	Div.	Mem.	t	Div.	Mem.	t				
Observed reading Total correction Corrected temp	063	200	332 (40.046)	064	278	314 (39.994)	041	244	290 (39.863)	077	294	331 (39.868)	20.0094	01130	01530	pt 40.1793
	82			86			36			82			10.00236	01145	01530	t 39.8008
			118 + 250 = 40.047			213 + 249 = 39.985			206 + 252 = 39.862			214 + 251 = 39.865	.00033	01200	01540	t pt. 3697
	053	186	305 (40.054)				61	280	312 (39.890)	080	304	334 (39.894)	.00033	01175	01535	
			302						316				1.00083	.01355 (-2)		
Observed reading Total correction Corrected temp			135 + 250 = 40.054			39.997			219 + 253 = 39.887			224 + 254 = 39.888	32.90973	32.90973		
			40.0505			39.991			39.8945			39.8945		32.92326		
			-.2397			-.1815			-.0760			.0759				
			39.8108			39.8095			39.8085			39.8106				8 1.844
		40.056				40.000			39.894			39.896		01245	01630	pt 40.1903
Observed reading Total correction Corrected temp		.057			.002			.896			.897		01250	01640	t 39.8203	
		.058			.003			.897			.899		01300	01640	t pt. 3699	
		.061			.004			.898			.900		01300	01670		
		40.0580			40.0022			39.8962			39.8960		01274	01645		
		-.2397			-.1815			-.0760			.0759		.01460 (2)			
Corrected temp		39.8103			39.8207			39.8202			39.8221		32.90973			8 1.844

Tunnelot 4334				Tunnelot 4335				Tunnelot 4336				Colls	r_1/r_2	r_2/r_1	
Div.	Men.	Div.	t	Div.	Men.	Div.	t	Div.	Men.	Div.	t				
40.064		40.009		39.905		39.907							01400	01795	pt 40.2050
.072		.010		.908		.908							01405	01795	t 39.8311
.073		.012		.908		.910							01430	01790	t-pt .3759
.074		.014		.908		.910							01420	01800	
													01414	01795	
40.0708		40.0112		39.9068		39.9068							.01604 (-2)		
-.2397		-.1815		-.0760		-.0759							32.90973		
39.8311		39.8297		39.8308		39.8329							32.92575		8 1.860
Observed reading				Div. Men. Div. t				Div. Men. Div. t							
Total correction				025 071 272 (40.021)				085 124 338 (39.916)				050 104 298 (39.919)			
Corrected temp				24				89				53			
				47 + 248 = 40.019				37 + 251 = 39.915				53 + 247 = 39.921			
				070 142 320 (40.027)				034 092 283 (39.924)				104 162 352 (39.921)			
				320				283				352			
				72 + 250 = 40.029				58 + 249 = 39.923				58 + 248 = 39.923			
Observed reading				40.083				39.919				39.922			
Total correction				-.2397				-.0760				-.0759			
Corrected temp				39.8433				39.8430				39.8461			
Calibration				-.1120				+.0180				+.0280			
Ext. pressure				-.0045				-.0046				-.0046			
Int. pressure				+.0450				+.0450				+.0461			
Zero				+.0301				+.0094				-.0054			
Fund. interval				-.0286				-.0354				-.0340			
H. scale corr.				-.1070				-.1070				-.1070			
Supercorrection				-.0014				-.0014				+.0010			
				-.1815				-.0760				-.0759			

Divisions back

Corrections	8 1.847
-------------	---------

Divisions
backCorrec-
tions

TABLE III.

Values of δ .

Thermometer A					Thermometer B				
Point	t-pt	δ	Observed minus mean	Residual in temp. C	Point	t-pt	δ	Observed minus mean	Residual in temp. C
30°	.3264	1.560	+.006	0°.0013	30°	.3276	1.565	+.009	0°.0019
40°	.3711	1.548	-.006	.0014	40°	.3707	1.547	-.009	.0021
50°	.3890	1.556	+.002	.0005	50°	.3897	1.559	+.003	.0008
60°	.3722	1.550	-.004	.0010	60°	.3742	1.558	+.002	.0005
70°	.3269	1.554	.000	.0000	70°	.3259	1.550	-.006	.0013
Mean		1.554	.004	0°0008	Mean		1.556	.006	0°0013

II. THE TRANSITION TEMPERATURE OF SODIUM SULPHATE.

5. INTRODUCTORY.

Since it has been shown that the relation between platinum temperature and the International Hydrogen Scale in the interval 0° to 100° may be given by a single constant (δ) it is evident that this constant might be equally well determined at a single temperature (in addition to the two fixed points) if the determination were made with sufficient accuracy. The work of Richards⁸ and of Richards and Wells⁹ has shown that the transition point of sodium sulphate serves to fix a definite temperature which would be suitable for such determinations. Crystals of $Na_2SO_4 \cdot 10 H_2O$ melt at about $32^{\circ}4$. The presence of the anhydrous salt Na_2SO_4 apparently does not affect the equilibrium other than by taking up any water in excess of the water of crystallization. In other words, the phases $Na_2SO_4 + Na_2SO_4 \cdot 10 H_2O + \text{water} + \text{vapor}$ are in equilibrium at a temperature which has been determined by a number of observers as being very near $32^{\circ}38$.

The very careful work of Richards and Wells and the small probable error deduced by them from their observations require that anyone attempting to repeat their work should have exceptional facilities for obtaining accurate results. On the other hand, it is very desirable to secure a further check on a point so important, in view of its proposed use in the calibration of calorimetric thermometers.

It was decided to use the two platinum thermometers, calibrated as above described, in a redetermination of this temperature. While this is an indirect method, it has certain advantages over a direct determination with mercurial thermometers. In the first place, the

⁸ Am. J. Sci. 6, p. 201; 1898.

⁹ Proc. Am. Acad. of Arts and Sciences 38, p. 431; 1902.

platinum temperature could be determined to about 0.001 by a single observation so that the errors in referring the point to the platinum scale become almost negligible and the whole problem is reduced to the relation between the platinum scale and the hydrogen scale, i. e., the accuracy of mercurial thermometers.

Since the four thermometers used at each point have not only been calibrated at the International Bureau, but have been compared with a number of other primary standards¹⁰ also calibrated there and some of them directly compared with the standards of the International Bureau, they offer a means of reproducing the international hydrogen scale with the highest accuracy.

Moreover, the mercurial thermometers were used under conditions most favorable to attaining accurate results, i. e., with total immersion, reading at principal calibration points where their indications are most reliable and reading with a rising meniscus. This avoids one of the largest sources of error in using mercurial thermometers, i. e., the determination of a fixed point. In addition, as has been shown, the hydrogen scale represented by these platinum thermometers was obtained from the mean of observations taken at 30° , 40° , 50° , 60° , and 70° , so that in fact the determination of the transition temperature 32.38 rests upon observations of four thermometers at each of five points.

6. METHOD OF EXPERIMENT.

The arrangement of the apparatus is shown in the accompanying figure, Fig. 4. (*B*) is a cylindrical brass case 6 cm in diameter and 24 cm long, which was immersed to the point (*D*) in a water bath ordinarily used for testing clinical thermometers. By this means the external temperature was regulated and could be maintained constant to 0.1 for any length of time. The salt was contained in a strong test tube (*N*) 3 cm in diameter and 15 cm long holding about 100 grams. Smaller tubes holding about 50 grams were also used in some of the preliminary work. The head of the thermometer (*T*) was held in place by a cork (*C*) fitted into a removable sleeve (*S*). The salt was thus entirely enclosed in an air space at the temperature of the surrounding water.

¹⁰ Loc. cit., note 5.

The first experiments were made on a sample of "Kahlbaum" sodium sulphate. Following the procedure of Richards, the crystals were first broken up rather fine in a mortar, then melted in the test tube by dipping in warm water until the mass could be readily stirred. The tube was then placed in position, the thermometer inserted, and readings taken at intervals during several hours. These observations showed that no very definite equilibrium had been established. Stirring the contents of the tube and small changes in the external temperature produced changes amounting to over 0.01 even after two hours. The procedure was therefore slightly modified, as follows: As the crystals seemed rather moist, a small quantity of the anhydrous salt was first added and the crystals were almost entirely melted. Recrystallization was then started by running cold water over the tube. In this way a perfectly definite temperature was secured within five minutes after starting the experiment. It was found later that the addition of anhydrous salt was unnecessary, as enough was formed during the process of preparing and melting. The procedure just indicated was followed in all subsequent experiments.

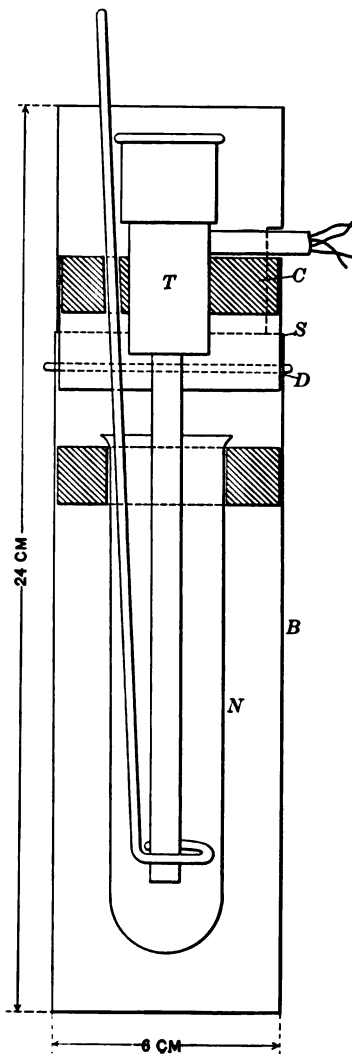


Fig. 4.

7. THE EFFECT OF EXTERNAL CONDITIONS.

In one experiment the external temperature was kept at 33° for an hour and the temperature of the salt was observed at intervals.

These observations showed variations of about 0.0005 . The temperature was then dropped to 32° for an hour, but no larger variations were observed. When the temperature was dropped to 23° and maintained for about forty-five minutes, until the salt could no longer be stirred, no changes exceeding 0.001 were observed in the transition temperature. On raising the external temperature to 43° a rise of about 0.003 was observed after thirty minutes, but it was noted that the salt was almost completely melted. The temperature was again lowered to 33° and the original value was found. The experiment of lowering the temperature was tried a number of times with the same result. Richards found that lowering the outside temperature two or three degrees lowers the temperature of the salt by about 0.01 . It seems probable that the procedure followed in the present instance, whereby new crystals are formed by sudden cooling, makes the transition temperature more definite and less dependent on external temperature. In this connection it should be pointed out that when the cooling was omitted the variations were not larger than the accidental errors of mercurial thermometers. These irregular observations, moreover, fell on both sides of the final value.

To test further the influence of external conditions a smaller test tube was used. The salt was first melted in a large tube and the temperature observed; some of it was then transferred to the smaller tube, where the same value was again found. This experiment further indicates that the effects of superheating, conduction down the thermometer stem, etc., were negligible. When the smaller tube was used, however, the external temperature had to be more carefully regulated, as a difference of 5 degrees from the transition point, if maintained for some time, began to affect the readings. Stirring the salt usually produced a small rise of less than 0.001 , which disappeared in two or three minutes.

8. IMPURITIES.

It seemed desirable to determine directly the error which might be introduced by the presence of impurities in the salt. Since the transition temperature must be determined with a mixture of crystals and solution, and the effect of the impurity except in some special cases, varies with the relative amount of the two present, it

is evident that this effect will not remain constant. The values quoted in the table below are therefore only approximate, but serve to show what effect and range of variation may be expected in the transition temperature.

TABLE IV.
 Effect of impurities.

	Lowering of transition point for the following impurities—					
	0.2% NaCl	0.08% NaCl	0.05% NaCl	0.01% NaCl	0.1% K ₂ SO ₄	0.1% (NH ₄) ₂ SO ₄
	0°1885	0°0873	0°0505	0°0147	0°0620	0°0937
	0°1942	0°1019	0°0486	0°0147	0°0607	0°0922
	0°1942	0°1019	0°0490?	0°0147		0°0911
Mean	0°193	0°100	0°049	0°015	0°061	0°092

From the preliminary determinations it may be concluded (a) the transition temperature of sodium sulphate is definite to 0°001; (b) within a considerable range the transition temperature is not affected by external conditions (temperature and pressure); (c) the presence of 0.001 per cent of other salts may lower the transition temperature by 0°001. From the latter consideration it appears that if samples made by different methods give different values the higher is to be chosen as representing the purer salt.

9. FINAL DETERMINATION.

For the final determination four samples prepared by the chemical division of the Bureau of Standards, were used. No. 1 was prepared from Kahlbaum sodium sulphate three times recrystallized. No. 2 was the same salt five times recrystallized. No. 3 was prepared by neutralizing the carbonate and four times recrystallized. No. 4, was the same, five times recrystallized. Table V shows values obtained for the different samples with the two thermometers.

The agreement of the two thermometers and the constancy of the values for each sample taken at different times show that the differences observed represent real differences in the transition temperature for the different samples. There is reason to believe that

sample No. 2, which gives a low value, had been accidentally contaminated with some of the solution used in cleaning the tubes. It may be noted that the constancy of the indications for the various samples with varying conditions is not necessarily a proof of their purity, since any impurities which were not removed by recrystallization would have an effect independent of the relative amount of crystals and solution present. Samples prepared from the carbon-

TABLE V.

Date	Therm.	Temp.	Mean	Date	Therm.	Temp.	Mean
" Kahlbaum "				No. I			32°384
1-4-07	A	32°3812	32°380	1-12-07	A	32°3832	
				B	°3841		
1-8-07	A	32°3798		No. III			
	B	°3789		1-12-07	A	32°3812	
3-7-07	A	32°3798		B	°3818		
	B	°3773	No. IV			32°383	
No. II			32°381	1-5-07	A		32°3855
1-7-07	A	32°3798			B		°3860
	B	°3796		1-8-07	A		32°3855
1-8-07	A	32°3819			B	°3834	
	B	°3813		1-12-07	A	32°3812	32°384
1-12-07	A	32°3799			B	°3818	
	B	°3806					

ate are less likely to contain impurities of this kind. Since differences between different samples are probably due to impurities which lower the transition temperature, the higher values may be taken as more nearly representing pure salt. The mean value from samples Nos. 1, 3, and 4 is 32°383.¹¹ But, for the reasons given above, the higher values are entitled to greater weight so that the most probable value from these observations is 32°384.

¹¹This value was found by Richards and Wells.

10. CONCLUSION.

A special form of resistance thermometer, intended for calorimetric work, has been found applicable for general temperature measurement in the interval 0° to 100° . Two of these thermometers have been compared with eight primary standard mercurial thermometers, representing the mean temperature scale of the Bureau of Standards, at the temperatures 30° , 40° , 50° , 60° , and 70° . This calibration has shown that the Callendar formula $t - pt = \delta(t/100 - 1)t/100$ may be used to define the relation between the platinum scale of these resistance thermometers and the mean scale of the Bureau of Standards to within $0^{\circ}.002$. By using these calibrated thermometers the transition temperature of sodium sulphate has been determined and found to be $32^{\circ}.384$ for the purest salt. Resistance thermometers for use in the interval 0° to 100° may be calibrated to the highest degree of accuracy by using the Callendar formula and determining the constant δ from the transition temperature of pure sodium sulphate, $32^{\circ}.384$. These thermometers may be used to reproduce the international hydrogen scale of temperature to within $0^{\circ}.002$ or $0^{\circ}.003$.

In conclusion, the authors wish to express their obligations to Dr. C. W. Waidner for many valuable suggestions throughout the course of the present investigation, to Dr. F. A. Wolff, for a very careful calibration of the resistance bridge, and to Dr. Helen Isham, who prepared the samples of sodium sulphate used in determinations of the transition temperature.

WASHINGTON, June 21, 1907.

ON THE STANDARD SCALE OF TEMPERATURE IN THE INTERVAL 0° TO 100° C.

By C. W. Waidner and H. C. Dickinson.

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I. OBJECT OF INVESTIGATION.

The standard scale of temperature in the interval -35° to $+100^{\circ}$, very generally adopted for scientific and technical work, is the scale of the constant volume hydrogen gas thermometer, defined in the following resolution of the International Committee on Weights and Measures, adopted October 15, 1887:

"The International Committee on Weights and Measures adopts as the standard thermometric scale for the international service of

weights and measures the centigrade scale of the hydrogen thermometer, having as fixed points the temperature of melting ice (0°) and of the vapor of distilled water boiling (100°) at standard atmospheric pressure, the hydrogen being taken at an initial manometric pressure of one meter of mercury—that is to say, $\frac{1.000}{1.3158} = 1.3158$ times the standard atmospheric pressure.”

The scale adopted in this resolution refers to the scale defined by the hydrogen gas thermometer set up by Chappuis at the International Bureau of Weights and Measures, at Sèvres, near Paris, and compared by him with the scale¹ defined by four primary standard mercurial thermometers of French hard glass (*verre dur*), made by Tonnelot and bearing the Nos. 4428, 4429, 4430, and 4431. The relation between the mean scale defined by these four thermometers, which scale is known as the *mean verre dur scale*, and the Hydrogen Scale was thus established. From this known relation it is now therefore possible to express on the Hydrogen Scale, temperatures measured with a *verre dur* mercurial thermometer, provided the glass of which the thermometer is constructed is identical in its physical properties with the *verre dur* of the four standards that were directly compared with the hydrogen gas thermometer. This condition can not, however, be fully realized, as experience has shown that thermometers made up even out of the same ingot of glass differ somewhat in chemical composition and in physical properties, and therefore in the temperature scales which they define. While the magnitude of the variations in the scales, due to these causes, is quite small, it is yet definite and measureable, and for thermometers constructed of *verre dur* the deviation of the temperature scales from the *mean verre dur scale* may be as much as, or even exceed, ± 0.01 in the interval 0° to 100° . To express temperatures on the International Hydrogen Scale with the highest attainable precision it is therefore necessary to apply to the corrected readings of a *verre dur*

¹ By the term “scale defined by a (mercurial) thermometer” is meant the temperatures found after correcting the observed readings for the variations in caliber of the tube, for the variation in the external pressure on the bulb from standard atmospheric pressure (760 mm), for the effect of the internal pressure on the bulb due to the column of mercury above the center of the bulb, for zero (which is the observed reading in melting ice, corrected for calibration, external and internal pressure, this corrected reading being applied, with changed sign, to the observed reading), and for the variation in the value of the fundamental interval from 100° .

thermometer further small supercorrections, which are the amounts by which the scale of the thermometer differs from the *mean verre dur scale* at the temperatures in question. These supercorrections must be determined either by direct comparison with the four primary standards of the International Bureau or by comparison with other primary standards whose relation to the *mean verre dur scale* has been so determined.

The Bureau of Standards was in possession of 16 primary standard mercurial thermometers, all constructed of *verre dur*, the calibration corrections, pressure coefficients, and fundamental intervals of which have been determined at the International Bureau at various times from 1885 to 1903. As the temperature scales defined by these thermometers would very likely differ slightly from one another, one of the objects of the present investigation was to determine the magnitude of these differences; in other words, to determine the *supercorrections* that must be applied at different temperatures to the scale defined by each thermometer to reduce to temperatures on the mean scale of all the thermometers, and thus to establish a standard scale of temperature for future use and reference with as high a degree of precision as possible.

The advantages resulting from the use of the same standard scale of temperature are of course at once obvious. Only a few years ago, before the introduction of the International Hydrogen Scale, investigators frequently spent far more time in setting up a gas thermometer and establishing a temperature scale to which to refer their measurements than in carrying out their experiments. A familiar illustration is furnished by the classical determinations by Rowland of the mechanical equivalent of heat and the capacity for heat of water. A further great advantage in a single standard scale at once available to every investigator, aside from the great saving in time, is that all measurements expressed on that scale, made by different methods and by different observers, can be compared with one another with the greatest advantage, and outstanding differences very often explained. Thus a comparison of the best determinations of the mechanical equivalent of heat, carried out by mechanical and electrical methods, showed a difference amounting to about 1 part in 400, and the two methods gave a different variation of the capacity for heat of water with the temperature. When,

however, the temperature scale used by Rowland, based on his air thermometer, was reduced to the International Hydrogen Scale, which was used by Griffiths and by Schuster and Gannon in their experiments, the curves expressing the relation between the capacity for heat of water and the temperature, found by the two methods, were parallel and showed a constant difference of about 1 part in 400 in the interval 15° to 25° . An analysis of the factors entering into the determinations at once pointed to a possible error in the electrical units in terms of which the energy measurements were expressed, which was almost simultaneously confirmed by the redeterminations of the electromotive force of the standard Clark cell. When the electrical determinations were then corrected by using the newly found value of the electromotive force of the Clark cell, the results found by the two methods were brought into agreement to within about 1 part in 2000.

It is therefore highly desirable that the standard scale of temperature to be used by this Bureau should be as nearly as possible identical with the International Hydrogen Scale. By intercomparisons of the standard thermometers at a number of temperatures, and thus determining their supercorrections to the mean scale of all of them, it was possible to fix a very definite temperature scale. As several of these thermometers have been directly compared with the standards of the International Bureau it was therefore possible to determine any outstanding difference between the mean scale defined by these thermometers and the mean *verre dur* scale defined by the standards of the International Bureau, which scale serves to fix and to reproduce the International Hydrogen Scale.

The results of the redetermination of the constants of these thermometers and their intercomparisons in various groups at every 10° in the interval 0° to 100° , may be briefly summed up in the statement that the mean scale defined by the 16 primary standard thermometers included in the investigation is in agreement with the mean *verre dur* scale of the International Bureau and therefore serves to reproduce the International Hydrogen Scale to within about 0.002, which is probably within the limits of accuracy at present attainable in mercurial thermometry.

II. MERCURIAL THERMOMETRY.

The instrument most generally used for temperature measurement within the interval 0° to 100° (or even 500°) is the mercury-in-glass thermometer, the two fixed points of which, 0° and 100° on the centigrade scale, are defined respectively as the temperature of melting pure ice and the temperature of the vapor of boiling pure water, both at standard atmospheric pressure.¹ The scale is then fixed by dividing the volume included between these two fixed points into 100 equal parts.

The scale of temperature thus defined is found to vary (*a*) with the composition of the glass, such slight changes in composition as are brought about in working the bulbs in the blast flame being often sufficient to produce a noticeable effect on the temperature scale defined by the thermometers, and (*b*) even with glass of the same composition in different physical conditions, as to strains. The mercury-in-glass scale is therefore different for every different kind of glass used. Thermometers made up at different times, of glass of the same trade name, are found to differ by as much as 0.02 in the temperature scales which they define.²

The only scale of temperature independent of the physical properties of a particular substance is the absolute thermodynamic scale introduced into physical science by Lord Kelvin. The scale defined by a perfect gas would be in agreement with this scale. The scale defined by the more permanent gases is a very close approximation to the absolute scale. For this reason, as well as on account of the large coefficient of expansion of gases and the large range throughout which they can be used, the scale of the gas thermometer is adopted as the standard of reference in all temperature measurements. Our knowledge of the slight departure of this scale from the absolute thermodynamic scale is based on experiments showing the departure of gases from the laws of a perfect gas. The first experimental realization of the absolute scale is due to Thomson and Joule³ in

¹ I. e., a pressure of 760 mm of mercury at 0° C, latitude 45° , sea level.

² Grützmacher: *Wiss. Abhandl. der Phys.-Tech. Reichsanstalt*, **3**, p. 256; 1900.

³ Thomson and Joule: *On the Thermal Effect of Fluids in Motion*, *Phil. Trans. Roy. Soc.* **148**, p. 357; **1853**: **144**, p. 321; **1854**: **152**, p. 579; 1862.

their porous plug experiments on the cooling (or heating) of a gas accompanying its free expansion.

The departure of the scales of mercury-in-glass thermometers, made of the different kinds of glasses that are now very generally used (verre dur, Jena 16^m, Jena 59^m, and English-Kew glasses), from the scale of the standard gas thermometer has been determined by a number of investigators,⁵ so that it is now possible to express on the gas scale, temperatures measured with mercurial thermometers made of these glasses. A thermometer so constructed that it is capable of defining within itself a scale of temperature is called a *primary standard*. Experience has shown that the length of one degree which best satisfies all the requirements for a primary standard in the interval 0° to 100° is about 6 or 7 mm (not less than 5). This length of degree divided into 0°:1 intervals, when used with a small telescope magnifying 6 to 10 times, permits of eye estimates to 0.01 or 0.02 of such an interval (i. e. 0°:001 or 0°:002). An increase in the length of a degree and a finer subdivision would increase the accuracy of reading, which, however, would be more than offset by other sources of error; for to increase the length of a degree a larger bulb would be necessary, which would increase the lag of the thermometer, unless a thinner wall bulb or a finer capillary were used. In both of the latter cases the effect of the variable capillary pressure of the mercury meniscus would give rise to greatly increased uncertainties in the indications of the thermometer. For

⁵ For the French hard glasses: Chappuis: Trav. et Mem. du Bur. Int. des Poids et Mesures. 6; 1888.

For the Jena 16^m normal, 59^m borosilicate, and 122^m baryt-borosilicate glasses; Wiebe and Böttcher: Zs. für Instrumentenkunde, 10, p. 245; 1890. Grützmaker: Wied. Ann., 68, p. 769; 1899. Thiesen, Scheel, und Sell: Wiss. Abhandl. der Phys.-Tech. Reichsanstalt, 2, pp. 1-71; 1895; Zs. für Instrumentenkunde, 15, p. 433; 1895. Mahlke: Wied. Ann., 58, p. 965; 1894. Scheel: Wied. Ann., 58, p. 168; 1896. Lemke: Zs. für Instrumentenkunde, 19, p. 33; 1899. Grützmaker: Zs. für Instrumentenkunde, 15, p. 250; 1895; also Greiner and Friederich's "Resistenzglas."

For the older Thüringen glasses and English-Kew glass: Grunmach: Metron. Beiträge der Kais. Normal-Aich. Kommission, Berlin, No. 3, p. 54; 1881. Marek: Zs. für Instrumentenkunde, 10, p. 283; 1890. Wiebe: Zs. für Instrumentenkunde, 10, p. 438; 1890. Grützmaker: Wiss. Abhandl. der Phys.-Tech. Reichsanstalt, 3, p. 231. Balfour Stewart: Phil. Trans. Roy. Soc., 153, p. 425; 1863. Chree: Phil. Mag., 45, p. 225; 1898. Guillaume: Determination du Rapport du Yard au Mètre, Bur. Int. des Poids et Mes.; 1896. Schloesser: Zs. für Instrumentenkunde, 21, p. 296; 1901. Harker: Proc. Roy. Soc., A, 78, p. 225; 1906.

higher temperatures the length of a degree should be less. In order not to unduly increase the length of the thermometer the scale need only include the region in which it is intended for use, e. g., 0° to 50° , 100° to 200° , etc. It must, however, be so constructed with suitable auxiliary reservoirs in the stem that it contains the two fixed points 0° and 100° , and further, that the volume of any part of the stem can be referred to the fundamental volume between the 0° and 100° marks.

In the practical construction of thermometers it is not possible to realize the ideal conditions, viz, exact location of the 0° and 100° marks on the stem, and subdivision of the included volume into 100 equal parts. In the construction of laboratory thermometers this is approximately done by calibrating the tube by moving mercury threads along the capillary and observing their lengths in various positions and thus fitting the graduation to allow for irregularities of the bore. By this method good laboratory thermometers can be constructed accurate to a few hundredths of a degree. For primary standards, however, this method of construction is not permissible, for here the corrections for variation in caliber must be known to a far higher degree of accuracy and must be determined by the most careful *calibration of the tube*.⁶ The amount of work involved in this calibration is greatly increased if the graduations are irregularly spaced. The tube of which the thermometer is made must be carefully selected by a preliminary calibration and must be very uniform in cross section. The greatest difference of the calibration corrections should not exceed 0.2 or 0.3 at most. After the positions of the fixed points are determined the space between must be divided

⁶ Full details of the methods of calibration of mercurial thermometers will be found in the following references: Marek: Carl's Rep., 15, p. 300; 1879; adaptation of method of Hansen, Abhandl. d. math.-phys. Classe der K. Sächs. Gesellschaft der Wiss.; 1874. Thiesen: Carl's Rep., 15, p. 285, p. 678; 1879. Report of Committee of the British Association for the Advancement of Science, B. A. Report, pp. 145-204; 1882. Benoit: Trav. et Mem. du Bur. Int. des Poids et Mes., 2, p. C35; 1883. Broch: Ibid, 5, p. 3; 1886. This and the preceding reference relate to the calibration of scales. Guillaume: Ibid, 5, Etudes Thermometriques, p. 5; 1886; Thermometrie de Precision, p. 40, Gauthier-Villars et Fils, 1889. Pernet, Jaeger und Gumlich: Wiss. Abhandl. der Phys.-Tech. Reichsanstalt, 1; 1894. Pernet: Viertel Jahrschrift der Naturforsch. Gesellschaft, Zurich, 41, p. 128; Anm. 1. Grützmacher: Wiss. Abhandl. der Phys.-Tech. Reichsanstalt, 3, p. 245; 1900.

into parts of equal length. For this purpose a dividing engine should be used whose screw has been very carefully studied for progressive and periodic errors. As the length of a degree is generally 5 to 7 mm the accidental errors of ruling should be kept within a few thousandths of a millimeter. The graduation marks on the stem should be very fine and clear, and their thickness should not be greater than 0.05 of the smallest scale interval and should preferably be less.

As the indications of a thermometer are influenced by the pressure on the outside of the bulb (atmospheric pressure and the pressure of the medium in which the bulb is immersed) and by the pressure from within due to the mercury column and capillary forces, it is necessary to determine the *external* and *internal pressure coefficients* so that the indications of the thermometer can be reduced to standard conditions, i. e., to an external pressure of 760 mm and an internal pressure of zero (really the somewhat variable pressure of the mercury meniscus) corresponding to the horizontal position of the thermometer.

As it is not possible for the maker to locate without error the fixed points, the value of the interval included between these two points must be determined with the greatest care by observing the reading of the thermometer in steam and the corresponding barometric pressure; and immediately after, before any appreciable recovery of the zero shall have taken place, the reading in melting ice. This gives the necessary data for determining the *fundamental interval* of the thermometer, i. e., the number of scale degrees¹ of the thermometer between the temperatures defined by the two fixed points, 0° and 100°. The error in the fundamental interval should not exceed 0°1.

The stem of the thermometer should be transparent, so that errors of parallax can be avoided by taking the mean of readings with scale before and with scale behind the mercury column, the line of sight remaining unchanged; for this reason the use of an enamel-back stem is not desirable in a primary standard thermometer. In thermometers of the inclosed scale type errors of parallax are

¹ I. e., corrected for variation in caliber and for internal and external pressure.

avoided by adjusting the line of sight so that the portion of the graduation seen through the fine capillary stem is continuous with the graduation seen on either side of it.

To better understand the methods adopted in the intercomparisons described in this paper it will perhaps be worth while to briefly consider the thermal properties of glasses and especially the effects of thermal hysteresis on the indications of thermometers.

Glasses exhibit in varying degree the phenomena of *thermal hysteresis*. When a thermometer is heated to a definite temperature it expands to its final equilibrium condition in a few minutes for the better thermometric glasses and in an hour or so for the inferior glasses. When the thermometer is again cooled to its original temperature it takes from many days to many months for the glass to again contract to the equilibrium condition corresponding to that temperature. These phenomena manifest themselves in the so-called *depression of the zero*⁸ and *recovery of the zero*.

The glasses⁹ used before the introduction of the verre dur, Jena 16^m normal, and Jena 59^m borosilicate glasses, nearly all showed large zero depression, ranging from a few tenths of a degree to a degree or more. The above glasses have depressions of about 0°10, 0°08, and 0°03, respectively. The Jena 59^m borosilicate glass is the best thermometric glass yet brought out. In addition to the small zero depression after heating, the scale of temperature which it defines nowhere differs from the scale of the hydrogen thermometer in the interval 0° to 100° by more than 0°03. This glass can also be used for temperatures as high as 500°. The investigations of R. Weber¹⁰ and of Wiebe¹¹ have shown that glasses in which are present the oxides of both sodium and potassium show relatively

⁸ As a measure of the zero depression is taken the difference in the ice point corresponding to long continued (some weeks) exposure at 0°, and the ice point taken immediately after the thermometer has been at 100°, before any appreciable recovery of the zero shall have taken place.

⁹ For a complete discussion of the thermometric behavior of some of the older glasses reference is made to Grützmacher: *Wiss. Abhandl. der Phys.-Tech. Reichsanstalt*, 3, p. 231; 1900.

¹⁰ R. Weber: *Sitzungsberichte d. K. Preuss. Akad. d. Wiss.*, Dec. 13; 1883.

¹¹ Wiebe: *Ibid.*, July 17, 1884; Nov. 12, 1885. *Zs. für Instrumentenkunde*, 6, p. 167; 1886.

large depressions in comparison with glasses in which only one of these oxides is found, and especially so if the two are present in about equal proportions. The same statement holds for the elastic hysteresis exhibited by the glasses after undergoing mechanical deformation of any kind.¹²

On account of these phenomena of thermal hysteresis, the observed reading of a thermometer depends not alone on its temperature, but also on its previous thermal history. The effect of the consequent variations of zero on the observed readings is eliminated by the procedure due to Pernet,¹³ which consists in using as the zero from which the temperature is to be reckoned the reading found in melting ice, immediately after the temperature measurement in question, the fundamental interval being determined in the same way by using the ice point observed immediately after the steam-point observations.

Other changes which affect a mercury-in-glass thermometer may be broadly grouped under the head of *secular* and *annealing changes*.

There is a slow contraction of the glass and consequent rise of the ice point which goes on for years. With thermometers of suitable glass that have been properly annealed this change is very small, and should not exceed 0.01 in some years for thermometers that are used at ordinary temperatures. The larger part of this change takes place in the first few months after the thermometer is made up.

A discussion of the annealing changes and experimental data relating thereto will be found in a paper by one of the authors,¹⁴ so that it will only be necessary to briefly refer to this subject here. The strains set up in the glass in making the thermometer, which ordinarily only slowly disappear, are relieved by a suitable annealing at high temperatures, resulting in a contraction of the glass and consequent rise of the ice point. Unless the thermometer has been properly annealed a small rise of the ice point will be observed after each heating. Hence in experiments on the zero depression

¹² G. Weidmann: Dissert., Jena; 1886. Wied. Ann., 24, p. 214; 1886.

¹³ Pernet: Trav. et Mem. du Bur. Int. des Poids et Mesures, 1, p. B.12; 1881.

¹⁴ Dickinson: Bureau of Standards Bulletin, 2, p. 189; 1906.

it is necessary to work with well-annealed glass, or the phenomena will be partly masked by the rise of the ice point. Schott¹⁵ has investigated the effect of annealing and has found that it acts by relieving the strains, as was shown by the almost complete disappearance of double refraction in glass after annealing. Thermometers intended for use at high temperatures should be annealed at a temperature of about 450° for a period of 200 hours, which treatment should be followed by a period of slow cooling extending over a day or two. The total rise of the ice point for this process of annealing will be about 25° or 30°. The contraction of the glass, which results from the annealing of a thermometer, takes place in the stem as well as in the bulb, and produces an increase in the fundamental interval. As the volume included between the two fixed points is approximately one-sixtieth of the volume of the bulb, it follows that, if the stem undergoes the same proportional change in volume as the bulb, the increase in the fundamental interval should be about one-sixtieth of the observed rise in the ice point. Experiment shows that the change in the fundamental interval is more nearly one-thirtieth of the observed rise of the ice point. This is very probably due in part to the fact that the strains present in the thick glass of which the stem is made are greater than in the thin wall tubing of which the bulb is made, so that the change in volume, on annealing, is relatively greater. However, an increase in the fundamental interval is observed when only the bulb is annealed. The coefficient of expansion of glass is greater in the strained than in the unstrained condition.¹⁶ When the strains which are present in the glass are relieved by the annealing the coefficient of expansion is diminished, which results in an increase in the fundamental interval of the thermometer. When, however, a thermometer has been well annealed the subsequent change in the fundamental interval is very small, even when used at high temperatures, and when used in the interval 0° to 100° does not much

¹⁵Schott: *Zs. für Instrumentenkunde*, 11, p. 330; 1891.

¹⁶Schott: *Vortrag im Verein zur Beförd. d. Gewerbfl. in Berlin*; 1892. Hovestadt: *Jenaer Glas*, p. 235, published by Gustav Fischer, Jena, 1900; this book contains very complete information on the physical and chemical properties of the various Jena and some other glasses.

exceed the limits of experimental error within which this constant can be determined.¹⁷

For primary standards used within this interval the relative changes in the caliber of the tube are almost within the limits of precision to which the calibration corrections can be determined.¹⁸ Long continued use apparently results in a very small decrease in the volume of the small auxiliary reservoir relatively to the remainder of the capillary tube, as is to be expected from the strained condition of the glass in the region of this reservoir.

III. CONSTANTS AND STANDARDIZATION OF THERMOMETERS.

The primary standard thermometers available for these intercomparisons were:

Tonnclot Nos.	{ 433 ¹	433 ²	4334	4335
	4336	4623	4624	
Baudin Nos.	{ 15282	15555	15583	
	15962	15963		
	16016	16017	16018	
Tonnclot No.	11801			

Full details of the types, ranges, and dimensions of the thermometers are given in Fig. 8 and in Table XXI in the Appendix to this paper.

These thermometers are all made of French verre dur, a lead-free glass containing about 11 per cent of the oxide of sodium (Na_2O), and only a small trace (0.3 or 0.4 per cent) of the oxide of potassium (K_2O).¹⁹

The series of Tonnclot thermometers Nos. 4331-4336 were made by Tonnclot, of Paris, in 1884. Of this series, 4333 was broken before the beginning of this investigation. Nos. 4623 and 4624 were made in 1888. This set of eight thermometers accompanied the Interna-

¹⁷ Guillaume: Trav. et Mém. du Bur. Int. des Poids et Mesures. 5; 1886.

¹⁸ Guillaume: loc. cit., pp. 63-67; and Grützmacher: Wiss. Abhandl. der Phys.-Tech. Reichsanstalt, 3, p. 242; 1900.

¹⁹ Tornöe: Trav. et Mém. du Bur. Int. des Poids et Mesures, 5; 1886.

tional Prototype Meters Nos. 21 and 27, and the International Prototype Kilograms Nos. 4 and 20, which were awarded by lot to the Government of the United States. These thermometers are graduated into $0^{\circ}1$ and include on their scales the interval 0° to 50° , above which are an auxiliary reservoir and a small region of the scale including the 100° point. The length of 1° varies for the different thermometers from 6.56 mm to 7.02 mm. The thickness of the graduation marks, which are sharp and clear, varies from about 30 to 45 microns (0.04 to 0.06 of an interval), being less in the more recently constructed thermometers.

Thermometers Nos. 15282, 15555, 15583, 15962, 15963, 16016, 16017, and 16018 were made by Baudin, of Paris, at various dates from April, 1900, to November, 1903. All of these thermometers are graduated into $0^{\circ}1$, and, as regards fineness of graduations and perfection of workmanship, leave nothing to be desired. Nos. 15282, 15962, and 15963 include the range 0° to 100° , the length of 1° being 5.913, 5.914, and 5.878 mm, respectively. Nos. 15555 and 15583 include the range 0° to 50° , the length of 1° being 8.252 and 7.954 mm. Nos. 16016, 16017, and 16018 include the range 50° to 100° , the length of 1° being 7.260, 7.236, and 7.196 mm, respectively. The thickness of the graduation marks is about 18 microns (0.02 to 0.03 of an interval).

Tonnelot No. 11801, which was kindly loaned for this investigation by Professor J. S. Ames, of the Johns Hopkins University, was made by Tonnelot in 1895, and includes the range 0° to 100° . The length of 1° is 5.858 mm.

All of these thermometers have been submitted to an exhaustive study for errors of graduation, calibration corrections, external and internal pressure coefficients, and fundamental interval corrections at the International Bureau of Weights and Measures under the supervision of Dr. Ch. Ed. Guillaume. Tonnelot No. 11801 and Baudin Nos. 15962, 15963, 16016, 16017, and 16018 have in addition been compared with the primary standard mercurial thermometers of that Bureau in order to determine the small deviation of the verre dur scale which they define from the mean verre dur scale of the International Bureau, which scale fixes the International Hydrogen Scale of Temperature.

Tonnélet No. 11801 has also been compared by Dr. W. S. Day²⁰ and by Mr. F. Mallory and one of the authors²¹ with the thermometers used by Rowland in his determinations of the mechanical equivalent of heat, as well as with a Callendar-Griffiths platinum resistance thermometer, so that the scale defined by Tonnélet 11801 is directly connected with the gas scales defined by the air thermometers of Rowland and of Callendar and Griffiths, as well as with the International Hydrogen Scale.

Calibration corrections.—The details of the methods employed in the calibration of the thermometers at the International Bureau, together with the tables of the calibration corrections, are given in the Appendix, pp. 721–728.

We have not deemed it necessary to repeat in detail the calibration of these thermometers, as any small errors in the calibration corrections would be taken care of in the supercorrections found from the intercomparisons. As the calibrations of some of the Tonnélet thermometers were made nearly twenty years ago, it seemed possible that measurable changes might have taken place in the relative volumes of different portions of the stem. If such were the case, one would expect to find the greatest changes in the calibration correction at 50° for those thermometers having a 50° auxiliary reservoir, as the strains called into existence in the construction of the thermometers in the region of this enlargement are probably greater than, or at least different from, the strains present in the remainder of the stem, so that the disappearance of these strains might be expected to affect the volumes of the tube and the reservoir differently. Furthermore, on account of the difference in area of the two 50° intervals, any surface changes on the interior walls would be most pronounced for such a thermometer. Some check determinations of the calibration corrections at 50° for thermometers of this type gave results that differed from the values given in the certificates by amounts which were within the limits of experimental error (0°002).

²⁰W. S. Day: *Phys. Rev.*, 6, p. 193; 1898. *Phil. Mag.*, 46, p. 1; 1898.

²¹Waidner and Mallory: *Phys. Rev.*, 8, p. 193; 1899. *Phil. Mag.*, 48, p. 1; 1899.

Pressure coefficients.—The values of the pressure coefficients of the thermometers, as given in the certificates, are summarized in Table I, where

β_e = external pressure coefficient = change of reading in scale degrees produced by a change in pressure of 1 mm of mercury.

β_i = internal pressure coefficient = change of reading in scale degrees produced by a change in pressure of 1 mm of mercury.

The method employed in these determinations at the International Bureau is to inclose the thermometers in a glass tube, containing sufficient mercury to cover the bulb of the thermometer, the remainder of the tube being filled with glycerine. Observations are then made with slowly rising temperature, as the pressure within the tube is varied by alternately connecting it with an exhausted receiver and opening it to the atmosphere. This gives the necessary data for the determination of the external pressure coefficient, β_e . The internal pressure coefficient is obtained from the relation

$$\beta_i - \beta_e = 0.000\ 015\ 4.$$

In our own determinations of the fundamental intervals, described later, the thermometers were read in steam in both horizontal and vertical positions, from which and the known dimensions of the thermometers the internal pressure coefficients were directly obtained. The values of β_e obtained in this way,²² by use of the above relation, are included in Table I.

²² In all of the computations of the values of β_e from the observations in steam, the necessary corrections were made for the variation in the calibration corrections and for the difference of level of the bulb in the horizontal and vertical positions of the thermometer in the steam-point apparatus.

TABLE I.
Pressure coefficients.

Thermometer	From Certificates of the Bureau International	From the observations in steam
Tonnolot No. 4331.....	0.0001145	0.0001163
" " 4332.....	.0001080	.0001058
" " 4334.....	.0001192	.0001214
" " 4335.....	.0001202	.0001167
" " 4336.....	.0001219	*.0001215
" " 4623.....	.0001282	.0001255
" " 4624.....	.0001228	.0001202
" " 11801.....	.0001159	.0001147
Bandin " 15282.....	.0001536	.0001525
" " 15555.....	.0001483	.0001463
" " 15583.....	.0001336	.0001316
" " 15962.....	.0001238	*.0001248
" " 15963.....	.0001345	(†)
" " 16016.....	.0001199	*.0001216
" " 16017.....	.0001266	.0001265
" " 16018.....	.0001217	.0001240

* The external pressure coefficient, β_e , was determined directly for 4336, 15962, and 16016, in an apparatus similar, in all essential particulars, to that used at the Bureau International. The bulb of the thermometer was immersed in mercury, the remainder of the tube being filled with water. The two independent determinations by Mr. E. F. Mueller and one of the authors have given the following values:

Observer.	No. 4336	No. 15962	No. 16016
H. C. D.	0.0001211	0.0001244	0.0001216
E. F. M.	.0001219	.0001253	.0001216
Mean	0.0001215	0.0001248	0.0001216

The values found from the observations in steam, for these three thermometers, were not entirely satisfactory, and differed by several per cent from the above values, being lower for 4336 and higher for 15962 and 16016.

† No. 15963 was broken during the intercomparisons at 80°.

The agreement between the values of β_e , as given in the certificates and as found from the observations in steam, is very satisfactory. Our values of the coefficients are, on the average, smaller by 0.0000006 (about 0.5 per cent), which corresponds to less than 0.0004 in the correction for internal pressure at 100°. In this con-

nection it is interesting to note that Thiesen, Scheel, and Sell²³ have shown that the heating and cooling of the liquid in which the thermometer is immersed, produced by the variations in pressure, will in general give high values for the pressure coefficient. This effect is greater for glycerine than for mercury or water, and vanishes for water at 4° C. These investigators found that when the determinations are made with the bulb immersed in mercury the value of the pressure coefficient is increased, due to the above cause, by about 0.0000014 (i. e., about 0.8 per cent), and recommend that the experiments on the determination of external pressure coefficient be carried out in a tube filled with water, in which case the correction due to the above cause is negligible.

In the observations in steam the readings are taken first in the horizontal and then in the vertical position (see below), so that the reading in vertical position is obtained under the disadvantageous condition of falling meniscus. For this reason, as well as on account of the great number of readings that can be obtained in the direct determination of β_0 in a short time, the determinations of this constant with the well-known Marek pressure apparatus, as used at the International Bureau, is entitled to greater weight than the values given by the observations in steam. We have therefore used the values of the pressure coefficients given in the certificates, and which are most satisfactorily corroborated by our own determinations, the maximum difference only in one case amounting to 0.00015 in the value of the internal pressure correction at 100°.

Determination of fundamental intervals.—The fundamental intervals of the thermometers were determined by observations in steam, with divisions before and divisions back, in horizontal and vertical positions, followed by similar observations in ice in vertical position.

Observations in steam.—The observations in steam were made in the International Bureau type of boiling-point apparatus devised by Chappuis, and shown in Fig. 1, which consists of a double wall steam jacket, the steam being led from the boiler, through the axis on which the apparatus turns, up the central tube in which the thermometer is found and down the outer annular space into the condenser, whence the condensed steam is returned to the boiler. The

²³Thiesen, Scheel, and Sell: *Wiss. Abhandl. der Phys.-Tech. Reichsanstalt*, 2, p. 7; 1895.

excess of pressure within the apparatus was measured by the small water manometer. By a careful preliminary study with several such manometers, inserted in the thermometer space from above as well as in the axis, with the inner opening of the manometers in different directions, it was made certain that the slight excess of pressure within the apparatus as measured by the manometer (which rarely attained 3 mm of water) was a true measure of the excess of pressure in the region in which the bulb was found, and that the readings were not appreciably affected by the dynamic action of the flowing steam. The action of the manometer was also tested by mounting within the apparatus a sensitive platinum resistance thermometer, and varying the excess of pressure from a fraction of a millimeter to 10 mm of water. The resistance thermometer at once responded to every change in pressure by the calculated amount, to within 0.001, thus further indicating that there was no appreciable superheating of the steam even for quite rapid steaming.

The top of the hypsometer was closed with a thin rubber disc, through a hole in which the thermometer was inserted. It was thus possible to expose nearly all the mercury thread to the steam except a few tenths of a degree, in and above the rubber, to permit of reading. With this arrangement, however, the mercury distills from the end of the column into the cooler region above it, so that the reading changes with time. The error from this source is in part eliminated, as the reading in ice, which is taken immediately after the observations in steam, will also be lowered. To avoid this source of error Pernet, Jaeger, and Gumlich²⁴ mounted above the hypsometer a small glass bell jar so that the thermometer was entirely inclosed in steam. To avoid the difficulties of reading the thermometer through the wet glass, we have sought to attain the same ends by adding to the apparatus a simple device which may be termed the *emergent stem heater* shown diagrammatically in Fig. 2 and in position on the hypsometer in Fig. 1. It consists of a brass tube, A, closed at the top with a removable cover, B, through which the rod R passes, and from which the thermometer is suspended. A current of air from the laboratory pressure system is passed

²⁴Pernet, Jaeger, and Gumlich: *Wiss. Abhandl. der Phys.-Tech. Reichsanstalt*, 1; 1894.

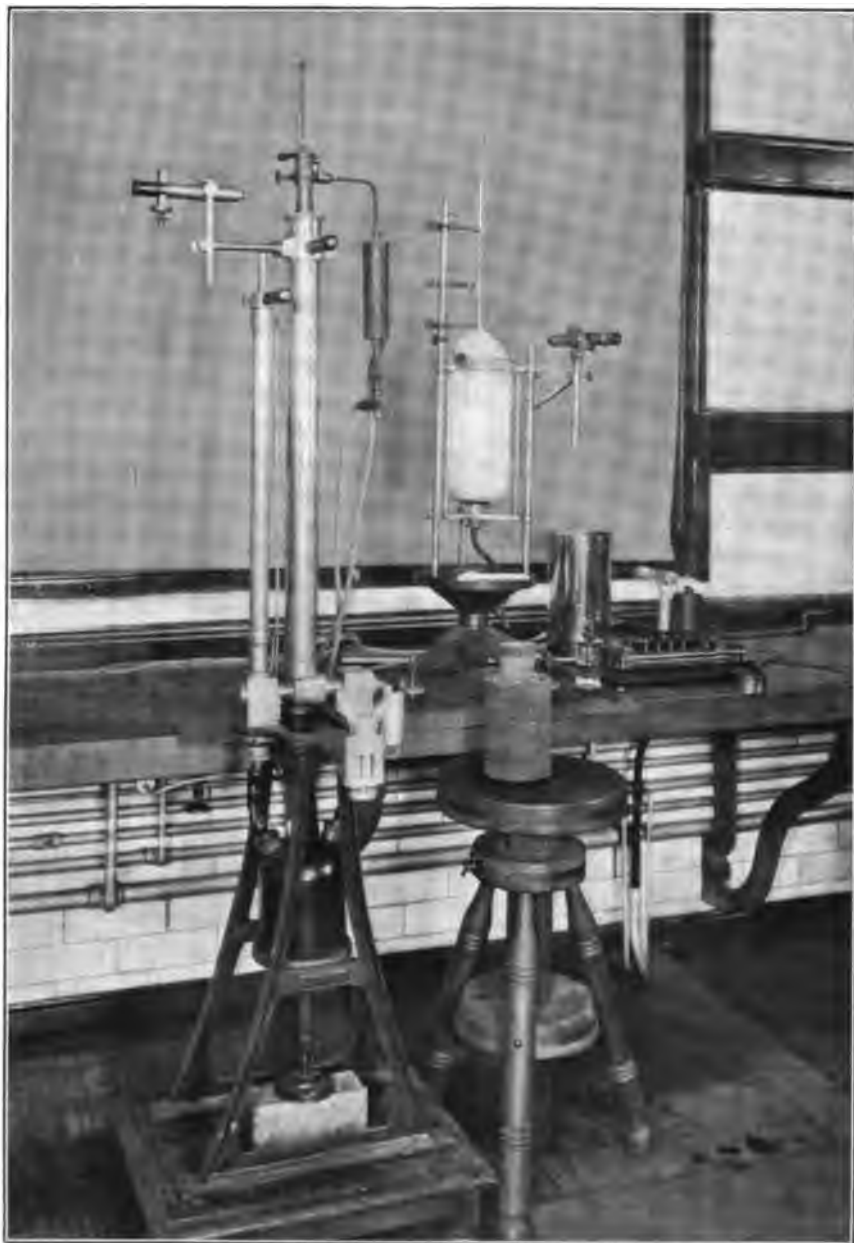


Fig. 1.—*Steam-Point and Ice-Point Apparatus.*

through the coiled copper pipe C and heated to the temperature of 100° to 105° by means of the Bunsen burner Bb, and is then discharged into the space A, inclosing the emergent stem of the thermometer, T, the temperature of the air being indicated by a small thermometer, t , and being controlled by valves in the air and gas supply pipes. When once adjusted, with only occasional attention the temperature of the air could be kept at practically 100° for any length of time, and all sources of error incident to evaporation were avoided. The standard thermometer, T, was read by means of a telescope through the front plane glass window, W_1 , and the scale of the thermometer was illuminated by light reflected through the rear window, W_2 , by means of a piece of milk glass.

Pure water was used in the steam-point apparatus; a number of small quartz crystals in the bottom of the boiler served as nuclei to facilitate smooth boiling. Previous to being placed in position in the steam bath, the thermometers were exposed to a temperature of 100° in a Regnault hypsometer of the usual laboratory form for a period of 10 to 15 minutes.

The observations were taken first with the thermometer in horizontal and then in vertical position. This sequence of observations is necessary in the usual method of determining the steam point, where distillation is present, in order to avoid the possible error due to the distilled mercury which would be reunited with the column if the thermometer were turned from vertical to horizontal position. This procedure has one disadvantage, that the reading in vertical position is made with falling meniscus. This source of error may, however, be avoided by withdrawing some 10° or 15° of the stem from the steam bath for an instant and then pushing the thermometer back into position. When the emergent stem heater was used and distillation was avoided, it would therefore have been better to have taken the readings in vertical position first.

The position of the mercury meniscus was estimated twice by each observer, alternately; the readings were written on a small slip of paper and passed to a third party who entered the results on the record sheet, along with the time corresponding to each reading. This procedure was followed throughout, in the observations in ice as well as in steam, in order that each observer might be entirely

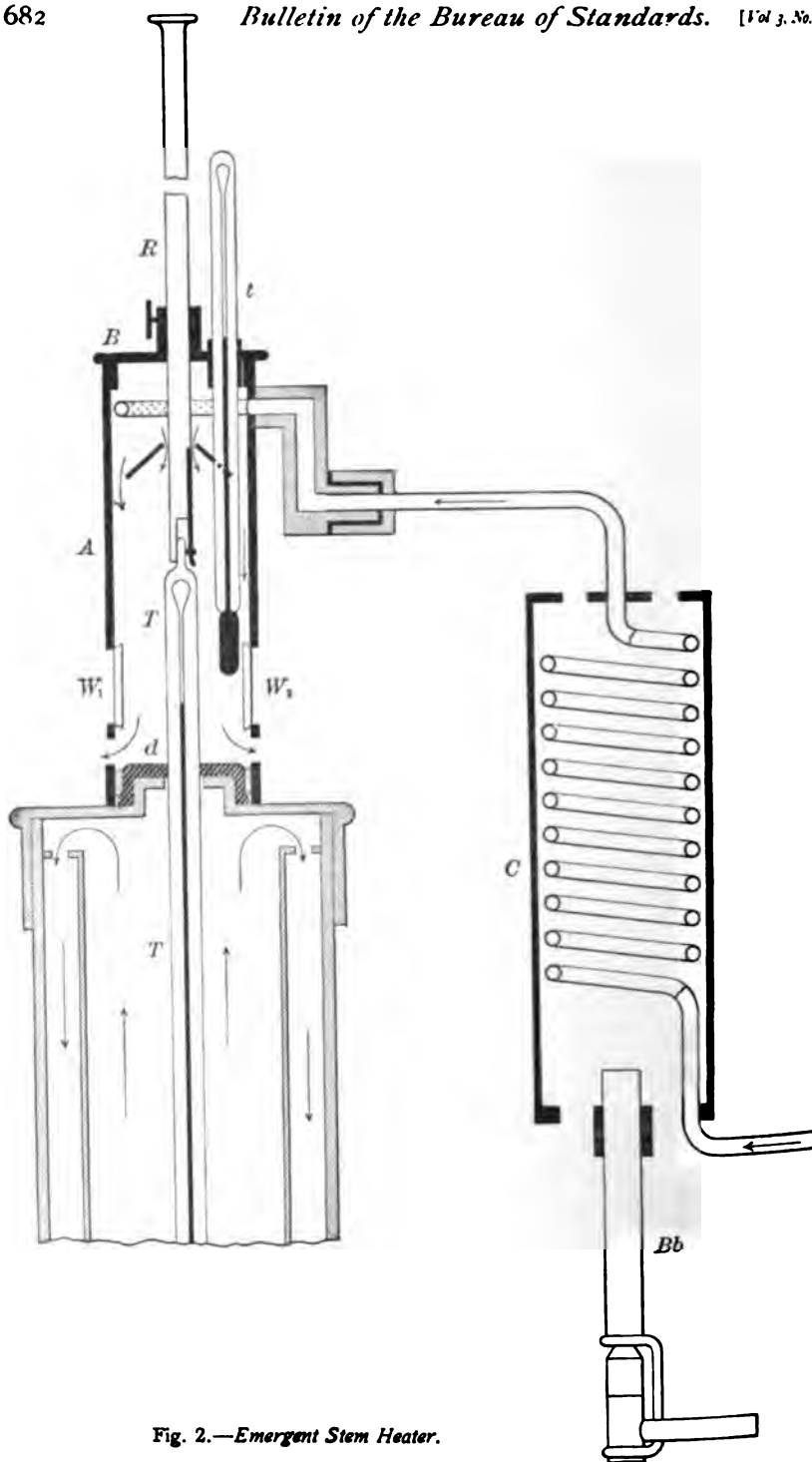


Fig. 2.—Emergent Stem Heater.

free from the inevitable influence exerted by a knowledge of the other's reading. The mean of the estimates by the two observers only very rarely differed by as much as $0^{\circ}003$. In order to check the possible development of a systematic error in estimation, frequently throughout the work, micrometric determinations accompanying the estimates were made, and the order of agreement was always found to be within $0^{\circ}001$ or $0^{\circ}002$.

The temperature of the steam, corresponding to observed readings of the thermometers, was deduced from readings of the standard barometer.

Barometers.—While the observations in steam were being made, two other observers were making a series of observations on the Feuss primary standard barometer, which is shown removed from its usual wall mounting, in Fig. 3 (on the left). In this barometer, which consists of a combination of the cistern and siphon types, the difference in height of the mercury in the two glass tubes of equal diameter (14 mm) is measured by means of a vernier, graduated to 0.02 mm, which slides over a silvered brass tube, inclosing the barometer tubes, and on which the scale is engraved. The temperature of the mercury column is given by a thermometer, mounted within and read through a slot in the brass tube. The position of the mercury meniscus could be varied by means of the capstan wheel at the lower end of the barometer, so that, knowing the volume of the space above the mercury in the closed limb, it was possible to determine the pressure of the residual gas in this space. This "correction for residual gas" was determined at frequent intervals throughout the work. The corrections to the scale readings of the thermometer attached to the barometer have been determined at different times. By comparison with a meter bar, the corrections to which were known, it was found that the errors of graduation on the scale of the barometer were less than 0.02 mm. The errors of reading with this barometer were within 0.02 or 0.03 mm for different observers. Each determination of the barometric pressure was accompanied by a measurement of the height of the upper and lower meniscus. The observed barometer heights were reduced²⁵ to 0° and

²⁵ By means of the table given in Guillaume's *Thermometrie de Precision*, p. 323.

latitude 45° , sea level.²⁶ Corrections were also applied for the slight amount of residual gas in the closed limb of the barometer, and for the differential pressure of the upper and lower menisci. From a time chart of reduced barometric pressures, the pressure corresponding to the time of observation on the standard thermometer in steam was found, and from this pressure the corresponding temperature of the steam.²⁷

Some of the earlier determinations of the fundamental intervals apparently showed a systematic departure from the values given in the certificates and found at the International Bureau. This discrepancy at once suggested a possible difference in the barometric measurements and led to an extended investigation of the barometer. After a redetermination of the scale errors, residual gas correction, and correction to attached thermometer of the Feuss barometer, it was compared with two laboratory barometers of the Fortin type, made by H. J. Green, of Brooklyn, N. Y. The order of agreement between these three barometers was within 0.1 mm, the limit of accuracy to which the Green barometers could be used. The desirability of a direct comparison with another standard barometer led to the construction of the barometer shown in Fig. 3. This consists of a U tube, siphon type, barometer, 21 mm internal diameter. It was filled after being kept exhausted for some days by a mercury pump. The positions of the menisci could be varied by running mercury into or out of the open limb of the barometer by means of the simple device shown in the illustration, so that the correction for residual gas could be determined. The distance between the upper and lower menisci was measured by means of a Société Générale cathetometer and a standard bronze meter bar, graduated into millimeters on a silver strip in the neutral axis of the bar. This meter bar, together with the table of corrections to its scale, was kindly placed at our disposal by Mr. L. A. Fischer, of the Division of Weights and Measures.

²⁶ By means of the equation giving the acceleration of gravity at any latitude L and elevation A (in meters) above sea level, in terms of the acceleration at latitude 45° , sea level, as computed by Broch (*Trav. et Mem. du Bur. Int.* 1, p. A9; 1881)

$$\frac{g_{L,A}}{g_{45,0}} = (1 - 0.00259 \cos 2L) (1 - 0.000000196A).$$

²⁷ From the tables given by Broch (*loc. cit.*), based on a recomputation of the experiments of Regnault (see also Guillaume, *Thermometrie de Precision*, pp. 115 and 327).

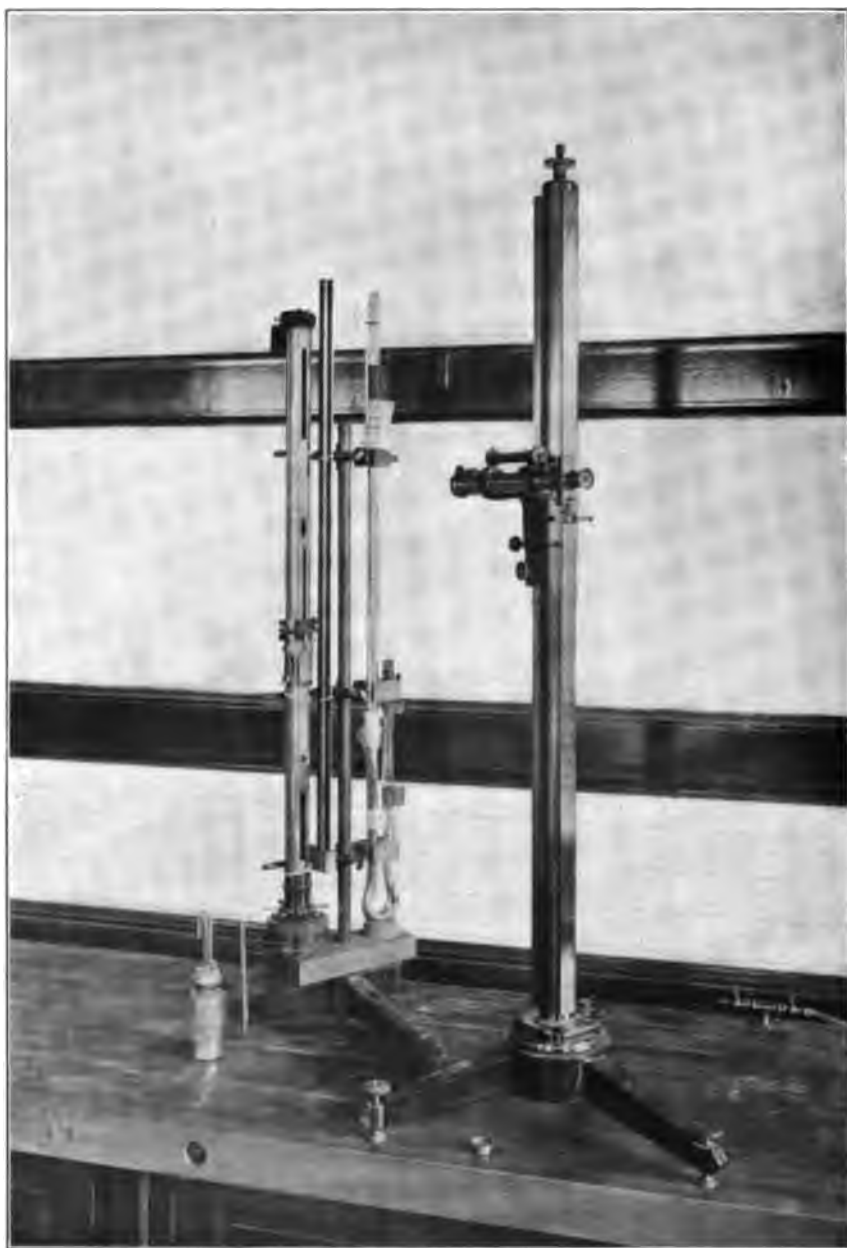


Fig. 3.—*Standard Barometers.*

The position of the top of the meniscus was sharply defined by surrounding the barometer tube with a black paper cylinder, the lower edge of which was brought almost down to the plane of the meniscus, and using an illuminated white background. This method effectively cuts off stray reflections so that settings of the cross hair of the telescope could be readily made on the plane of the upper surface of the meniscus to within 0.02 or 0.03 mm. The temperature of the mercury column was measured by two thermometers with bulbs in contact with the barometer tube and protected against external influences by wrapping with cotton.

The illustration, Fig. 3, shows (from right to left) the standard barometer, the standard meter, and the Feuss barometer, and nearby the cathetometer. These barometers have been intercompared

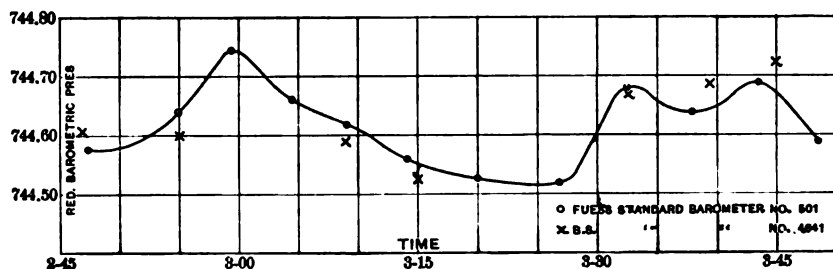


Fig. 4.—Intercomparison of Barometers.

many times and no certain difference as great as 0.02 mm has been found. The accompanying curve, Fig. 4, will serve as an illustration of the intercomparisons.

After the completion of these intercomparisons an opportunity presented itself for a comparison of the standard Feuss barometer with another standard barometer by the same maker; the difference between the two instruments being less than 0.02 mm as given by the mean of four intercomparisons.

Observations in ice.—Immediately after completion of each observation in steam, the thermometer was removed from the steam bath, rapidly cooled to 50° by stroking with a slightly moistened cloth, and then plunged into the super-cooler for a period of about 15 seconds, where it was cooled to a temperature of -2° or -3° , when it was rapidly transferred to the ice bath. Observations, consisting of

alternate readings (divisions front and back) by each observer, were in general begun within $1\frac{1}{2}$ to 2 minutes after the thermometer was removed from the steam bath, and were completed at the end of about 4 or 5 minutes, where eye estimates alone were made, and in about 6 minutes where micrometer measurements were made in addition. When the thermometer was withdrawn from steam, the recorder started a stop watch, and in this way entered with each observation the time that had elapsed since the thermometer was removed from the steam.

The zero begins to recover (rise) as soon as the thermometer is removed from steam; the true ice-point reading corresponding to the temperature of the steam could only be obtained by instantly cooling the thermometer from 100° down to 0° and reading before any appreciable recovery had taken place, or, knowing the rate of recovery, by applying a correction to find the reading corresponding to the time when the thermometer was removed from the steam. Under practical working conditions the ice-point determination can be begun within 2 and completed in about 4 minutes. We have therefore reduced all our ice-point determinations to a time of 3 minutes after removal from steam, using as the rate of recovery per minute 0.0011 . The time required for a thermometer to attain its lowest reading in the ice bath can be greatly reduced by the use of a small stirrer made of thin glass rod, the circular arc of the stirrer surrounding the bulb of the thermometer. This procedure was followed in the latter half of the work.

The observations were made in the Ice-Point Apparatus, shown in Fig. 1, which consists of two concentric glass vessels surrounded by the polished metal cylinder shown to the right of and removed from the apparatus. The two glass vessels are filled with finely divided ice, the ice in the inner vessel being thoroughly saturated with distilled water. The latter precaution is very important, as otherwise very significant errors may arise entirely aside from the degree of purity of the ice used.²⁸ The ice was heaped up around the thermometer, as shown in the illustration, and a narrow deep channel was made to permit of reading. The thermometer was held in position by an adjustable spring, which pressed it against two

²⁸ Pernet: *Trav. et Mem. du Bur. Int.*, 1, p. B. 12, 1881.

V-shaped rests. A brass tube, with a fine wire suspension running through it and carrying a plumb bob at the bottom, of the same diameter as the tube, permitted of rapid adjustment of the thermometer supports in a vertical position. The telescope was adjusted to horizontal position by means of a striding level.

Throughout the work every precaution was taken to insure the purity of the ice. The ice used was a commercial artificial product made from filtered and twice-distilled water, and was very clear and pure. The ice itself and everything brought into contact with it was thoroughly washed with distilled water. After every series of determinations the resistances of several samples of water drawn from the ice bath were measured by means of the Kohlrausch Conductivity Bridge and Nernst Electrolytic Cell, shown in Fig. 1. The specific resistance after a complete series of ice-point determinations (7 thermometers) was in nearly every instance almost or quite as high as the specific resistance of the distilled water furnished by a block-tin condenser, and the resistance of the water obtained by melting the ice directly was considerably higher.

In the usual method of making the observations in ice the reading is obtained under the condition of falling meniscus, to avoid which the thermometer was supercooled to -2° or -3° , as mentioned above, before putting it into the ice.²⁹ The *Supercooler* used for this purpose, shown on the tripod stand in Fig. 1, consisted of a bottle, well covered with felt, containing a suitable mixture of salt and ice, into which dipped a test tube containing sufficient mercury to cover the bulbs of the thermometers.

Sample F. I. determination.—A complete series of observations and computations included in two determinations of the fundamental interval of Tonnelot No. 4332 are given in Table II. Under the columns W and D are given the independent readings by two observers, the two numbers included in each bracket being readings with divisions in front of and with divisions back of the mercury thread, respectively. The temperature of the steam was obtained from a time chart of reduced barometric pressures, after adding the small excess of pressure given by the water manometer, in the

²⁹ The authors have since found that a similar procedure was used by Grützmacher; Abhandl. der Phys.-Tech. Reichsanstalt, 8, p. 252, 1900.

manner already indicated. The times given under "Observations in Ice" refer to the time that has elapsed since the thermometer was removed from steam. Z_1 is the ice point reading reduced to a time of 3 minutes after removal from steam.

TABLE II.
Fundamental Interval Determination.

Nov. 10, 1906.

Tonnelot No. 433a.

Observations in Steam							Observations in Ice						
Horizontal				Man.	Vertical				Man.				
Time	W	Time	D		Time	W	Time	D		Time	W	Time	D
11:21:00	99.641 .642	11:20:00	99.638 .637	2.0 1.8	11:24:30	99.577 .577	11:23:30	99.583 .582	2.6 1.8	2:15	{ -.095 -.092	2:40	{ -.090 -.090
11:22:30	99.641 .638	11:22:00	99.639 .638	2.1 2.0	11:26:00	99.579 .580	11:25:10	99.580 .580	2.3 2.2	3:15	{ -.091 -.088	3:30	{ -.086 -.082
2:59:00	99.595 .598	2:58:00	99.596 .597	2.0 2.0	3:01:45	99.542 .548	3:01:00	99.543 .545	3.0 3.0	2:40	{ -.096 -.091	2:05	{ -.094 -.088
3:00:00	99.598 .597	2:59:15	99.596 .597	2.1 2.2	3:03:45	99.538 .547	3:02:00	99.544 .544	2.6 2.8	3:30	{ -.094 -.085	2:55	{ -.087 -.094
11:21:45	99.640	11:21:00	99.638	2.0	11:25:15	99.578	11:24:20	99.581	2.2	2:45	-.091	3:05	-.090
Cal. cor.	-.010		-.010			-.011		-.011			.000		.000
E. P. cor.	+.001		+.001			+.001		+.001			+.001		+.001
I. P. cor.	.000		.000			+.056		+.056			+.007		+.007
Z. cor.	+.083		+.082			+.083		+.082					
	99.714		99.711			99.707		99.709			-.083		-.082
Steam T.	= 99.625		99.625			99.624		99.625			$Z_1' = -.083$		$Z_2' = -.082$
F. I.	= 100.089		100.086			100.083		100.084					
2:58:30	99.597	2:59:37	99.596	2.1	3:02:45	99.544	3:01:30	99.544	2.8	3:05	-.091	2:30	-.091
Cal. cor.	-.011		-.011			-.012		-.012			.000		.000
E. P. cor.	+.001		+.001			+.001		+.001			+.001		+.001
I. P. cor.	.000		.000			+.056		+.056			+.007		+.007
Z. cor.	+.083		+.082			+.083		+.082					
	99.670		99.668			99.672		99.671			-.083		-.083
Steam T.	= 99.580		99.580			99.581		99.582			$Z_1' = -.083$		$Z_2' = -.082$
	100.090		100.088			100.091		100.089					

Results of fundamental interval determinations.—The fundamental intervals of all the thermometers were determined after the conclusion of the intercomparisons, in the manner already described. The results of the several series of determinations carried out at different times are given in Tables III–V. In Table VI are given the mean value of the fundamental interval of each thermometer as found by the authors, the value found at the Bureau International, and the final value adopted. Two additional series of 18 determinations of the fundamental interval of Tonnelot 11801 have given $99^{\circ}.9986$. The observations on Baudin 16017 were interrupted at the conclusions of the series on November 20 on account of the appearance of a small crack in the glass in the neighborhood of the auxiliary reservoir at 50° . An independent series of 18 observations taken on this thermometer at an earlier date, along with Tonnelot 11801, gave $100^{\circ}.0046$, which is practically identical with the mean of the 8 determinations ($100^{\circ}.0049$) given in Table V.

The determinations here summarized show that the average difference (irrespective of sign) between the independent but simultaneous determinations by the two observers is $0^{\circ}.002$, the values found by one observer being quite consistently higher throughout by about $0^{\circ}.001$. In only 11 of the 216 determinations do the two observers differ by as much as $0^{\circ}.005$, yet it will be seen that the range throughout which the fundamental intervals of the thermometers varies at different times is about $0^{\circ}.015$. The authors are of the opinion that these variations are due in part to a sticking of the meniscus and resulting variations in capillary pressure, as the observations in both steam and ice are taken with a practically static meniscus, and in part to slight variations in the coefficient of expansion of the glass, depending on the state of strains present. The average difference between the fundamental intervals found by the authors and the values given in the certificates of the International

TABLE III.
Fundamental Interval Determinations.

Date	Ob- server	Thermometers No.						
		4331	4332	4334	4335	4336	4523	4524
1905								
Oct. 23	W.		100.071	100.072	100.090	100.082	100.061	100.073
	D.		.067	.073	.086	.080	.064	.068
"	W.	100.067	100.079	100.076	100.089	100.088	100.067	100.079
	D.	.062	.075	.076	.089	.088	.067	.079
Oct. 24	W.	100.066	100.076	100.076	100.083	100.083	100.061	100.073
	D.	.063	.076	.073	.082	.081	.061	.072
"	W.	100.072	100.087	100.072	100.090	100.093	100.065	100.073
	D.	.069	.083	.071	.088	.084	.065	.073
"	W.	100.074	100.083	100.071	100.096	100.093	100.070	100.073
	D.	.071	.084	.075	.093	.093	.070	.074
"	W.	100.080	100.085	100.069	100.094	100.091	100.072	100.080
	D.	.083	.084	.074	.094	.088	.072	.079
Nov. 6	W.	100.071	100.084	100.073	100.088	100.089	100.066	100.072
	D.	.068	.083	.072	.088	.086	.067	.073
"	W.	100.064	100.078	100.072	100.086	100.079	100.061	100.071
	D.	.059	.076	.072	.087	.079	.059	.071
Nov. 10	W.	100.069	100.079	100.073	100.084	100.088	100.066	100.068
	D.	.073	.078	.071	.084	.088	.064	.066
"	W.	100.066	100.082	100.074	100.081	100.089	100.061	100.066
	D.	.066	.081	.073	.081	.085	.063	.065
Mean F. I.		100.0690	100.0796	100.0729	100.0876	100.0863	100.0651	100.0724

TABLE IV.

Fundamental Interval Determinations.

Date	Ob- server	Thermometers No.						
		4331	4332	4334	4335	4336	4423	4524
1906								
Nov. 6	W.	100.071	100.073	100.063	100.067			
	D.	.071	.072	.064	.067			
Nov. 8	W.	100.076	100.084	100.076	100.089	100.083		
	D.	.072	.081	.080	.089	.084		
"	W.	100.069	100.077	100.073	100.091	100.082	100.062	100.073
	D.	.064	.075	.066	.088	.084	.065	.072
Nov. 10	W.	100.063	100.086	100.076	100.089	100.087	100.073	100.076
	D.	.063	.085	.076	.087	.086	.071	.077
"	W.		100.082	100.069	100.095	100.090	100.066	100.075
	D.		.081	.069	.095	.086	.065	.076
Nov. 12	W.		100.076	100.065	100.087	100.089	100.067	100.072
	D.		.073	.063	.088	.085	.065	.072
"	W.		100.080	100.071	100.083	100.083	100.064	100.074
	D.		.076	.071	.081	.081	.062	.075
Nov. 13	W.		100.075	100.070	100.087	100.083	100.065	100.072
	D.		.076	.069	.088	.085	.061	.070
"	W.		100.076	100.071	100.082	100.092	100.076	100.069
	D.		.074	.069	.078	.088	.072	.066
"	W.		100.080	100.071	100.081	100.086	100.065	100.071
	D.		.080	.069	.082	.084	.062	.071
"	W.		100.081	100.075	100.092	100.090	100.078	100.080
	D.		.080	.072	.089	.089	.077	.078
Mean F. I.		100.0686	100.0783	100.0704	100.0670	100.0853	100.0676	100.0733

TABLE V.
Fundamental Interval Determinations.

Date	Ob- server	Thermometers No.						
		1560a	1560a	15555	15593	16016	16017	16018
1906								
Nov. 19	W.	99.915	99.895	100.020	99.997	100.028	100.010	100.000
	D.	.914	.897	.020	100.000	.026	.007	99.999
"	W.	99.908	99.896	100.015	99.995	100.026	100.007	99.997
	D.	.904	.895	.015	100.003	.026	.007	.998
Nov. 20	W.	99.920	99.899	100.022	99.996	100.026	100.002	100.004
	D.	.920	.899	.020	.997	.025	.000	.002
"	W.	99.913	99.893	100.009	99.991	100.021	100.003	99.996
	D.	.909	.890	.008	.991	.018	.003	.996
Nov. 22	W.	99.913	99.893	100.008	99.993	100.022		99.995
	D.	.911	.894	.004	.990	.019		.996
Nov. 23	W.	99.917	99.893	100.020	100.001	100.020		99.998
	D.	.914	.890	.022	99.999	.017		100.000
"	W.	99.913	99.894	100.018	99.999	100.024		100.001
	D.	.911	.894	.016	100.001	.023		99.994
Nov. 26	W.	99.915	99.894	100.019	100.009	100.026		99.995
	D.	.912	.890	.020	.009	.026		.995
Nov. 27	W.	99.912	99.884	100.018	99.992	100.018		99.991
	D.	.910	.885	.021	.991	.019		.990
Nov. 28	W.	99.919	99.895	100.016	99.988	100.020		99.995
	D.	.915	.894	.021	.990	.017		.995
Nov. 30	W.	99.912	99.891	100.017	99.993	100.026		99.993
	D.	.908	.889	.016	.991	.022		.996
"	W.	99.913	99.884	100.016	99.986	100.021		100.002
	D.	.910	.889	.015	.986	.020		99.998
Dec. 1	W.	99.909	99.892	100.013	99.983	100.020		99.994
	D.	.907	.891	.010	.984	.017		.991
Mean F. I.		99.9124	99.8922	100.0161	99.9944	100.0220	100.0049	99.9966

TABLE VI.
Final Résumé of Fundamental Intervals.

Thermometer	Fund. Int. found by authors	Fund. Int. given in certifs. of Bur. Int.	Fund. Int. adopted
Tonnelet No. 4331	100.0688	100.0632	100.0660
“ 4332	100.0789	100.0760	100.0774
“ 4334	100.0716	100.0800	100.0716
“ 4335	100.0873	100.0896	100.0884
“ 4336	100.0858	100.0844	100.0851
“ 4623	100.0664	100.0668	100.0666
“ 4624	100.0729	100.0710	100.0720
“ 11801	99.9986	99.9968	99.9977
Baudin No. 15962	99.9124	99.9071	99.9098
“ 15963	(*)	100.0856	100.0856
“ 15282	99.8922	99.8937	99.8930
“ 15555	100.0161	100.0199	100.0180
“ 15583	99.9944	99.9890	99.9917
“ 16016	100.0220	100.0171	100.0196
“ 16017	100.0049	99.9945	99.9997
“ 16018	99.9966	99.9912	99.9939

* Thermometer No. 15963 broke during the intercomparisons, and before the beginning of the fundamental interval determinations.

Bureau is ± 0.002 . In two instances the differences attain nearly 0.010 , namely, for Tonnelet 4334 and Baudin 16017. On account of the very satisfactory agreement in the two series of determinations with Tonnelet 4334, taken about a year apart, the mean value given by these two series has been adopted as the final value; in all other instances the final value adopted is the mean of the determinations by the authors and the values found at the International Bureau. When the crack appeared in the neighborhood of the auxiliary reservoir of 16017 it seemed that the difference between the values found by the authors and those found at the International Bureau might possibly be explained by a change in the volume of the auxiliary reservoir, due to strains which were evidently present there. A redetermination of the calibration corrections at 50° , however, gave the identical value given in the certificate.

By a correlation of the data contained in the present paper and some subsequent intercomparisons of small groups of these thermometers, it seems highly probable that the fundamental intervals of some of the thermometers at the close of this work are slightly different from the values at the beginning. It would have been better if some of the fundamental interval determinations had been carried out previous to as well as at the conclusion of the intercomparisons. In general an interval of some days between successive determinations is preferable.

Emergent stem and distillation corrections.—The amount by which the readings of thermometers in steam are lowered, due to the emergent stem above the rubber diaphragm, was determined by means of a Mahlke³⁰ "thread thermometer," the bulb of which consisted of a capillary tube 102 mm long, and stem of finer bore. Micrometric measurements gave the mean temperature of the equivalent length of stem (102 mm, which corresponds to from 13° to 18° on the standard thermometers) to about 0°01, which corresponds to only a few ten-thousandths of a degree in the total stem correction. The values of the stem correction found in this way for a thermometer, for which the length of 1° = 7 mm is given in Table VII.

TABLE VII.
Corrections for Emergent Stem.

No. of degrees emergent	S. C. Stem Correction Observed	S. C. Stem Correction Computed	Obs.—Comp.
—0°3	0°0002		
+ .3	.0013	0°0016	—0°0003
.3	.0018	.0016	+ .0002
.7	.0033	.0034	— .0001
.7	.0036	.0034	+ .0002
1.05	.0053	.0052	+ .0001
1.45	.0078	.0075	+ .0003
1 45	.0081	.0075	+ .0006
2.15	.0134	.0125	+ .0009
2.90	.0192	.0190	+ .0002

³⁰ Mahlke, Zs. für Instrumentenkunde, 18, p. 58; 1893.

The measurements are expressed, to a sufficient degree of accuracy, by the equation

$$\text{Stem correction} = 0.0006 + 0.0032n + 0.00108n^2$$

where n = number of degrees emergent above the sheet-rubber diaphragm. For the standard thermometers used the length of 1° ranges from about 6 to 8 mm; with 1° emergent the stem correction for these degree lengths varies only between 0.0050 and 0.0053 . If there is 0.5° emergent, the mean temperature of the last degree is about 85° .

Two independent series of observations, micrometer and estimated, on Baudin No. 15555 in steam, with 0.3° emergent, gave as the rate of distillation of the mercury 0.0006 per minute. Accompanying observations of the ice point gave an identical result. For the observations in steam it generally took from 5 to 6 minutes for a complete series of measurements (two estimates alternately by each observer) in horizontal and vertical positions, with divisions front and back. The total distillation of mercury during this time was therefore about 0.0035 . The ice point determined immediately after was therefore also lowered by this amount, so that the resulting fundamental interval is too high by about 0.0017 . Now the stem correction under these conditions is 0.0016 . It will be seen therefore that the usual procedure in the determination of the fundamental interval, viz, reading the thermometer with about 0.3° emergent, gives the same value for the F. I. as is obtained by avoiding (or correcting for) both the stem correction and the distillation off the end of the thread. There still remains however the advantage in avoiding distillation that the relative changes of the ice and steam points in a series of determinations may be compared.

Zero depression and recovery.—Experiments on the depression of the ice point of thermometers made of French *verre dur*, Jena 16^{III} normal, and Jena 59^{III} borosilicate glasses have been made by several investigators.³¹

³¹ Guillaume: Trav. et Mem. du Bur. Int., 5, p. 49; 1886. Böttcher: Zs. für Instrumentenkunde, 8, p. 409; 1888.

The results of the experiments of Guillaume for *verre dur* in the interval 0° to 100° are represented by the equation—

$$Z_0 - Z_t = 0.000\ 888\ 6t + 0.000\ 001\ 084t^2,$$

where Z_0 and Z_t are the corrected ice-point readings of the thermometer after 0° and after t° , respectively.

Thiesen, Scheel, and Sell³² find for *verre dur*—

$$Z_0 - Z_t = 0.001\ 003\ 6t + 0.000\ 000\ 928t^2$$

A recomputation of their experiments by Scheel, omitting some of the more uncertain observations near 0° led to the equation—

$$Z_0 - Z_t = 0.001\ 199t - 0.000\ 000\ 52t^2$$

In the present investigation the thermometers were placed in ice and kept at approximately 0° for a period of 4 to 8 weeks preceding each series of intercomparisons and fundamental interval determinations.³³ When removed from the ice box care was taken to prevent any appreciable rise of temperature until the ice-point reading was taken; the thermometers were then quickly transferred to the comparator and after the intercomparisons at 10° , 20° , etc., the corresponding ice-point readings were taken. This gave the necessary data for the determination of the relation between the ice-point reading and the temperature to which the thermometer had been exposed. The results obtained in this way are summarized in Table VIII. The values given in the table, corresponding to any temperature, are the mean values for all the thermometers, taken several times, so that they represent the mean of a large number of ice-point observations.

³² Thiesen, Scheel, and Sell: *Zs. für Instrumentenkunde*, **16**, p. 58; 1896. Scheel: *Zs. für Instrumentenkunde*, **17**, Beiblatt No. 13, p 98, 1897.

³³ In some instances an exposure of 4 to 6 weeks in ice, after the thermometers had been heated to 100° , seemed insufficient to bring the glass to an equilibrium condition corresponding to 0° , while at other times under apparently similar conditions a considerably shorter exposure seemed sufficient.

TABLE VIII.

Zero Depressions.

t° Temperature	$Z_0 - Z_t$ Observed Depression	$Z_0 - Z_t$ Comp. Depression	Obs.-Comp.
10°	0°0099	0°0094	+0°0005
20°	.0186	.0191	— .0005
30°	.0295	.0291	+ .0004
40°	.0390	.0393	— .0003
50°	.0498	.0497	+ .0001
60°	.0599	.0605	— .0006
70°	.0723	.0715	+ .0008
80°	.0825	.0827	— .0002
90°	.0944	.0942	+ .0002
100°	.1063	.1060	+ .0003

The above observations satisfy the equation—

$$Z_0 - Z_t = 0.000930t + 0.000001300t^2$$

The ice-point readings correspond to the observed readings taken immediately after each temperature in question and reduced to a time of 3 minutes after the thermometer was removed from the bath.

For the purposes of comparison the depression curves for *verre dur* found by Guillaume, by Thiesen, Scheel, and Sell, and by the authors are given in Fig. 5.

The observations in ice gave 0°0015 as the recovery per minute, at 0°, 3 minutes after the thermometer is removed from the bath, and 0°0011 as the recovery corresponding to 4 minutes. Guillaume has found that the rate of recovery during the first few minutes in ice is practically independent of the temperature to which the thermometer has just previously been exposed. This is in agreement with our observations, as will be seen from the accompanying table obtained from the ice-point determinations of one of the series of intercomparisons.

TABLE IX.

Rate of Zero Recovery.

Temperature	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°	Mean
Rate of recovery after 3 minutes.....	.0004	.0016	.0013	.0018	.0018	.0015	.0011	.0012	.0018	.0018	0.0015

After 10° the rate of recovery is the same to within the limits of error of experiments of this kind.

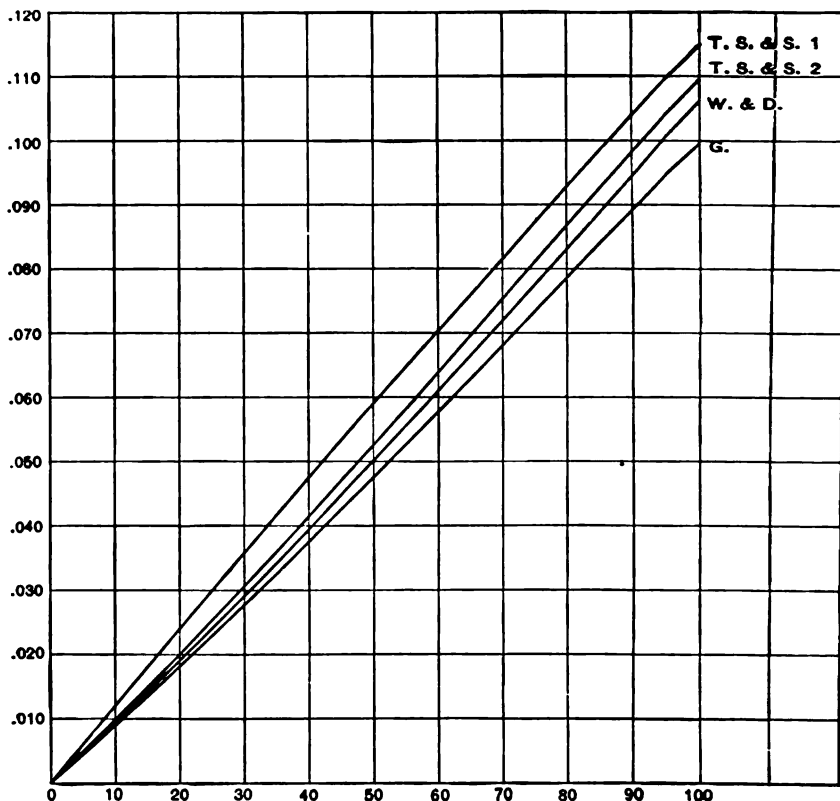


Fig. 5.—Depression of Zero for Verre Dur.

IV. INTERCOMPARISON OF THERMOMETERS.

The intercomparisons of the sixteen primary standard thermometers available for this investigation were carried out in a specially designed *Comparator* which has admirably fulfilled every require-

ment of this work, as well as that of testing the large number of laboratory and special thermometers that are submitted to the Bureau each year for comparison with its standards.

Comparator.—The comparator, shown diagrammatically in section in Fig. 6, and by photographic reproduction in Fig. 7, is a triple wall water bath, which can be used in either horizontal or vertical position, with water coil for rapid heating and cooling and electric heating coil for precise temperature regulation, and with a rapid circulation and thorough stirring of the water produced by an electrically driven propeller. The upper part of the tank is made of three concentric brass tubes and contains within the inner one a holder with places for fourteen thermometers, as ordinarily used in testing, two primary standards and twelve thermometers to be compared. Between the outer two of these tubes is a packing of hair felt, and between the inner two the downward circulation of water which comes upward around the thermometers in the inner space. Plate glass windows, W, W, 60 cm long and about 8 cm wide, are let into opposite sides to permit of reading the thermometers. These windows, being opposite each other, afford excellent direct illumination, especially when an incandescent lamp, with diffusing screen interposed, is placed in front of the window away from the observer.

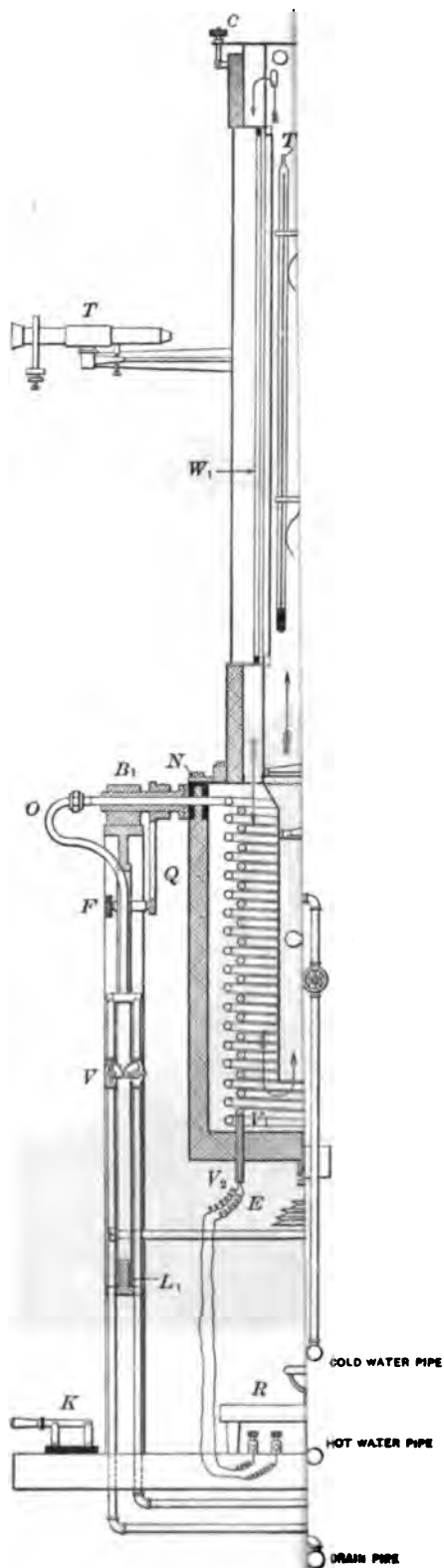
The thermometers are read by means of the microscope, T, moved by rack and pinion, on a vertical support, in front of the window. The thermometer holder, H, may be conveniently rotated to bring the thermometers successively into the field of the microscope by means of a belt running from the pulley, P, in the water-tight cover, over idlers, down to a hand wheel, placed on the side of the tank, which is operated by the right hand of the observer, while he grasps the hand wheel of the telescope rack and pinion movement with his left hand, and thus rapidly raises or lowers the telescope to bring the mercury meniscus of the thermometer into the center of the field.

The construction of the holder, H, will be understood from Figs. 6 and 7. The thermometers are pressed outward against V-shaped rests and held in position by the phosphor bronze springs, s, s. The holder, partly loaded with thermometers, is shown alongside the comparator.

The lower portion of the comparator consists of an inner brass chamber covered with a heavy layer of felt and a removable outer

brass casing. In the inner chamber are contained the means for securing temperature regulation and circulation of water. First, there is a three-eighths inch copper coil, W, making sixteen turns, with inlet, I, and outlet, O, through the shaft on which the comparator is supported. Through this coil either hot or cold water may be circulated. For temperatures below that of the room, water from an ice-water tank is used; at higher than room temperatures hot water is used only for rapid heating, and for exact regulation the electric heating coil, E, furnishes the energy. This coil, which is of a type made by the Simplex Electric Heater Company for the trade, was found to fulfill the requirements of this apparatus very satisfactorily. It consists of a three-eighths inch brass tube, in which are inserted two asbestos covered coils, combinations of which give three rates of heating for any E. M. F., and which, in conjunction with the rheostat, R, permit regulation from a few watts to 800 watts. Stirring is effected by means of a propeller, S, driven by the 75-watt motor, M₁, belted to a wheel, P₂, on the shaft of the comparator, from which another belt runs over idlers, P₃, to the pulley, P₁, on the propeller shaft, which runs through the stuffing box, G. The object of this arrangement was to permit of one of the most convenient features of the comparator, i. e., its adaptation to horizontal as well as vertical comparison. By drawing the spring key, F, tipping the comparator to a horizontal position and releasing the key, the comparison of a set of thermometers may be continued in horizontal position without any change whatever in the working of any part of the comparator.

At temperatures below that of the room it is important, in order to secure satisfactory temperature regulation, that the cold water be delivered at the inlet, I, at a constant temperature, which would not be the case if there were a standpipe connecting the comparator to the cold-water reservoir. To secure this condition a constant stream of cold water is kept circulating to the comparator through the pipe marked Ice Water Pipe, past the inlet I, and back to the cold-water reservoir (or to drain) through the pipe marked Return Pipe, the amount of resistance to the flow of water in this circuit, and therefore the effective pressure available at the inlet I being controlled by the valve V₄. This circulation is maintained by means of a small centrifugal pump, C, driven by a 75-watt motor, M₂. From



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Fig. 7.—*Thermometer Comparator.*

this cold-water circulation the requisite amount of water to maintain the desired temperature is admitted through a sensitive stopcock, I, into the copper coil W, after passing through which it may either be returned to the cold-water reservoir or passed to the sewer by turning the geared valves V.

Hot water, drawn from the hot-water distribution system in the laboratory, for rapidly raising the temperature from one test point to another, is admitted into the coil W through the valve V_s . A constant circulation of the hot water, and thus a steady temperature condition, is provided for by the valve V_s .

With this comparator and equipment and with reasonable constancy in the temperature of the room, any temperature within the range 5° to 90° may be quickly attained at the rate of 2° to 4° per minute, and may be held constant to within a few thousandths of a degree for many minutes or made to vary at any desired rate.

Method of Intercomparison.—The intercomparisons were made at every 10° , in the interval 0° to 100° , in nine series, with the thermometers mounted in the holder of the comparator in the order indicated in Table X.

Before each series of intercomparisons the thermometers were kept in ice for a period of 4 to 8 weeks.

TABLE X.

Series of Intercomparisons.

Series No.	Thermometers Intercompared							Range
I	4331	4332	4334	4335	4336	4623	4624	0° – 50°
II	11801	15962	15963	15282	15555	15583		0° – 50°
III	4331	4332	4334	4335	4336	4623	4624	0° – 50°
IV	11801	15962	15963	15282	15555	15583		0° – 50°
V	11801	15962	15963	4331	4332	4623		0° – 50°
VI	11801	15962	15963	15282	16016	16017	16018	50° – 100°
VII	11801	15962	15555	4331	4332	4623		0° – 50°
VIII	11801	15962	15282	16016	16017	16018		50° – 100°
IX	15962	15282	16016	16017	16018			50° – 100°

In each series six or seven thermometers were intercompared at one time. This was done for several reasons, not alone on account of the large number of thermometers involved which would have greatly extended the time required for intercomparisons in groups of two or three thermometers, but also because of the important advantage that in reading this number of thermometers the observer does not keep in mind the last reading of any particular thermometer and is thus free from the unconscious but ever present influence of his last reading. The advantages of working with only two or three thermometers, viz, that the effect of irregularities in the rate of rise of temperature of the bath is of less importance and that the meniscus of the thermometers can more easily be prevented from disappearing behind a graduation mark in the course of a set of readings, are much minimized by the construction of the comparator. First, because of the extremely sensitive temperature control made possible by the electric heating and the constant and thorough stirring, a very uniform rise of temperature is readily obtainable, and if the thermometers are read at equal time intervals forward and backward, the first source of error is extremely small; and second, if in the course of a set of readings the meniscus of one or more of the thermometers should come behind a graduation mark the observer may turn the thermometer to the right or left, so that the meniscus may be viewed off the edge of the graduation or the telescope can be raised or lowered and the meniscus viewed from above and from below the graduation mark, when it is easy to estimate from the observed motion of the telescope whether the meniscus is above or below the center of the graduation. As the thermometers were turned in succession into the field of view of the telescope they occupied identical positions in the comparator while being read, so that the effect of any small constant difference of temperature in different parts of the section of the tank in which the bulbs are found does not enter.³⁴ It was therefore deemed permissible to mount the thermometers in the holder in the same serial order in the several series where the same thermometers were intercompared, as this was a great convenience in comparing two series and in checking the corrections that had been taken from the tables.

³⁴Some experiments made during the work showed that the maximum differences in temperature between the top and bottom of the space inclosing the thermometers was less than 0.001.

The comparator was so adjusted that the thermometers were held in a vertical position. The line of collimation of the reading telescope on the comparator was maintained horizontal with the aid of a striding level. To eliminate any outstanding errors of parallax the thermometers were always read with divisions before, then with divisions back, and again with divisions before the mercury thread. The curvature produced in the thermometers by the slight spring pressure of the holder³⁵ was entirely too small to give rise to any appreciable error.

The supply of electrical energy, or of cold water, to the comparator was carefully adjusted before every set of observations to give a slowly rising temperature, as readings with falling meniscus are liable to very considerable error. Indeed in those sets of observations, where the temperature was apparently constant or the rise was very small, the irregularities were nearly always greater than for a more rapid rise. The rate of rise used at different times varied from 0.005 to 0.009 per minute.

One of the most serious sources of error in mercurial thermometers, and which more than any other factor limits the precision attainable (now that glasses with small zero depression are available, and that methods of using thermometers have been found which almost completely eliminate the effects of thermal hysteresis) arises from the irregular variations of the capillary forces of the mercury meniscus, which produce a variable internal pressure, equivalent to some centimeters of mercury, depending on the form of the meniscus, the angle of contact, and the varying dimensions of and condition of surface of the capillary tube. These variable forces, which may give rise to irregularities amounting to many thousandths of a degree, manifest themselves in the calibration of thermometers, where it is frequently observed that the differential pressure exerted by the two menisci of the calibrating thread is sufficient to displace or hold in equilibrium considerable lengths of thread. Pernet³⁶ has shown that for a thermometer of very fine bore the difference between the indications for rising and falling temperatures may

³⁵ A holder has since been constructed in which the pressure of the springs against the thermometer is directly in line with the V-shaped supports against which the thermometer is held.

³⁶ Pernet: *Zs. für Instrumentenkunde*, 6, p. 382; 1886.

attain 0.07°C . It is frequently observed that one thermometer may indicate an absolutely constant temperature while those on either side of it, in the same bath, show a very decided rise, amounting to 0.01 or more. Micrometric measurements of the position of the meniscus are, therefore, not to be recommended, as the order of precision of estimation for a trained observer considerably exceeds the accuracy to which a single indication of the thermometers can be relied upon. As has been previously stated, a large number of micrometric and estimated readings has shown that the two observers rarely differed by as much as 0.003 . It was, therefore, deemed best to dispense with time-consuming micrometric measurements, and to obtain a larger number of estimated readings, to better eliminate the irregularities in the behavior of the thermometers.

The thermometers were read at intervals of 10 seconds forward and backward, until each had been read six times, when a similar set of observations were made by the other observer, beginning at the other end of the set. The thermometers were then turned in the holder so that the divisions were back of the mercury thread, when two similar sets of six readings each were made alternately by each observer, after which the thermometers were again turned to their former position with divisions before the mercury thread, and another set of readings taken by each observer. At every intercomparison temperature each thermometer was therefore read 24 times by each observer.

Observations in ice.—Before each series of intercomparisons the thermometers were kept at 0° for a period of 4 to 8 weeks, and the ice-point reading corresponding to long-continued exposure at 0° (Z_0) was determined.

Immediately after completion of the intercomparisons at any temperature, the zero readings corresponding to that temperature were determined in melting ice. In these determinations the thermometer was quickly transferred from the comparator to the supercooler (see p. 687) and then to the ice-point apparatus. Two independent estimates were made by each observer. The mean time of reading was generally very near to 3 minutes after removal from the comparison bath. The zero was always reduced to that corresponding to 3 minutes after removal from the bath, by making proper allowance for the zero recovery as previously described.

After completion of the observations in ice the thermometer was returned to the comparison bath, the temperature of which all the while had been maintained constant. After all the thermometers had been observed in ice the series of ice-point observations was repeated. In all comparisons below room temperature care was taken that the thermometer was not heated appreciably above the temperature of the comparison bath in the process of transference from the bath to the supercooler. After the completion of the series of intercomparisons (at the highest temperature) the thermometers were exposed to a temperature of 100° (steam) for a period of about 20 minutes and the ice-point readings corresponding to 100° (Z_{100}) determined. All the precautions previously described were taken to insure the purity of the ice used in these determinations.

The observed ice-point readings were then corrected for calibration, external pressure,³⁷ internal pressure, and fundamental interval. These corrected ice-point readings,

$$Z_0 \quad Z_{10} \quad Z_{20} \quad \quad Z_{100}$$

corresponding 0° , 10° , 20° , etc., were then plotted, which gave a complete *depression curve* for that series of intercomparisons. From this depression curve the *mean* Z_0 was computed and thus the reduced zeros corresponding to 10° , 20° , 30° , etc. It was found that the ice-point readings reduced in this way gave more concordant results than the use of the ice point determined only once after the particular temperature in question. The ice point is one of the most difficult points to determine. It is possible to multiply readings indefinitely at any temperature in the comparison bath, but the accuracy gained in this way is lost unless the ice point is determined with equal accuracy. We have, therefore, adopted the practice, even if a single temperature only is to be accurately measured, of determining the ice point at a number of temperatures and deducing the *mean zero* corresponding to that temperature from the observed depression curve. The *mean corrected ice-point* reading correspond-

³⁷ Including pressure of column of water of height equal to the distance from center of bulb to the zero mark as well as excess (or deficiency) of reduced barometric pressure over (or below) standard atmospheric pressure (760 mm).

ing to each temperature, reduced in the way indicated above, was then applied, with its sign changed, as the *zero correction* to the observed reading of the thermometer.

Sample intercomparison.—The laboratory record of the observations and reductions of an intercomparison of six thermometers at one temperature (80°) is reproduced, as an illustration, in Tables XI and XII, the asterisk (*) indicating the first thermometer read in each group of observations. The mean of the 24 observed readings of each thermometer is corrected for calibration, external²⁰ and internal pressure, zero, and fundamental interval. The temperatures thus found are temperatures on the *verre dur scales* defined by the several thermometers. In the last line of each table are given the *supercorrections* resulting from the intercomparisons, i. e., the correction that must be applied to the scale defined by each thermometer at the given temperature to reduce to temperature on the mean scale defined by all of the thermometers included in the intercomparison.

²⁰ Including pressure of column of water in the comparator above the center of the bulb of thermometer as well as the excess (or deficiency) of the reduced barometric pressure above (or below) standard atmospheric pressure.

TABLE XI.
Intercomparison Record.

Observer, C. W. W. Computations by C. W. W. Checked by H. C. D.

Position of Therm.	Tonnelet No. 11801	Baudin No. 15962	Baudin No. 15282	Baudin No. 16016	Baudin No. 16017	Baudin No. 16018
Divs. Before	79.806	79.886	79.899	79.973	79.952	80.051*
	.809	.893	.904	.983	.962	.067
	.828	.900	.916	.996	.970	.067
	.832	.901	.928	80.000	.985	.085
	.845	.913	.941	.005	.992	.087
	.848	.922	.946	.015	80.000	.100
	79.8280	79.9025	79.9223	79.9953	79.9968	80.0762
Divs. Back	*79.850	79.922	79.950	80.023	80.003	80.102
	.873	.942	.958	.035	.007	.102
	.872	.945	.962	.040	.020	.118
	.892	.951	.974	.046	.026	.118
	.893	.953	.986	.051	.040	.134
	.900	.969	.994	.056	.042	.136
	79.8800	79.9470	79.9707	80.0418	80.0230	80.1183
Divs. Before	79.798	79.860	79.895	79.959	79.946	80.042*
	.798	.867	.896	.967	.949	.045
	.803	.886	.901	.969	.950	.046
	.805	.884	.902	.978	.951	.051
	.811	.888	.909	.983	.956	.053
	.813	.894	.911	.992	.962	.061
	79.8047	79.8798	79.9023	79.9747	79.9523	80.0497
Divs. Before	*79.805	79.890	79.905	79.988	79.963	80.065
	.830	.900	.918	.997	.974	.068
	.835	.902	.937	80.002	.993	.093
	.852	.929	.946	.015	.996	.095
	.852	.929	.952	.028	80.007	.107
	.880	.950	.962	.044	.011	.108
	79.8423	79.9167	79.9367	80.0123	79.9907	80.0893
Mean Obs. Rds . .	79.8388	79.9115	79.9330	80.0060	79.9857	80.0834
Calibration Cor. . .	+.0746	-.0219	-.0750	+.0160	-.0159	-.0683
Ext. Pres. Cor. . . .	-.0052	-.0056	-.0069	-.0054	-.0057	-.0055
Int. Pres. Cor. . . .	+.0691	+.0737	+.0918	+.0467	+.0486	+.0468
Zero Cor.	+.0228	-.0335	-.0256	-.0505	-.0527	-.0657
Cor. to F. I.	+.0011	+.0721	+.0856	-.0157	+.0002	+.0048
Temp. Verre dur Scale.	80.0012	79.9963	80.0029	79.9970	79.9920	79.9955
Cor. to Mean Scale.	-.0037	+.0012	-.0054	+.0005	+.0055	+.0020

TABLE XII.

Intercomparison Record.

Observer H. C. D. Computations by H. C. D. Checked by C. W. W.

Position of Therm.	Tonnelot No. 11801	Baudin No. 15962	Baudin No. 15282	Baudin No. 16016	Baudin No. 16017	Baudin No. 16018
Divs. Before	*79.752	79.824	79.848	79.923	79.902	80.003
	.752	.840	.856	.928	.904	.003
	.763	.843	.860	.944	.918	.020
	.780	.853	.871	.946	.921	.020
	.783	.855	.880	.956	.939	.040
	.799	.868	.895	.957	.940	.041
	79.7713	79.8472	79.8683	79.9423	79.9207	80.0212
	79.806	79.874	79.900	79.970	79.951	80.048*
	.806	.875	.900	.972	.954	.050
	.813	.894	.909	.980	.956	.051
Divs. Back	.815	.895	.910	.986	.964	.059
	.833	.900	.922	.994	.966	.061
	.834	.903	.927	80.000	.978	.074
	79.8195	79.8902	79.9113	79.9837	79.9615	80.0572
	*79.770	79.846	79.862	79.938	79.916	80.014
	.777	.850	.868	.944	.920	.016
	.778	.850	.873	.948	.924	.022
	.788	.856	.881	.955	.930	.025
	.788	.856	.883	.953	.931	.033
	.795	.858	.885	.957	.936	.030
Divs. Before	79.7823	79.8527	79.8753	79.9492	79.9262	80.0233
	79.754	79.823	79.850	79.916	79.898	80.000*
	.754	.826	.852	.923	.904	.003
	.763	.838	.855	.927	.905	.004
	.764	.844	.860	.942	.921	.024
	.783	.852	.872	.946	.925	.023
	.784	.855	.878	.952	.942	.042
	79.7670	79.8397	79.8612	79.9343	79.9158	80.0160
	79.7850	79.8575	79.8790	79.9524	79.9311	80.0294
	Calibration Cor. . . .	+ .0752	— .0215	— .0750	+ .0160	+ .0159
Ext. Pres. Cor.	— .0052	— .0056	— .0069	— .0054	— .0057	— .0055
Int. Pres. Cor.	+ .0691	+ .0737	+ .0917	+ .0467	+ .0486	+ .0467
Zero Cor.	+ .0228	— .0335	— .0256	— .0505	— .0527	— .0657
Cor. to F. I.	+ .0011	+ .0721	+ .0856	— .0157	— .0002	+ .0048
Temp. Verre dur Scale	79.9480	79.9427	79.9488	79.9435	79.9374	79.9414
Cor. to Mean Scale.	— .0044	+ .0009	— .0052	+ .0001	+ .0062	+ .0022

Results of intercomparisons.—The results of the nine series of intercomparisons and the additional intercomparison of the seven Tonnelot thermometers at 20° are given in Tables XIII–XVII.

The temperatures there given are the final reduced temperatures on the *verre dur* scale of each thermometer. Each temperature was found in the manner previously explained and illustrated in Tables XI and XII. The two nearly equal temperatures recorded for each intercomparison point are the results of independent intercomparisons by each observer. The supercorrections to the mean scale of the thermometers included in each separate intercomparison are given below each temperature in smaller print.

TABLE XIII.

Intercomparisons—Series I and III.

Tunnelot No. 4331	Tunnelot No. 4332	Tunnelot No. 4334	Tunnelot No. 4335	Tunnelot No. 4336	Tunnelot No. 4623	Tunnelot No. 4624
10.0768 +.0041	10.0801 +.0008	10.0850 -.0041	10.0776 +.0033	10.0817 -.0008	10.0850 -.0041	10.0804 +.0005
10.0936 +.0003	10.0897 +.0042	10.0987 -.0048	10.0905 +.0034	10.0928 +.0011	10.0945 -.0006	10.0972 -.0033
20.0586 +.0037	20.0633 -.0010	20.0677 -.0054	20.0640 -.0017	20.0605 +.0018	20.0628 -.0015	20.0592 +.0031
20.1025 +.0041	20.1066 .0000	20.1110 -.0044	20.1093 -.0027	20.1051 +.0015	20.1078 -.0012	20.1038 +.0028
30.0405 -.0005	30.0371 +.0029	30.0432 -.0032	30.0396 +.0004	30.0389 +.0011	30.0433 -.0033	30.0375 +.0025
30.0444 -.0006	30.0434 +.0004	30.0480 -.0042	30.0437 +.0001	30.0400 +.0038	30.0461 -.0023	30.0409 +.0029
40.0692 +.0007	40.0684 +.0015	40.0731 -.0032	40.0704 -.0005	40.0703 -.0004	40.0708 -.0009	40.0673 +.0026
40.1036 +.0011	40.1029 +.0018	40.1086 -.0039	40.1050 -.0003	40.1049 -.0002	40.1084 -.0037	40.0994 +.0053
50.0548 +.0005	50.0553 .0000	50.0553 .0000	50.0531 +.0022	50.0557 -.0004	50.0593 -.0040	50.0537 +.0016
50.1049 +.0008	50.1058 -.0001	50.1043 +.0014	50.1038 +.0019	50.1071 -.0014	50.1106 -.0049	50.1035 +.0022
10.1201 +.0004	10.1173 +.0032	10.1264 -.0059	10.1173 +.0032	10.1194 +.0011	10.1189 +.0016	10.1238 -.0033
10.1756 -.0017	10.1709 +.0030	10.1781 -.0042	10.1703 +.0036	10.1733 +.0006	10.1730 +.0009	10.1761 -.0022
20.0634 -.0057	20.0541 +.0036	20.0594 -.0017	20.0575 +.0002	20.0520 +.0057	20.0602 -.0025	20.0574 +.0003
20.0900 -.0034	20.0861 +.0005	20.0881 -.0015	20.0862 +.0004	20.0808 +.0058	20.0896 -.0030	20.0852 +.0014
30.0544 -.0025	30.0485 +.0034	30.0553 -.0034	30.0508 +.0011	30.0474 +.0045	30.0554 -.0035	30.0512 +.0007
30.1268 -.0046	30.1198 +.0024	30.1256 -.0034	30.1212 +.0010	30.1155 +.0067	30.1254 -.0032	30.1213 +.0009
40.1406 -.0002	40.1400 +.0004	40.1429 -.0025	40.1404 .0000	40.1349 +.0055	40.1454 -.0050	40.1386 +.0018
40.0913 -.0013	40.0909 -.0009	40.0929 -.0029	40.0896 +.0004	40.0854 +.0046	40.0921 -.0021	40.0877 +.0023
50.0783 +.0011	50.0796 -.0002	50.0777 +.0017	50.0820 -.0026	50.0798 -.0004	50.0828 -.0034	50.0754 +.0040
50.0379 +.0020	50.0419 -.0020	50.0413 -.0014	50.0412 -.0013	50.0385 +.0014	50.0422 -.0023	50.0365 +.0034

TABLE XIV.

Intercomparisons—Series II and IV.

Tonnelet No. 11801	Baudin No. 15962	Baudin No. 15963	Baudin No. 15982	Baudin No. 15555	Baudin No. 15583
9.8199 - .0062	9.8082 + .0055	9.8133 + .0004	9.8134 + .0003	9.8102 + .0035	9.8171 - .0034
9.9072 - .0061	9.8964 + .0047	9.9003 + .0008	9.9002 + .0009	9.8967 + .0044	9.9057 - .0046
19.9965 - .0097	19.9794 + .0074	19.9854 + .0014	19.9872 - .0004	19.9799 + .0069	19.9926 - .0058
20.0641 - .0100	20.0473 + .0068	20.0534 + .0007	20.0545 - .0004	20.0470 + .0071	20.0583 - .0042
29.9679 - .0088	29.9560 + .0031	29.9566 + .0025	29.9575 + .0016	29.9522 + .0069	29.9642 - .0051
30.0183 - .0099	30.0049 + .0035	30.0059 + .0025	30.0081 + .0003	30.0024 + .0060	30.0111 - .0027
39.9317 - .0095	39.9153 + .0069	39.9171 + .0051	39.9258 - .0036	39.9158 + .0064	39.9276 - .0054
39.9499 - .0084	39.9376 + .0039	39.9374 + .0041	39.9454 - .0039	39.9335 + .0080	39.9454 - .0039
49.9921 - .0076	49.9770 + .0075	49.9803 + .0042	49.9897 - .0052	49.9809 + .0036	49.9872 - .0027
50.0247 - .0075	50.0129 + .0043	50.0110 + .0062	50.0200 - .0028	50.0132 + .0040	50.0214 - .0042
9.9038 - .0057	9.8934 + .0047	9.8943 + .0038	9.8980 + .0001	9.8957 + .0024	9.9034 - .0053
9.9587 - .0042	9.9514 + .0031	9.9519 + .0026	9.9534 + .0011	9.9518 + .0027	9.9600 - .0055
20.0367 - .0074	20.0253 + .0040	20.0267 + .0026	20.0277 + .0016	20.0234 + .0059	20.0361 - .0068
20.0470 - .0036	20.0384 + .0050	20.0406 + .0028	20.0441 - .0007	20.0389 + .0045	20.0514 - .0080
29.9452 - .0064	29.9327 + .0061	29.9365 + .0023	29.9361 + .0027	29.9346 + .0042	29.9478 - .0090
29.9743 - .0051	29.9645 + .0047	29.9688 + .0004	29.9664 + .0018	29.9640 + .0052	29.9774 - .0082
40.0185 - .0062	40.0057 + .0066	40.0158 - .0035	40.0105 + .0018	40.0031 + .0092	40.0201 - .0078
40.0896 - .0063	40.0769 + .0064	40.0868 - .0035	40.0808 + .0025	40.0760 + .0073	40.0899 - .0066
49.9764 - .0011	49.9683 + .0070	49.9847 - .0094	49.9770 - .0017	49.9662 + .0091	49.9794 - .0041
50.0224 - .0002	50.0155 + .0067	50.0289 - .0067	50.0232 - .0010	50.0175 + .0047	50.0258 - .0036

TABLE XV.

Intercomparisons—Series V and VII.

Tonnelet No. 11801	Baudin No. 15968	Baudin No. 15963	Tonnelet No. 4331	Tonnelet No. 4332	Tonnelet No. 4623
9.9845 -.0024	9.9756 +.0065	9.9818 +.0003	9.9819 +.0002	9.9818 +.0003	9.9870 -.0049
10.0388 -.0015	10.0295 +.0078	10.0384 -.0011	10.0414 -.0041	10.0351 +.0022	10.0405 -.0032
19.9396 +.0020	19.9319 +.0097	19.9419 -.0003	19.9442 -.0026	19.9447 -.0031	19.9473 -.0057
20.0297 +.0013	20.0215 +.0095	20.0314 -.0004	20.0340 -.0030	20.0329 -.0019	20.0363 -.0053
29.9136 -.0014	29.9068 +.0054	29.9159 -.0037	29.9120 +.0002	29.9110 +.0012	29.9138 -.0016
29.9425 +.0029	29.9369 +.0085	29.9513 -.0059	29.9484 -.0030	29.9450 +.0004	29.9485 -.0031
39.9970 -.0021	39.9879 +.0070	39.9974 -.0025	39.9951 -.0002	39.9971 -.0022	39.9950 -.0001
40.0584 -.0004	40.0515 +.0065	40.0600 -.0020	40.0598 -.0018	40.0582 -.0002	40.0604 -.0024
50.0591 -.0009	50.0511 +.0071	50.0602 -.0020	50.0577 +.0005	50.0616 -.0034	50.0594 -.0012
50.0223 -.0024	50.0123 +.0076	50.0223 -.0024	50.0183 +.0016	50.0234 -.0035	50.0207 -.0008
10.0378 -.0054	10.0361 -.0037	10.0268 +.0056	10.0307 -.0017	10.0288 +.0036	10.0340 -.0016
9.9832 -.0064	9.9798 -.0030	9.9713 +.0055	9.9727 +.0041	9.9745 +.0023	9.9792 -.0024
19.9984 -.0023	19.9941 +.0020	19.9892 +.0069	19.9965 -.0004	19.9996 -.0035	19.9987 -.0026
19.9482 -.0032	19.9422 +.0028	19.9379 +.0071	19.9468 -.0018	19.9469 -.0019	19.9482 -.0032
29.9946 -.0016	29.9873 +.0057	29.9884 +.0046	29.9974 -.0044	29.9936 -.0006	29.9968 -.0038
30.0331 -.0006	30.0259 +.0066	30.0287 +.0038	30.0361 -.0036	30.0356 -.0031	30.0355 -.0030
40.0550 -.0026	40.0427 +.0097	40.0468 +.0058	40.0564 -.0040	40.0567 -.0043	40.0568 -.0044
40.0058 +.0003	39.9965 +.0096	40.0016 +.0045	40.0111 -.0050	40.0107 -.0046	40.0110 -.0049
50.0954 +.0009	50.0801 +.0162	50.0919 +.0044	50.1037 -.0074	50.1041 -.0078	50.1024 -.0061
50.0489 -.0009	50.0362 +.0118	50.0451 +.0029	50.0522 -.0042	50.0531 -.0051	50.0523 -.0043

TABLE XVI.
Intercomparisons—Series VI and VIII.

Tonnelet No. 11801	Baudin No. 1596a	Baudin No. 15963	Baudin No. 1528a	Baudin No. 16016	Baudin No. 16017	Baudin No. 16018
49.9902 - .0035	49.9827 + .0040	49.9943 - .0076	49.9888 - .0021	49.9818 + .0049	49.9865 + .0002	49.9829 + .0038
50.0424 - .0059	50.0321 + .0044	50.0431 - .0066	50.0381 - .0016	50.0305 + .0060	50.0369 - .0004	50.0324 + .0041
60.0058 - .0079	59.9954 + .0025	60.0060 - .0081	60.0008 - .0029	59.9899 + .0080	59.9947 + .0032	59.9927 + .0052
60.0500 - .0063	60.0405 + .0032	60.0517 - .0080	60.0487 - .0050	60.0353 + .0084	60.0401 + .0036	60.0399 + .0038
70.0037 - .0024	69.9973 + .0040	70.0087 - .0074	70.0069 - .0056	69.9953 + .0060	69.9996 + .0017	69.9978 + .0035
70.0096 - .0029	70.0026 + .0041	70.0142 - .0075	70.0139 - .0072	70.0013 + .0054	70.0031 + .0036	70.0021 + .0046
79.9123 - .0033	79.9082 + .0008	79.9151 - .0061	79.9132 - .0042	79.9050 + .0040	79.9028 + .0062	79.9063 + .0027
79.9657 - .0033	79.9622 + .0002	79.9673 - .0049	79.9658 - .0034	79.9610 + .0014	79.9563 + .0061	79.9583 + .0041
90.0734 - .0024	90.0722 - .0012	Broken.	90.0730 - .0020	90.0725 - .0015	90.0673 + .0037	90.0674 + .0036
90.1257 - .0016	90.1244 - .0003		90.1281 - .0040	90.1249 - .0008	90.1207 + .0034	90.1207 + .0034

Tonnelet No. 11801	Baudin No. 1596a	Baudin No. 1528a	Baudin No. 16016	Baudin No. 16017	Baudin No. 16018
50.1124 - .0086	50.1012 + .0026	50.1053 - .0015	50.1010 + .0028	50.1012 + .0026	50.1016 + .0022
50.0669 - .0074	50.0579 + .0016	50.0612 - .0017	50.0559 + .0036	50.0573 + .0022	50.0577 + .0018
60.0566 - .0058	60.0461 + .0047	60.0527 - .0019	60.0487 + .0021	60.0493 + .0015	60.0516 - .0008
59.9994 - .0052	59.9906 + .0036	59.9974 - .0032	59.9914 + .0028	59.9919 + .0023	59.9947 - .0005
70.0555 - .0046	70.0444 + .0065	70.0575 - .0066	70.0492 + .0017	70.0477 + .0032	70.0509 + .0000
70.0227 - .0033	70.0149 + .0045	70.0262 - .0068	70.0174 + .0020	70.0158 + .0036	70.0191 + .0003
80.0012 - .0037	79.9963 + .0012	80.0030 - .0055	79.9971 + .0004	79.9920 + .0055	79.9955 + .0020
79.9480 - .0044	79.9427 + .0009	79.9488 - .0052	79.9435 + .0001	79.9374 + .0062	79.9414 + .0022
90.1057 - .0007	90.1066 - .0016	90.1080 - .0030	90.1066 - .0016	90.1012 + .0038	90.1022 + .0028
90.0795 - .0023	90.0769 + .0003	90.0818 - .0046	90.0784 - .0012	90.0726 + .0046	90.0743 + .0029

TABLE XVII.

Intercomparisons—Series IX and 20°.

Tonnelet No. 11801.	Baudin No. 1596a	Baudin No. 1528a	Baudin No. 16016	Baudin No. 16017	Baudin No. 16018	
	70.0044 +.0018	70.0095 -.0033	70.0075 -.0013	70.0060 +.0002	70.0035 +.0027	
	69.9732 -.0009	69.9756 -.0033	69.9715 +.0008	69.9715 +.0008	69.9695 +.0028	
79.9151 -.0006	79.9108 -.0037	79.9197 -.0052	79.9153 -.0008	79.9122 +.0023	79.9139 +.0006	
79.9896 -.0013	79.9826 +.0057	79.9936 -.0053	79.9901 -.0018	79.9858 +.0025	79.9882 +.0001	
	80.0315 -.0014	80.0302 -.0001	80.0308 -.0007	80.0293 +.0008	80.0285 +.0016	
	79.9697 -.0015	79.9686 -.0004	79.9691 -.0009	79.9669 +.0013	79.9667 +.0015	
	90.0660 +.0009	90.0652 +.0017	90.0694 -.0025	90.0687 -.0018	90.0652 +.0017	
	90.0078 +.0007	90.0070 +.0015	90.0114 -.0029	90.0095 -.0010	90.0067 +.0018	
Tonnelet No. 4331	Tonnelet No. 4332	Tonnelet No. 4334	Tonnelet No. 4335	Tonnelet No. 4336	Tonnelet No. 46a3	Tonnelet No. 46a4
19.9706 +.0023	19.9723 +.0006	19.9765 -.0036	19.9704 +.0025	19.9764 -.0035	19.9790 -.0061	19.9648 +.0081
20.0407 +.0026	20.0421 +.0012	20.0484 -.0051	20.0410 +.0023	20.0467 -.0034	20.0478 -.0045	20.0364 +.0069

By applying to the results of these separate intercomparisons of various groups of thermometers, methods of computation analogous to those employed in an incomplete thermometer calibration, the *final supercorrections to the thermometers to reduce their corrected²⁹ readings to the mean scale of all the thermometers* included in the intercomparisons at the given temperature were found as given in Table XVIII.

²⁹ The reading found by applying to the mean of the observed readings corrections for calibration, external pressure, internal pressure, zero, and fundamental interval (the observed ice point reading itself being corrected in a similar way and the resulting corrected ice point reading being applied, with its sign changed, as the zero correction to the observed reading of the thermometer).

TABLE XVIII.

Supercorrections to Thermometers.

Unit = 0.001.

	Tonnelet No. 4331	Tonnelet No. 4333	Tonnelet No. 4334	Tonnelet No. 4335	Tonnelet No. 4336	Tonnelet No. 4623	Tonnelet No. 4624	Tonnelet No. 11801
10°	+1	+2	-5	+3	0	-2	-2	-4
20°	-1	-1	-5	-2	0	-4	+2	-2
30°	-2	+1	-4	0	+3	-3	+1	-2
40°	-2	-1	-4	-1	+1	-3	+2	-2
50°	-3	-5	-3	-3	-4	-5	-1	-2
60°								-6
70°								-3
80°								-3
90°								-2

	Baudin No. 15962	Baudin No. 15963	Baudin No. 15962	Baudin No. 15555	Baudin No. 15583	Baudin No. 16016	Baudin No. 16017	Baudin No. 16018
10°	+3	+2	+1	+5	-4			
20°	+7	+2	+2	+7	-4			
30°	+6	-2	+2	+6	-6			
40°	+8	-1	+1	+8	-4			
50°	+8	-4	0	+5	-3	+8	+5	+7
60°	+4	-5	-3			+6	+3	+2
70°	+4	-4	-5			+3	+3	+3
80°	+2	-3	-4			+1	+4	+2
90°	0		-2			-2	+2	+3

The average difference between the supercorrections found in the different intercomparisons is 0.0025; the average difference between the supercorrections found by the two observers in the same series of intercomparisons is 0.001. The fact that the supercorrections found at different times differ considerably more than the average difference between the observations of the two observers is, no doubt,

in part due to small inconsistencies in the ice-point determinations, but also, to some extent at least, to the fact that a thermometer may define a slightly different scale at different times, depending on the strains present in the glass, due to its past thermal history, and which modify the coefficient of expansion of the glass. Further evidence in support of this view is furnished by the results of the determinations of the fundamental intervals, as previously discussed. The variations in the value of the supercorrections found in different series of intercomparisons sometimes considerably exceed the limits of experimental error, and can hardly be explained by variations in the ice-point determinations on different dates, for it was extremely rare that the independent ice-point observations by the two observers differed by as much as 0.003 . In view of the precautions taken it is reasonably certain that contamination of the ice could play no part; and furthermore the ice-point readings after each temperature were made in the same ice for the entire group of thermometers, so that all would have been affected nearly equally, due to this cause. Notwithstanding the agreement between the ice-point readings of the two observers, it is nevertheless true that this reading may still be in error, due to the sticking of the meniscus in the capillary tube. While we have sought to minimize this source of error by supercooling the thermometer several degrees below 0°C before plunging it into the ice bath, thus insuring a rising meniscus, by cautiously tapping the thermometer with the finger, and by again determining the zero for each temperature after returning the thermometer to the comparison bath, we nevertheless feel that the discrepancies sometimes observed between the results obtained on different days are partly due to this cause.

Several of the thermometers, after their calibration and the determination of their pressure coefficients and fundamental interval, were compared with the standards of the International Bureau under the supervision of Dr. Guillaume, before being delivered to the Bureau of Standards. The certificates accompanying the thermometers give the following details of these intercomparisons:

Tonnelot No. 11801 was compared with standards Nos. 4327 and 4330 in February, 1896, at temperatures near to 10° , 30° , and 50° . Baudin No. 15962 was compared with standards Nos. 15201, 15273, 14674, and 14675 in October and

November, 1903, at temperatures near to 10° , 20° , 30° , 40° , and 50° . Baudin Nos. 16016, 16017, and 16018 were compared with standards Nos. 14674 and 14675 in October, 1903, at temperatures near to 50° and 60° . The comparisons were carried out in a horizontal position, with twenty readings of each thermometer, ten readings with divisions above and ten with divisions below the mercury thread. The zeros of all the thermometers were determined before and after each intercomparison temperature.

The results of these intercomparisons are given in Table XIX. In the columns B. I. are given the supercorrections to the several thermometers included in these intercomparisons, i. e., the corrections that must be applied to the temperature scales defined by them to reduce to temperatures on the mean verre dur scale of the International Bureau. For purposes of comparison, the supercorrections to the mean scale of the thermometers of the Bureau of Standards, as

TABLE XIX.

Supercorrections to Mean Verre Dur Scale and to Mean Scale of 16 Standards Intercompared.

Temp.	Baudin No. 15962		Baudin No. 16016		Baudin No. 16017		Baudin No. 16018		Tonnelot No. 1180	
	B. I.	B. S.	B. I.	B. S.	B. I.	B. S.	B. I.	B. S.	B. I.	B. S.
10°	+0.002	+0.003							-0.005	-0.004
20°	+ .007	+ .007								
30°	+ .005	+ .006							- .006	- .002
40°	+ .008	+ .008								
50°	+ .005	+ .008	+0.002	+0.008	+0.002	+0.005	+0.005	+0.007	- .008	- .002
60°			+ .008	+ .006	+ .007	+ .003	+ .003	+ .002		

found in the present investigation, are given in the columns B. S. An inspection of the results of these two independent series of intercomparisons shows that the resulting supercorrections are quite small, that they have the same sign in every instance, and that the agreement is at least as good as could be expected, the average difference being only about $+0^{\circ}.0014$.

V. RÉSUMÉ AND CONCLUSIONS.

In view of the number of thermometers included in the present investigation and the order of agreement between the supercorrections to the mean scale of these thermometers and the supercorrections to the *mean verre dur scale* as found by the Bureau International, we believe that the conclusion is fully warranted that the standard scale of temperature of the Bureau of Standards, in the interval 0° to 100° C., as defined by these thermometers and the supercorrections resulting from this investigation, is in agreement with the Hydrogen Scale of Temperature of the Bureau International to within the limits of accuracy at present attainable in mercurial thermometry (which may be put at about 0.002). This conclusion is further confirmed by the many subsequent intercomparisons of these thermometers, in small groups of three or four, which have been made incident to the work of testing thermometers submitted to the Bureau for standardization, and especially by the intercomparisons of a number of these standards with two sensitive platinum resistance thermometers, by Mr. E. F. Mueller and one of the authors,⁴⁰ who found for the transition temperature of sodium sulphate⁴¹ 32.484 . The final result of Richards and Wells⁴² was 32.483 , temperatures in both cases being expressed on the International Hydrogen Scale.

The small values of and the quite systematic variations in the supercorrections throughout the scale bear convincing evidence to the fact that the small differences between the scales defined by the thermometers are to a great extent at least true differences due to slight variations in the chemical and physical properties of the glass. The authors can not too strongly express their appreciation of the painstaking work that has been done on these thermometers by the International Bureau at different times over a period of twenty years, as is shown by the systematic correlation of the standardizations of these thermometers.

⁴⁰ Dickinson and Mueller: This Bulletin, 3, p. 641; 1907.

⁴¹ Kahlbaum sodium sulphate recrystallized five times; and sodium sulphate made from the carbonate and recrystallized five times.

⁴² Richards and Wells: Proc. American Acad., 38, p. 431; 1902.

The constants of the thermometers (calibration corrections, external and internal pressure coefficients, and fundamental interval), which were determined at various times from 1885 to 1903 by the International Bureau, were again redetermined by the authors. The changes in the calibration corrections, for the thermometers examined, were found to be within the limits of experimental error. The values of the pressure coefficients found by the authors differed in the average from the values found at the International Bureau by about 0.5 per cent (0.0000006 per mm of mercury), and the values of the fundamental interval by 0.002.

A comparator, adapted to the intercomparison of thermometers in vertical or horizontal position, was designed by the authors and constructed in the instrument shops of the Bureau, which gave, by means of electric heating and cold-water circulation, every desired refinement in temperature control and convenience and quickness in manipulation for work in the interval 5° to 95°. A primary standard barometer was constructed, with which was compared at frequent intervals the Fuess standard barometer used in the steam-point determinations.

The sixteen primary standard thermometers were intercompared in groups of six or seven thermometers at intervals of 10°, from 10° to 90°, in nine series. The supercorrections to reduce the scale defined by each of the thermometers to the mean scale defined by all are given in Table XVIII, page 715.

From the ice-point observations, made after the thermometers had been at 0° for some weeks preceding each series of intercomparisons, after each intercomparison temperature, and at the end of each series of intercomparisons after exposure to steam, the *depression curve* for the glass (*verre dur*) of which these thermometers are made was found to be

$$Z_0 - Z_t = 0.000\,930t + 0.000\,001\,300t^2$$

where Z_0 = ice-point reading corresponding to long-continued exposure at 0°, and Z_t = ice-point reading corresponding to t° . The depression curve for *verre dur* resulting from these experiments is intermediate between that found by Guillaume and by Thiesen, Scheel, and Sell, the depression corresponding to 100° ($Z_0 - Z_{100}$)

being $0^{\circ}106$, the corresponding values of $Z_0 - Z_{100}$ found by the above observers being $0^{\circ}0997$ and $0^{\circ}1147$, respectively.

The rate of recovery of the zero was found to be $0^{\circ}0015$, 3 minutes after, and $0^{\circ}0011$ 4 minutes after the thermometer was removed from the water bath. As has been previously found by Guillaume, the rate of recovery of the zero for the first few minutes after the thermometer is plunged into the ice bath is practically independent of the temperature to which it has just previously been exposed, at least for temperatures in the interval 20° to 100° .

A series of experiments were made to determine the magnitude of the possible sources of error arising from distillation of mercury from the end of the column and from the slight cooling of the end of the mercury thread which projects a short distance through and above the thin sheet rubber diaphragm that closes the top of the steam-point apparatus. To completely avoid this distillation, a simple device, called "the emergent steam heater" was added to the steam-point apparatus, by which the entire emergent part of the stem was kept at 100° . About two-thirds of the steam-point observations in the determinations of the fundamental intervals were made with the emergent stem heater. The value of the fundamental interval found in this way, however, does not differ appreciably from that found by the usual procedure, as the experiments referred to above show that, in the time required for two observers to make the necessary observations in steam, the somewhat lower reading at the steam point, due to neglecting the small stem correction for the exposed thread, just about compensates for the lowering of ice point (observed immediately after steam point) which is caused by the distillation of mercury. However, the convenience and satisfaction in working afforded by the emergent stem heater, and the fact that the ice-point readings obtained at different times can be better intercompared when distillation is avoided, has, in our opinion, fully justified its use.

It is possible that a somewhat greater concordance might be attained by the use of thermometers constructed of Jena 59^{III} borosilicate glass which has a zero depression about one-fourth that of verre dur. The resulting gain in accuracy would, however, not be as great as might be supposed. A careful study of the behavior of the thermometers in these intercomparisons gives evidence that the

factor which most limits the accuracy attainable in mercurial thermometry is the action of capillary forces, which manifest themselves especially in the determinations of the fixed points, where the resulting errors are not always eliminated by the limited number of observations usually made. This source of error, which is so troublesome when a stationary meniscus is being observed, is such as to render observations on a falling meniscus entirely untrustworthy. As a consequence of these forces the meniscus advances in a series of small and somewhat irregular steps, so that time-consuming micrometric measurements should be dispensed with, especially as the accuracy of estimation, for thermometers having fine graduations and an interval (0.1) of 0.6 to 0.8 mm, is about 0.002.

In conclusion the authors desire to acknowledge their indebtedness to Mr. E. F. Mueller, Mr. E. F. Wenderoth, and Mr. J. J. Crowe for their painstaking assistance in the work of computation and for all the barometer observations and reductions involved in the determinations of the fundamental intervals of the thermometers.

VI. APPENDIX.

CONSTANTS OF THERMOMETERS.

In the following pages are given the constants of the thermometers described in the present paper. The data concerning the dimensions of the thermometers, the description of the methods of calibration, and the tables of calibration corrections for the divisions at which these corrections were determined (every 2°) are taken from the certificates accompanying the thermometers, as furnished by the International Bureau. The remaining constants, viz, the external and internal pressure coefficient and the value of the fundamental interval of each thermometer, are those finally adopted as a result of the data obtained in the course of the present investigation as well as that given in the certificates.

CALIBRATION.

Division.—The equidistance of the divisions was examined with a dividing engine and found satisfactory.

Tonnelot Nos. 4331, 4332, 4334, 4335, 4336, 4623, and 4624.—Division into two parts of the interval $[0^{\circ}-100^{\circ}]$ with three different threads (two threads with 4623 and 4624), observed 10 times in each of the positions $[0.50]$ $[50-100]$. Division of the interval $[0.50]$ into 5 parts by three (two for 4623 and 4624) independent calibrations, computed separately. Calibration every 2° of each 10° interval, prolonged 2° at each end (-2° to $+12^{\circ}$, $+8^{\circ}$ to $+22^{\circ}$, etc.; for 4623 and 4624, first interval *only* prolonged to -2°).

Baudin Nos. 15555 and 15583.—Division into two parts of the interval $[0^{\circ}-100^{\circ}]$ with two different threads of 50° , observed 6 times in each of the positions $[0.50]$ $[50-100]$. Division of the interval $[0.50]$ into 5 parts. Calibration every 2° of each 10° interval, the first and last interval being prolonged to -4° and $+52^{\circ}$, respectively.

Baudin Nos. 15282, 15962, 15963, and Tonnelot No. 11801.—Determination of corrections at 20° , 40° , 60° , and 80° by division into 5 parts, with threads of 20° , 40° , 60° , and 80° , observed 6 times in each position. Calibration every 2° of each 20° interval. Corrections at -4° , -2° , and $+102^{\circ}$, $+104^{\circ}$ determined by supplementary calibrations.

Baudin Nos. 16016, 16017, and 16018.—Division into two parts of interval $[0.100]$ with two different threads of 50° , observed 6 times in each of the positions $[0.50]$ $[50.100]$. Division of the interval $[50.100]$ into 5 parts. Calibration every 2° of each 10° interval, the first and last intervals being prolonged to 48° and 102° , respectively, the corrections for the divisions in the neighborhood of $[0]$ were determined by supplementary calibrations.

Tables.—The corrections for intermediate points were obtained by graphic interpolation. The probable error of the corrections does not exceed ± 0.001 .

PRESSURE COEFFICIENTS.

The values of the pressure coefficients (β_0 and β_1) found in the present investigation, as well as those given in the certificates of the International Bureau, are given in Table I, page 678.

FUNDAMENTAL INTERVALS.

The values of the fundamental intervals of the thermometers found in the present investigation, as well as those given in the certificates, are given in Table VI, page 693.

SUPERCORRECTIONS TO MEAN VERRE DUR SCALE.

The supercorrections to reduce the scale defined by the thermometers (Tonnelot 11801, Baudin 15962, 16016, 16017, and 16018) to the mean verre dur scale of the Bureau International, as taken from the certificates, are given in Table XIX, page 717.

SUPERCORRECTIONS TO THERMOMETERS.

The supercorrections to reduce the scale defined by each thermometer to the mean scale defined by the 16 standard thermometers, as found from the intercomparisons already described, are given in Table XVIII, page 715.

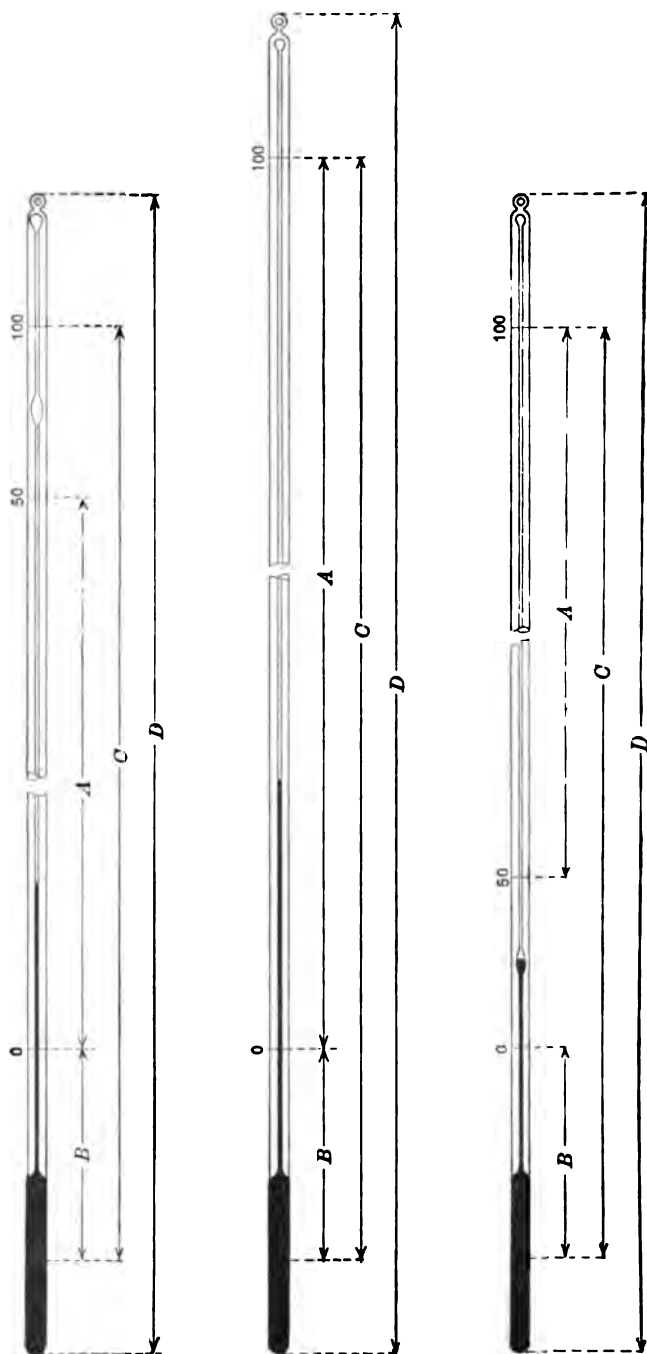


Fig. 8.—Types of Thermometers.

TABLE XXI.

Constants of Thermometers.

Therm. No.	Maker	Date of con- struction	Range	Dimensions *				
				A	B	C	Diam. of stem	Length of 1°
4331	Tonnelet..	July 1884	{ -4°3 to +51°8 and 94°2 to 104°0 C.	62.0	448.7	518.5	4.7	6.560
4332	"	"	{ -4°1 to +51°5 and 94°7 to 103°5 C.	60.0	461.3	527.0	5.0	6.908
4334	"	"	{ -4°1 to +51°5 and 94°3 to 103°2 C.	60.0	461.3	527.5	5.0	6.866
4335	"	"	{ -3°1 to +52°2 and 94°9 to 103°6 C.	55.0	461.0	529.0	4.8	6.926
4336	"	"	{ -3°8 to +51°3 and 94°2 to 103°4 C.	60.0	463.8	533.0	5.0	6.910
4623	"	June 1888	{ -3°3 to +52°0 and 95°1 to 102°6 C.	56.0	463.7	522.0	4.2	6.994
4624	"	"	{ -3°6 to +51°7 and 95°1 to 102°5 C.	57.0	464.3	522.0	4.3	7.024
11801	"	Nov. 1895	{ -4°7 to +103°1 C.	59.0	644.8	704.0	4.4	5.858
15282	Bandin ...	Apr. 1900	{ -5°0 to +104°0 C.	68.0	659.3	725.0	5.1	5.913
15555	"	Oct. 1901	{ -4°0 to +52°5 and 97°5 to 102°5 C.	63.0	539.7	606.0	4.7	8.252
15583	"	Dec. 1901	{ -4°1 to +52°6 and 97°4 to 102°6 C.	63.0	529.3	602.0	4.9	7.954
15962	"	July 1903	{ -5°1 to +104°1 C.	57.2	648.6	717.0	4.6	5.914
15963	"	"	{ -5°1 to +104°1 C.	56.8	644.6	712.5	4.9	5.878
16016	"	Nov. 1903	{ -2°6 to +2°6 and 47°4 to 102°6 C.	62.3	127.7	556.0	4.6	7.260
16017	"	"	{ -2°6 to +2°6 and 47°4 to 102°6 C.	62.2	125.8	551.5	5.1	7.236
16018	"	"	{ -2°6 to +2°6 and 47°4 to 102°6 C.	61.7	125.1	547.0	4.7	7.196

* The letters A, B, C refer to Fig. 8, on preceding page.

All of these thermometers are made of French hard glass (verre dur) and are graduated with equal distant divisions to $\frac{1}{10}$ ° C.

TABLE XXII.
Calibration Corrections.

Divs.	Tonnelot							Baudin	
	4331	4332	4334	4335	4336	4623	4624	15555	15553
-4								+0.0078	+0.0593
-2	+0.0292	-0.0039	-0.0047	-0.0545	-0.0052	+0.0119	+0.0253	+ .0025	+ .0305
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
+2	- .0118	+ .0018	+ .0074	+ .0449	+ .0067	- .0129	- .0091	+ .0039	- .0184
4	- .0135	+ .0058	+ .0138	+ .0797	+ .0091	- .0227	- .0147	- .0022	- .0204
6	- .0224	+ .0089	+ .0151	+ .0982	+ .0095	- .0404	- .0197	- .0160	- .0085
8	- .0373	+ .0069	+ .0082	+ .0971	+ .0072	- .0546	- .0260	- .0274	+ .0030
10	- .0481	+ .0085	- .0059	+ .0881	+ .0005	- .0646	- .0343	- .0244	+ .0291
12	- .0569	+ .0081	- .0155	+ .0891	- .0108	- .0687	- .0430	- .0153	+ .0500
14	- .0629	+ .0117	- .0200	+ .1057	- .0240	- .0728	- .0516	- .0140	+ .0478
16	- .0622	+ .0210	- .0247	+ .1251	- .0360	- .0774	- .0583	- .0085	+ .0449
18	- .0521	+ .0289	- .0335	+ .1408	- .0448	- .0755	- .0595	- .0006	+ .0368
20	- .0316	+ .0223	- .0425	+ .1490	- .0519	- .0669	- .0577	+ .0007	+ .0236
22	- .0144	+ .0013	- .0529	+ .1401	- .0534	- .0587	- .0505	- .0040	+ .0091
24	+ .0041	- .0172	- .0653	+ .1271	- .0455	- .0533	- .0458	- .0058	+ .0010
26	+ .0173	- .0364	- .0726	+ .1238	- .0339	- .0498	- .0396	- .0101	- .0138
28	+ .0320	- .0564	- .0881	+ .1197	- .0195	- .0486	- .0303	- .0209	- .0255
30	+ .0391	- .0768	- .1076	+ .1138	- .0041	- .0474	- .0173	- .0332	- .0257
32	+ .0417	- .0986	- .1202	+ .0995	+ .0145	- .0395	- .0043	- .0298	- .0047
34	+ .0400	- .1268	- .1276	+ .0850	+ .0300	- .0299	+ .0053	- .0193	+ .0325
36	+ .0420	- .1482	- .1272	+ .0695	+ .0364	- .0193	+ .0150	- .0003	+ .0791
38	+ .0460	- .1499	- .1231	+ .0411	+ .0328	- .0108	+ .0195	+ .0127	+ .1033
40	+ .0420	- .1417	- .1117	+ .0164	+ .0281	- .0028	+ .0204	+ .0222	+ .0873
42	+ .0272	- .1317	- .0925	- .0049	+ .0283	+ .0054	+ .0132	+ .0264	+ .0620
44	+ .0125	- .1140	- .0768	- .0132	+ .0336	+ .0132	+ .0065	+ .0198	+ .0378
46	- .0010	- .0820	- .0610	- .0151	+ .0334	+ .0145	+ .0005	+ .0209	+ .0232
48	- .0056	- .0422	- .0325	- .0148	+ .0252	+ .0139	- .0005	+ .0187	+ .0159
50	+ .0037	+ .0056	+ .0012	- .0065	+ .0077	+ .0131	+ .0027	+ .0105	+ .0030
52				+ .0065				- .0010	- .0079
96	+ .0888	- .1062	- .0680	- .0248	+ .0279	- .0156	+ .0177		
98	+ .0477	- .0492	- .0478	- .0124	+ .0126	- .0042	+ .0138	+ .0048	- .0232
100	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
102	- .0384	+ .0516	+ .0552	+ .0178	- .0100	- .0114	- .0203	+ .0096	+ .0166

TABLE XXIII.

Calibration Corrections.

Divs.	Ton.	Baudin					
	11801	15882	15962	15963	16016	16017	16018
- 4		-0.0162	-0.0333	-0.0501			
- 2	+0.0169	-.0077	-.0208	-.0181	+0.0193	+0.0646	-0.0669
0	.0000	.0000	.0000	.0000	.0000	.0000	.0000
+ 2	-.0154	+.0053	+.0183	+.0117	-.0160	-.0516	+.0541
4	-.0278	+.0118	+.0328	+.0104			
6	-.0267	+.0022	+.0422	+.0100			
8	-.0219	-.0158	+.0468	+.0053			
10	-.0092	-.0290	+.0428	-.0072			
12	-.0022	-.0294	+.0396	-.0318			
14	+.0067	-.0230	+.0317	-.0525			
16	+.0077	-.0175	+.0298	-.0697			
18	+.0096	-.0135	+.0264	-.0761			
20	+.0044	-.0101	+.0282	-.0801			
22	+.0033	-.0031	+.0270	-.0833			
24	+.0041	+.0117	+.0227	-.0859			
26	+.0043	+.0303	+.0200	-.0857			
28	+.0159	+.0440	+.0189	-.0820			
30	+.0169	+.0547	+.0203	-.0738			
32	+.0153	+.0495	+.0300	-.0640			
34	+.0020	+.0419	+.0450	-.0566			
36	-.0171	+.0366	+.0625	-.0455			
38	-.0437	+.0287	+.0813	-.0343			
40	-.0479	+.0179	+.0939	-.0262			
42	-.0477	+.0152	+.1042	-.0163			
44	-.0324	+.0118	+.1092	-.0149			
46	-.0102	+.0241	+.1088	-.0202			
48	+.0126	+.0413	+.1077	-.0152	+0.0386	+0.0511	-0.0395
50	+.0293	+.0581	+.1055	-.0099	+.0256	+.0258	-.0143
52	+.0392	+.0563	+.1004	-.0093	+.0170	+.0209	+.0026

TABLE XXIII—Continued.

Calibration Corrections.

Dive.	Ton.	Baudin					
	11801	15362	15962	15963	16016	16017	16018
48	+0.0126	+0.0413	+0.1077	-0.0152	+0.0386	+0.0511	-0.0395
50	+ .0293	+ .0581	+ .1055	- .0099	+ .0256	+ .0258	- .0143
52	+ .0392	+ .0563	+ .1004	- .0093	+ .0170	+ .0209	+ .0026
54	+ .0502	+ .0509	+ .0960	- .0220	+ .0075	+ .0202	+ .0100
56	+ .0538	+ .0403	+ .0939	- .0397	+ .0025	+ .0170	+ .0082
58	+ .0596	+ .0352	+ .0887	- .0595	- .0024	+ .0175	- .0084
60	+ .0636	+ .0301	+ .0790	- .0790	- .0010	+ .0132	- .0273
62	+ .0651	+ .0209	+ .0685	- .0944	+ .0067	+ .0158	- .0448
64	+ .0720	+ .0175	+ .0573	- .1028	+ .0130	+ .0190	- .0590
66	+ .0800	+ .0083	+ .0492	- .1035	+ .0143	+ .0218	- .0696
68	+ .0809	- .0094	+ .0415	- .0986	+ .0145	+ .0190	- .0762
70	+ .0794	- .0365	+ .0335	- .1039	+ .0198	+ .0195	- .0827
72	+ .0813	- .0590	+ .0248	- .1070	+ .0209	+ .0092	- .0821
74	+ .0802	- .0765	+ .0162	- .1049	+ .0194	+ .0084	- .0803
76	+ .0922	- .0758	+ .0055	- .1015	+ .0198	+ .0135	- .0739
78	+ .0922	- .0739	- .0079	- .1010	+ .0180	+ .0157	- .0722
80	+ .0733	- .0755	- .0222	- .1032	+ .0159	+ .0163	- .0682
82	+ .0611	- .0726	- .0339	- .1101	+ .0157	+ .0118	- .0639
84	+ .0492	- .0643	- .0373	- .1035	+ .0085	+ .0076	- .0622
86	+ .0323	- .0519	- .0367	- .1011	+ .0009	+ .0017	- .0581
88	+ .0244	- .0315	- .0349	- .1021	.0000	+ .0011	- .0437
90	+ .0166	- .0100	- .0303	- .0907	+ .0037	+ .0039	- .0298
92	+ .0108	+ .0057	- .0249	- .0794	+ .0136	+ .0035	- .0146
94	+ .0044	+ .0178	- .0199	- .0593	+ .0137	+ .0099	- .0058
96	- .0025	+ .0259	- .0159	- .0365	+ .0117	+ .0066	- .0058
98	- .0013	+ .0175	- .0116	- .0107	+ .0062	- .0009	- .0021
100	.0000	.0000	.0000	.0000	.0000	.0000	.0000
102	+ .0188	- .0072	+ .0153	+ .0016	- .0120	- .0026	+ .0014
104		- .0083	+ .0311	- .0108			

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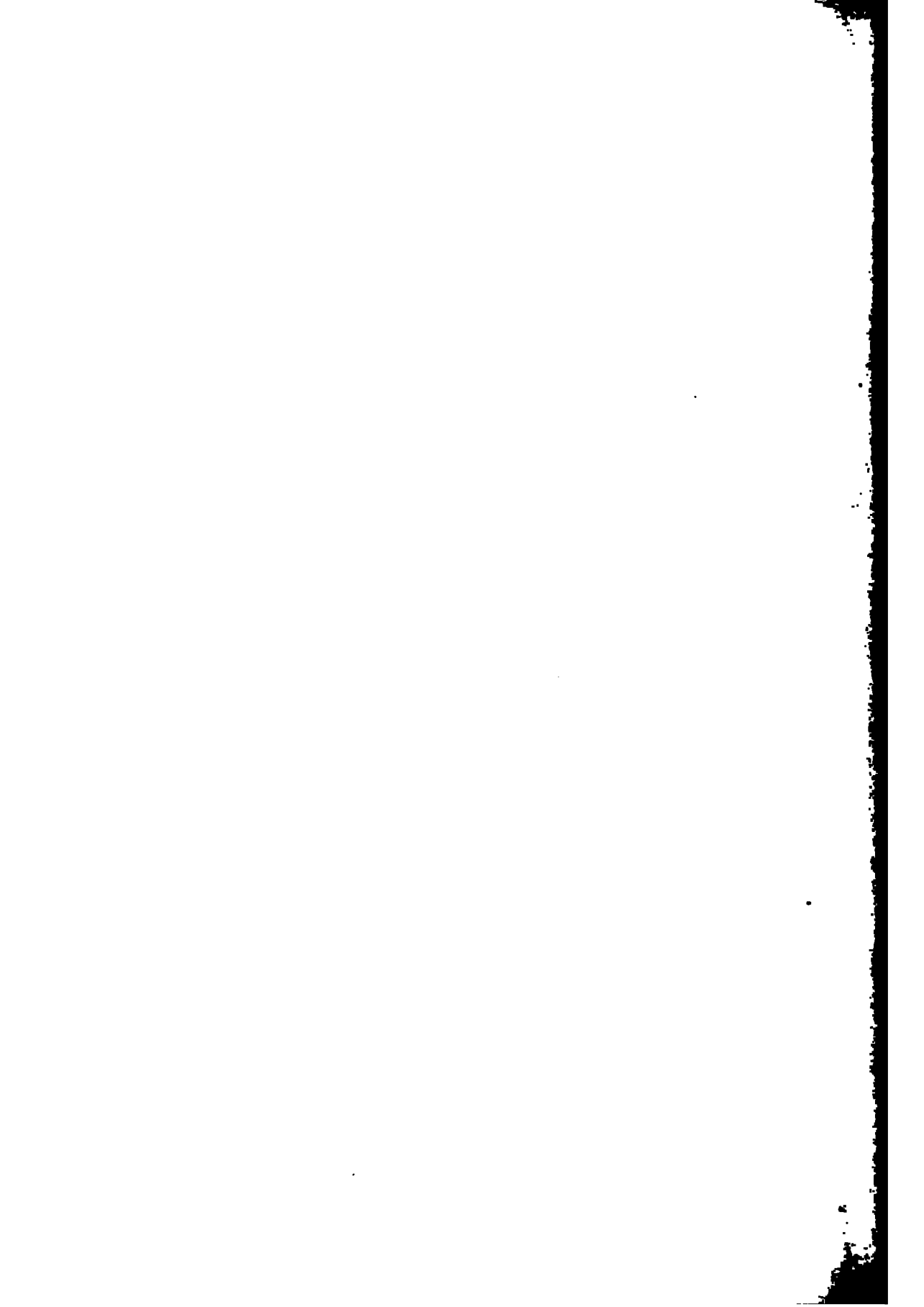
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